Static and Dynamic Properties of Bose-Einstein Condesates

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Abstract

The behaviour of superfluids is a fascinating topic in it's own right. I shall be discussing and analysing Bose-Einstein Condensates(BEC) which are macroscopic quantum objects that can be modelled as superfluids. I shall analysis the governing equation of BEC's the Gross Piteavskii Equation(GPE). I shall model the GPE in a harmonic trap, to find the critical parameter for the stabililty of the system. I shall then look at the dynamical properties of BEC to see how perturbations can effect the system. Which then leads into the behaviour of two superfluids and how they interact with each other depending on there different initial values of Phase and population difference in what are known as the Josephson Effects.

Chapter 1

An Introduction to Bose-Einstein Condensates

1.1 Introduction to Quantum Mechanics of Bose-Einstein Condensates(BEC)

Quantum Mechanics which generally applies to particles on a very small scale. The energy of the particles is a quantised amount. Quantum mechanics also treats the particle to have both particle-like and wave-like properties.
Max Planck was one of the Father of quantum theory and won a Nobel Prize in Physics in 1918. The Planck's Relation tells us that the Energy of a particle is quantised.

$$E = hv\,, \tag{1.1}$$

It follows that the particles at very low energy states are in bands. In a harmonic oscillator it is key feature in quantum mechanics. That where E is the energy of the particle, h is Planck's constant and v is the frequency of the particle. [3]
Louis de Broglie was a French physicist who wrote a PhD thesis on the wave properties of particles and won a Nobel Prize in Physics in 1929. The wave-like properties of matter are such that they have a wavelength defined using the de Broglie wavelength

$$\lambda = \frac{h}{p} \tag{1.2}$$

where λ is the wavelength, p is the momentum and h is Planck's constant. It tells us the probability of finding a particle in a given area as a wavefunction.

By then substituting momentum in terms of the kinetic energy $E_K = p^2/2m$ where E_K is the kinetic energy and m is the mass, we get

$$\lambda = \frac{h}{\sqrt{2mE_K}} \,. \tag{1.3}$$

When we then take the system to be an ideal gas at a specific temperature, we can set $E_K = \pi kT$ to give

$$\lambda = \frac{h}{\sqrt{2\pi mkT}} \,, \tag{1.4}$$

where k is the Boltzmann constant and T is the temperature of the gas. This infers $\lambda \propto 1/\sqrt{T}$. [16] Another key principle in the formation of BEC is is Heisenberg's uncertainty principle.

$$\Delta x \Delta p \geq \frac{\hbar}{2} \,, \tag{1.5}$$

This equation is named after the German theoretical physicist Werner Heisenberg who is considered to be one of the fathers of quantum mechanics. [7]

This equation tells us it is impossible to tell the exact postion and momentum of a particle simultaneously no matter how precise the measurement. This is known as 'quantum fuzziness'.

1.1.1 Schrödinger's Equation

An Austrian Physicist named Erwin Schördinger published the famous Schrödinger Equation in 1926. It in the fundamental equation of quantum mechanics. It describes how a quantum state of a physical system changes with time.

$$\imath\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi \tag{1.6}$$

It is derived by taking the conservation of energy $E = E_{kin} + E_{pot} = \frac{1}{2}mv^2 + V = p^2/2m + V$ then using a wavefunction for the description of the particles $\Psi(x,t)$ the using quantum operators the Schrödinger equation can be deduced.

1.2 Introduction to Bose-Einstein Condensates

The idea of Bose-Einstein condensates (BEC) came from an Indian Physicist Satyendra Nath Bose who wrote an article called 'Planck's Law and the hypotheses of Light Quanta'[4] which discussed how photons are indistinguishable from each other. He then told Albert Einstein about his discovered and he agreed with him and subsequently generalized the theory.

1.2.1 Formation of BEC

The formation of BEC can be described using the previously mentioned principles.
As previously shown $\lambda \propto 1/\sqrt{T}$. Thus as the temperature is lowers the wavelength gets very large. When they become sufficiently large such that the distance between the particles and the wavelength are similar, the wavelengths overlap. The particles become indistinguishable from each other and in the case of very colds bosons they are all in the same quantum state and act as one. The particles becomes a macroscopic quantum fluid. The theory of BEC was first published in 1924. Fritz London proposed in April 1938 however a pure BEC was not created experimentally until 1995 by Eric Cornell, Carl Wieman and Wolfgang Ketterle who used rubidium atoms to create the first BEC this was done at temperatures of 170 nanokelvin. [9] Cornell, Wieman and Ketterle went on to win the Nobel Prize in Physics in 2001. [5] The way they reached these temperatures was through a system of cooling which I shall describe later in the chapter.

Figure(1.1) shows the velocity-distribution of rubidium atoms when they are released from the trap where white is the slowest and red is the fastest. It tells us what energy state the particles were in which gives the required information to see the formation of a BEC. The three stages from left to right 1) $T > T_c$ the particles are in a thermal cloud with the particles spread over several different energy levels. 2) The temperature is around 185 nanokelvin $T \approx T_c$ and as we can see there is a large peak at the lowest energy level which is the formation of a BEC, however there are still a lot of particles in other energy states this is the beginning of the formation of the BEC. 3) This is at a temperature of 168 nanokelvin $T < T_c$ and as you can see almost all the particles are in the same energy state this is the formation of the BEC as all of the particles are acting collectively. [5]

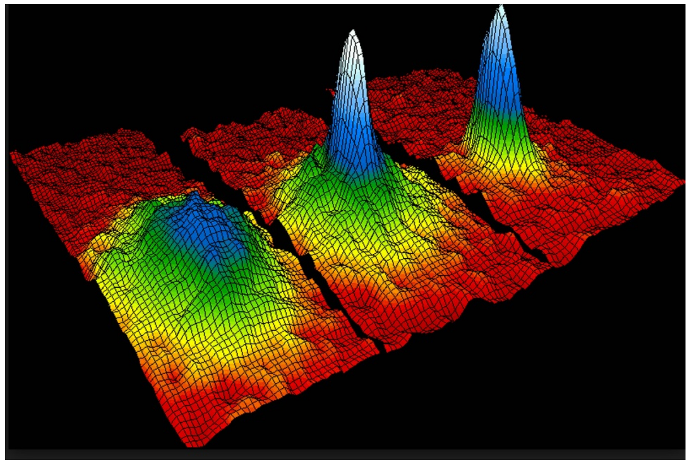
formation.jpg 

Figure 1.1: Velocity distribution of three different stages in the formation of a Bose-Einstein Condensates

[5]

1.3 Cooling Methods

The particles must be in a vacuum for the BEC to form. As previously stated the particles have to be extremely cold in order to get to this stage it takes three different processses.

1) Laser Cooling

This technique was developed in 1985 with the work of Steven Chu and others. It works by firing photons at the particles so that the particles absorb the photons and re-emit them in a random direction. The way it cools down the atoms using the doppler effect: if the atom is stationary then the photon passes through the atom and has not effect. If the atoms are travelling in the same direction as the photons then the atom sees the photon as red-shifted, the photon is not absorbed when this happens and will not effect particle. However when the photon is blue-shift and the particle is travelling towards the photon it will absorb it and re-emit it in a random direction, meaning that the net force of the absorbtion is in the opposite direction of the velocity of the particle thus slowing it down and cooling it down. This gets the particles to a temperature of 10^{-9} Kelvin. It can not get any lower than this as the re-imitation of the photon will cause the atom to move a

certain amount. [8]
2) TOP Trap
After the particles cooled by the laser. The next stage is to trap them in a TOP (time-averaged orbiting potential) trap. The trap has a pointy bottom which is rotated giving a rounded confining potential, which is parabolic. This creates a harmonic trap for the atoms. [6]
3) Evaporative Cooling
This cooling method is often described using the analogy of a cooling cup of coffee because when a cup of coffee cools it is mostly done by the particles will the most energy evaporating off the liquid then the coffee inside the cup rethermalizes, and has less energy in the cup and becomes colder. How this is done with the particles; the height of the harmonic trap is lowered and the particles will more energy are jumping out. The particles inside rethermalize and the average energy of the particles in the trap is less. This cools the atoms sufficiently to get to temperture that will create a BEC. [6]
Figure(1.2) shows how the particles are in a harmonic trap and the how it

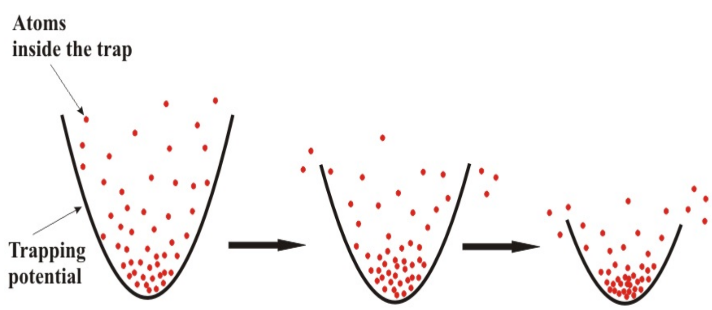
Cooling.jpg


Figure 1.2: Process of Evaporative Coolilng

is made shallower so the more energetic particles jump out, then the atoms re-thermalize with a lower average energy cooling the system.

1.4 Summary Of Report

In this report I shall discuss several different aspects of BEC:
1) Modelling the Behaviour of the BECs
Analysis of the Gross Pitaevskii Equation (GPE) and behaviour of BEC with variable interaction strength.
2) Static Properties and Stability of BECs

Using an Gaussian approximation to model the shape of the BEC and the analysis of equilibrium properties by using a variational method of modelling BECs.

3) Dynamics of BECs

Analysis of fluid properties of BECs. Analysis of perturbations within BEC in linearized systems.

4) Josephson Effects Looking into the interactions between two wealkly coupled BECs. Modelling the Josephson effects with variable phase and population.

Chapter 2

Static Properties of Bose-Einstein Condensates and Introduction to Gross-Pitaevskii Equation

2.1 Introduction to Time-Independent Gross-Pitaevskii Equation

The Gross-Pitaevskii equation(GPE) is the governing equation in the behaviour of Bose-Einstein Condensates(BEC). It was discovered by Eugene P. Gross and Lev Petrovick Pitaevskii. It describes bosons in a fully condensed state in their lowest state assuming the wave functions is a symmetrical product of the single-particle wave functions. As all the particles are in the same state and be treated as single a wavefunction [13];

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N) = \prod \phi(\mathbf{r}_i)\,, \tag{2.1}$$

where $\phi(\mathbf{r}_i)$is normalized by

$$\int d\mathbf{r}|\phi(\mathbf{r})|^2 = 1\,. \tag{2.2}$$

This wavefunction can be modelled using by the time-independent GPE which is the same as the previouly mentioned Schrödinger Equation (1.6) but with a interaction strength term $g|\Psi(\mathbf{r})|^2\Psi(\mathbf{r})$,

$$\mu\Psi(\mathbf{r}) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}) + g|\Psi(\mathbf{r})|^2\right)\Psi(\mathbf{r})\,. \tag{2.3}$$

where μ is the chemical potenial, $\hbar$ is Plank's constant $(6.626x10^{-34}J\dot{s})$, m is the mass of the boson, V is the external potential, $|\Psi(\mathbf{r})|^2$ is the density of the particles and $g = 4\pi\hbar^2 a_s/m$ where a_s is the scattering length. The scattering length tells us the effective interaction parameter. [11]

2.1.1 Time-Dependent Gross-Pitaevskii Equation

The time-dependent GPE it very useful is the development of the wavefunction with time. This is useful in analysing perturbations in the system seeing the behaviors and developments of the BEC.

$$\imath\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r},t) + g|\Psi(\mathbf{r},t)|^2\right)\Psi(\mathbf{r},t)\,. \tag{2.4}$$

2.1.2 Meaning of the Terms in the GPE

The $-\hbar^2/2m\nabla^2\Psi(\mathbf{r}$ is the kinetic energy term of the equation and can be worked out by

$$KE = mv^2/2 = \frac{p^2}{2m} \tag{2.5}$$

and the using the momentum operator $\hat{p} = -\imath\hbar\frac{\partial}{\partial x}$ and substiting in gives

$$-\frac{\hbar^2}{2m}\nabla^2\Psi(\mathbf{r}) \tag{2.6}$$

The $V(\mathbf{r})\Psi(\mathbf{r})$ is the potenial energy term. The first two terms give an equation is the Schrodinger equation however this does not account for the interaction actions between atoms which is the third term $g|\Psi(\mathbf{r})|^2\Psi(\mathbf{r})$ when $g > 0$ the particles have repulsive interactions and if $g < 0$ the particles have attractive interactions [**?**]

2.1.3 Dimensionless GPE

The GPE can be made dimensionless and this make the equation easy to work with when adding a perturbation. By setting $x = x'\xi, t = \frac{\hbar}{\mu}t', \psi =$

$\psi_\infty(g|\psi_\infty^2| = \mu), V = \mu V'$ and substuting these into the time-dependent GPE gives

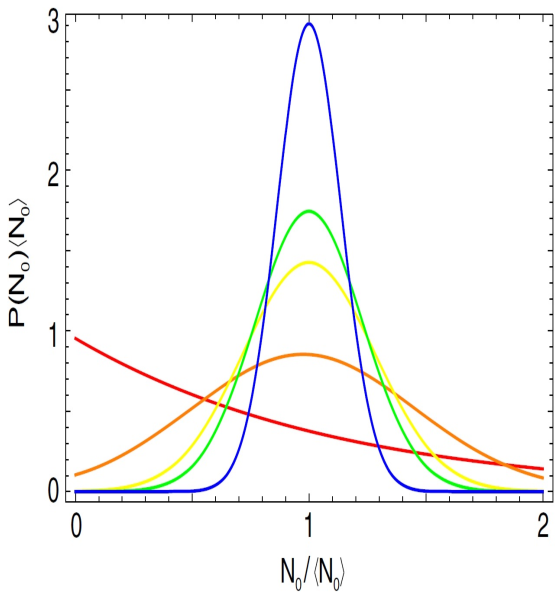
$$\imath\mu\Psi_\infty \frac{\partial \Psi_\infty \Psi'}{\partial t'} = \left(-\frac{\hbar^2}{2m}\frac{1}{\xi^2}\frac{\partial^2}{\partial x'^2} + \frac{V'}{\mu} + g|\Psi_\infty|^2|\Psi|^2 \right) \Psi_\infty \Psi' . \tag{2.7}$$

This shows simplifies to give

$$\imath \frac{\partial \Psi'}{\partial t'} = \left(-\frac{1}{2}\frac{\partial^2}{\partial x'^2} + V' + |\Psi'|^2 \right) \Psi' , \tag{2.8}$$

which is the Dimensionless GPE. [11]

2.1.4 Graphical Representation of Cloud Radius with variable Interaction Strength



radius.jpg

Figure 2.1: Radius of the Cloud with variable Interaction Strength

FIG. 2.1 (color online). 'Typical plot of the probability distribution for two-component bosons for a fixed interaction strength g 5 106 and different condensate fractions (from bottom to top) x red 0.04, x orange 0.28, x yellow 0.40, x green 0.45, and x blue 0.58.' Ref:Interaction Effects on Number Fluctuations in a Bose-Einstein Condensate of Light In all of the cases a BEC is formed meaning all of the particles are in the lowest energy state. Diagram(2.1) shows how the cloud radius decreases as the interaction strength increases. This is because when there is a repulsive interaction the particles are pushing away from each other making the cloud wider and shallower. As the interaction strength becomes more and more attractive the particles become closer and the distribution becomes taller and narrower. Ψ is normalised so the area under the curve is always the same as the number of particles in the BEC does not change based on the interaction strength.

Chapter 3

Variational Study of Gross-Pitaevskii Equation(GPE) and Modelling Stability

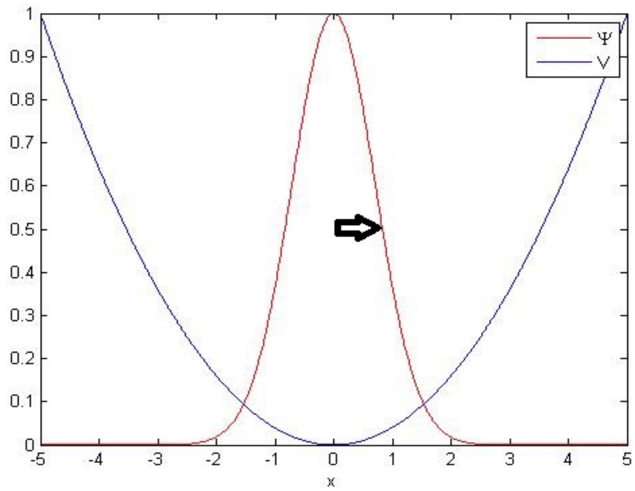
In this section I shall discuss the study of the equilibrium properties by the means of a variational method based the interaction strength of the cloud using a Gaussian approximation for the BEC.

3.1 Variational Method

As previously mentioned almost all of the atoms are in the ground state. The Thomas-Fermi approximation is an accurate representation for a sufficiently large cloud that is in the ground-state of a harmonic oscillator it neglects the kinetic energy term . I am using a Gaussian approximation with an effective width (as the variational parameter). So the motion of the particle in a harmonic potential and the wavefunction in 1D is

$$\psi(x) = \left(\frac{m\omega_x}{\pi\hbar}\right)^{1/4} e^{-m\omega_x x^2/2\hbar} . \tag{3.1}$$

[11] This Gaussian function is a good approximation of the ground state wavefunction. As the probability is greatest in the middle and decreases as it moves away as shown.



model.jpg

Figure 3.1: Radius of the Cloud with variable Interaction Strength

Where the arrow indicates the harmonic oscillator length $a_x = \sqrt{\hbar/m\omega_x}$. Diagram(3.1) shows how it has a very similar shape to what is expected from a BEC in the lowest state in a harmonic oscillator with a variable harmonic oscillator length.

3.2 3D GPE

I am now going to futher this approximation to a 3D variational approach to modelling a BEC in a trapped harmonic oscillator, in order to do this I have changed the system in the following ways $\mu = g|\psi(\mathbf{r})|^2 = gn$ where is the density of the atoms ($|\psi(\mathbf{r})|^2 = n$). The potential becomes

$$V(x, y, z) = \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)\,, \tag{3.2}$$

where ω_i is the oscillation frequency.
The wave function in 3D can be approximated using

$$\psi(\mathbf{r}) = \frac{N^{1/2}}{\pi^{3/4}(b_x b_y b_z)^{1/2}} e^{-x^2/2b_x^2} e^{-y^2/2b_y^2} e^{-z^2/2b_z^2}\,. \tag{3.3}$$

In order to get an energy of the state a energy functional can be found by using the effective Hamiltonian. The energy functional of the system is

$$E(\psi) = \int d\mathbf{r} \left[\frac{\hbar^2}{2m} |\nabla\psi(\mathbf{r})|^2 + V(\mathbf{r})|\psi(\mathbf{r})|^2 + \frac{1}{2} g |\psi(\mathbf{r})|^4 \right] . \tag{3.4}$$

[11]

3.3 Stability of BEC

In this section I shall use the variational approximation of a BEC in 3D to find the critical value for the interaction strength between the particles in the system for a stable BEC to exist.
As previously mentioned when $g > 0$ the interatomic interactions are repulsive and the BEC will be stable and when $g < 0$ the interatomic interactions are attractive. However there must be a value for $g < 0$ when a BEC does not form as the interatomic interactions are too strong and the cloud is no longer stable. When the wavefunction approximation(3.3) is substituted into the energy functional (3.4) it can be shown to give

$$E(b_1, b_2, b_3) = N \sum_{i=0} \frac{\hbar\omega_i}{2} \left(\frac{a_i^2}{4b_i^2} + \frac{b_i^2}{4a_i^2} \right) + \frac{N^2 g}{2(2\pi)^{3/2} a_1 a_2 a_3} , \tag{3.5}$$

where $a_i^2 = \hbar/m\omega_i$. By setting $x_i = b_i/a_i$ and then minimizing E with respect to each of the terms $(1,2,3) = (x,y,z)$

$$\frac{\partial E}{\partial x_1} = N\hbar\omega_1 \left(-\frac{1}{4x_1^2} + \frac{x_1^2}{4} \right) + \frac{N^2 g}{2(2\pi)^{3/2} a_1 a_2 a_3 x_1 x_2 x_3} = 0 . \tag{3.6}$$

By simplifying and setting

$$a_1 a_2 a_3 = \bar{a}^3 = \left(\frac{\hbar}{m} \right)^{3/2} \sqrt{\frac{1}{\omega_1 \omega_2 \omega_3}} = \left(\frac{\hbar}{m} \right)^{3/2} \sqrt{\frac{1}{\bar{\omega}}} = \left(\frac{\hbar}{m\bar{\omega}} \right)^{3/2} . \tag{3.7}$$

This gives

$$\frac{\hbar\omega_1}{2} \left(x_1^2 - \frac{1}{x_1^2} \right) - \frac{N^2 g}{2(2\pi)^{3/2} \bar{a} x_1 x_2 x_3} = 0 , \tag{3.8}$$

this is true for all $\imath$, we write more generally

$$\frac{\hbar\omega_i}{2} \left(x_i^2 - \frac{1}{x_i^2} \right) - \frac{N^2 g}{2(2\pi)^{3/2} \bar{a} x_1 x_2 x_3} = 0 . \tag{3.9}$$

Taking the system to be isotropic $b/a(= b_1/a_1 = b_2/a_2 = b_3/a_3)$ if you then expand (3.5) and then scaling to $\sigma = b/a$ gives

$$E(\sigma) = N\hbar\omega\left(\frac{3}{4\sigma^2} + \frac{3\sigma^2}{4}\right) + \frac{N^2 4\pi\hbar^2 a_s}{2(2\pi)^{1/2}a^3\sigma^3 m}\,. \tag{3.10}$$

$a_{osc} = \sqrt{\hbar/m\omega}$ this is the characteristic quantum-mechanical length scale for the harmonic oscillator. Subsituting these into the previous expression gives

$$E(\sigma) = N\hbar\omega\left(\frac{3}{4\sigma^2} + \frac{3\sigma^2}{4}\right) + \frac{1}{\sqrt{2\pi}}\left(\frac{Na_s}{a_{osc}}\right)\frac{1}{\sigma^3}\,. \tag{3.11}$$

To find the critical value for the interaction strength for a single particle we take the energy of given particle then take the first and second derivatives of the following expression

$$\frac{E}{N} = \hbar\omega\left(\frac{3}{4\sigma^2} + \frac{3\sigma^2}{4}\right) + \frac{1}{\sqrt{2\pi}}\left(\frac{Na_s}{a_{osc}}\right)\frac{1}{\sigma^3}\,. \tag{3.12}$$

The first derivative is the following,

$$\frac{\partial}{\partial\sigma}\left(\frac{E}{N}\right) = \hbar\omega\left(-\frac{6}{4}\sigma^3 + \frac{6\sigma}{4} - \frac{3}{\sqrt{2\pi}}\left(\frac{Na_s}{a_{osc}}\right)\right) = 0\,, \tag{3.13}$$

which simplifies to give

$$-\frac{\sigma}{2} + \frac{\sigma^5}{2} - \frac{1}{\sqrt{2\pi}}\left(\frac{Na_s}{a_{osc}}\right) = 0\,. \tag{3.14}$$

For the second dervative

$$\frac{\partial^2}{\partial\sigma^2}\left(\frac{E}{N}\right) = \hbar\omega\left(\frac{18}{4\sigma^4} + \frac{6}{4} + \frac{12}{\sqrt{2\pi}}\left(\frac{Na_s}{a_{osc}}\right)\frac{1}{\sigma^5}\right) = 0\,, \tag{3.15}$$

which simplifies to give

$$\frac{3}{2\sigma^4} + \frac{1}{2} + \frac{4}{\sqrt{2\pi}}\left(\frac{Na_s}{a_{osc}}\right)\frac{1}{\sigma^5} = 0\,. \tag{3.16}$$

If you then subsitute in for Na_s/a_{osc} and solve (3.16) and (3.14) simultaneously we get

$$\sigma = \pm\sqrt[4]{1/5}\,. \tag{3.17}$$

As we are looking $g < 0$ which the gives the critical value;

$$\frac{Na_s}{a_{osc}} = \frac{2\sqrt{2\pi}}{5^{\frac{5}{4}}} \approx -0.67\,. \tag{3.18}$$

[11]

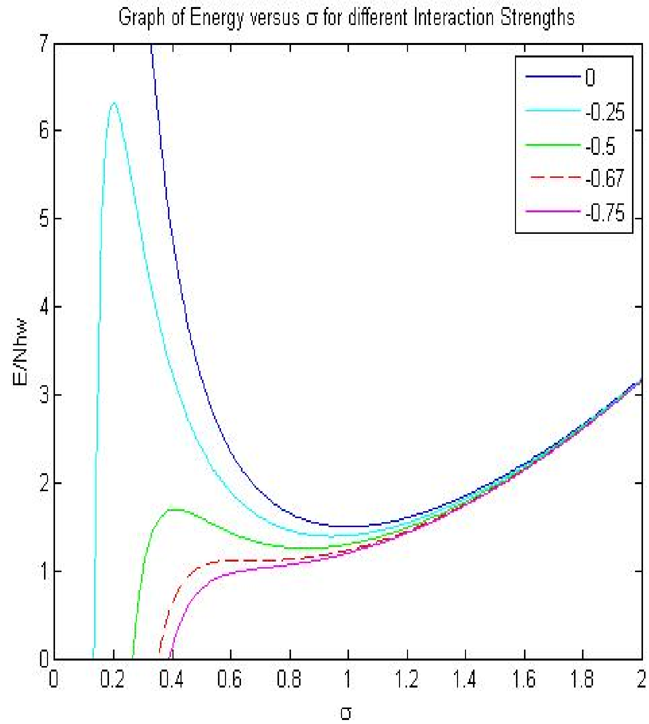


Figure 3.2: Graph of Energy Functional against variable σ

The key in this graph (3.2) gives different values Na_s/a_{osc}. It shows that when Na_s/a_{osc} at -0.67 there was a point of inflection there making this the critical value for the system. When $Na_s/a_{osc} > -0.67$ there is a clear minimum value which the BEC will be stable at. When $Na_s/a_{osc} < -0.67$ there is not a minimum value so the BEC will be unstable. It is intuitive that a system that has a repulsive interaction will have a stable solution as the particles are repelling each other and are unlikely to form molecules that will destroy the system. The system will be flatter and broader as the particles are repelling each other. When the particles are weakly attractive they can still be in a stable system as the 'quantum fuzziness' of the system at the ground state and will form a taller and narrower cloud. This continues to get narrower and thinner until the particles attraction is too great and the system collapses.

3.4 Experimental Evidence

An experiment was done on the controlled collaspe of a Bose-Einstein Condensate. It was studied using a stable Rubidium-85 condensate which can have the atom-atom interactions changed by using Feshbach resonance to vary the scattering length this is a magnetic field. It was predicted in this experiment that the BEC would be unstable for certain strength of attractive interaction. They found that the critical value for the system should be

$$\frac{Na_s}{a_{osc}} = 0.574\,. \tag{3.19}$$

They then ran the experiment with several different experiments with each dot representing a different experiment. Figure (3.3) shows a graph showing

value experiment.jpg

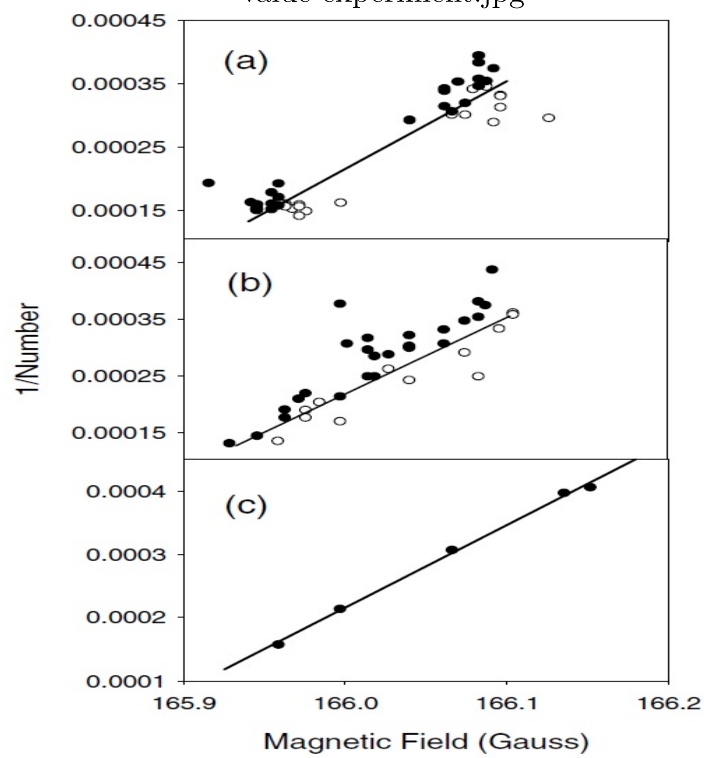


Figure 3.3: Graph Indicating the Point of collaspe for Interaction Strengths

many different interaction strength close to the critical values and the black dots show a stable BEC with no collaspe and the white dots show a collaspe of the BEC. It tell us the the critical values. From this experiment it observed that there are stable solutions with sufficiently small attractive interactions. The exact point of collaspe was measured to be $Na_s/a_{osc} = 0.459 \pm 0.012 \pm 0.054$ which is 25% less than predicted. [12]

Chapter 4

Superfluidity

Now we have looked at the static stability of the BEC I shall now look at the dynamic properties of the system and see what happens when a perturbation put into the system.

4.1 GPE vs Hydrodynamic Equations

To study link between the GPE and the widely known fluid equations, we can make use of the Madelung transformation.

$$\Psi(\mathbf{r}, t) = \sqrt{n(\mathbf{r}, t)} e^{\imath\theta(\mathbf{r}, t)} \,, \tag{4.1}$$

as the particles have wave-like properties so they have an amplitude $|\Psi(\mathbf{r}, t)| = \sqrt{n(\mathbf{r}, t)}$ and a phase of $\theta(\mathbf{r}, t)$. [15] Substiting this in to the time-dependent GPE

$$\imath\hbar \frac{\partial \Psi}{\partial t} = \left(-\frac{\hbar}{2m} \frac{\partial^2}{\partial x^2} + V + g|\Psi(\mathbf{r}, t)|^2 \right) \Psi(\mathbf{r}, t) \tag{4.2}$$

and splitting the Real and Imaginary parts gives

<u>Imaginary</u>

$$\sqrt{n}_t = -\frac{1}{2m} \left(\sqrt{n}\theta_{xx} + 2\sqrt{n}_x \theta_x \right) \,, \tag{4.3}$$

$$\frac{\partial \sqrt{n}}{\partial t} = \frac{1}{2\sqrt{n}} \frac{\partial n}{\partial t} \,, \tag{4.4}$$

$$m \frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(n\theta_x) = 0 \,, \tag{4.5}$$

and as the mass density $\rho = mn$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(n\theta_x) = 0\,. \tag{4.6}$$

This is the continuity equation which describes the movement of a conserved quantity of fluid.
<u>Real</u>

$$\theta_t = \frac{1}{2m}\frac{1}{\sqrt{n}}\sqrt{n}_{xx} - \frac{1}{2m}(\theta_x)^2 - V + gn\,, \tag{4.7}$$

and setting

$$\frac{\partial}{\partial x}\left(\frac{\partial \theta}{\partial t}\right) = \frac{\partial v}{\partial t}\,, \tag{4.8}$$

which gives

$$m\frac{\partial v}{\partial t} = -\nabla\left(\frac{1}{2}mv^2 + V + ng - \frac{\hbar^2}{2m\sqrt{n}}\nabla^2\sqrt{n}\right)\,. \tag{4.9}$$

Then setting the quantum pressure as $P = gn^2 - \frac{\hbar^2}{2m\sqrt{n}}\nabla^2\sqrt{n}\,.$

These equations are the Euler equation and the continuity equation. [15]

4.2 Excitation Spectrum

If we take the dimensionless time-dependent GPE

$$\imath\frac{\partial \Psi(\mathbf{r},t)}{\partial t} = \left[-\frac{\nabla^2}{2} + V + g|\Psi(\mathbf{r},t)|^2\right]\Psi(\mathbf{r},t)\,, \tag{4.10}$$

then add a perturbation to the system at rest the wavefunction becomes $\Psi_0 + \delta\Psi$ where Ψ_0 is the wave function at equilibrium and $\delta\Psi$ is the perturbation to the system. It can be shown

$$|\Psi|^2\Psi \approx |\Psi_0|^2\Psi_0 + 2|\Psi_0|^2\delta\Psi + \Psi_0^2\delta\Psi^*\,. \tag{4.11}$$

If this is substituted in the time dependent GPE and linearizing gives

$$\imath\frac{\partial(\delta\Psi)}{\partial t} = -\frac{1}{2}\frac{\partial^2(\delta\Psi)}{\partial x^2} + V\delta\Psi + 2g|\Psi_0|^2\delta\Psi + g\Psi_0^2\delta\Psi^* \tag{4.12}$$

There is a similar expression for $\delta\Psi^*$,
then by setting

$$\delta\Psi(x,t) = e^{-\imath\mu t}\left[u(x)e^{-\imath\omega t} + v^*(x)e^{\imath\omega t}\right], \tag{4.13}$$

$$\delta\Psi^*(x,t) = e^{-\imath\mu t}\left[u^*(x)e^{\imath\omega t} + v(x)e^{-\imath\omega t}\right] \tag{4.14}$$

and

$$\Psi_0(x,t) = e^{-\imath\mu t}\sqrt{n_0} \tag{4.15}$$

where n_0 is the density of the system at equilibrium.
When this is then substituted into the previous equation we get

$$(\mu+\omega)u = -\frac{1}{2}\frac{\partial^2 u}{\partial x^2} + Vu + 2gn_0 u + gn_0 v = 0 \tag{4.16}$$

and

$$(\mu-\omega)v^* = -\frac{1}{2}\frac{\partial^2 v^*}{\partial x^2} + Vv^* + 2gn_0 v^* + gn_0 u^* = 0\,. \tag{4.17}$$

If we set the superfluid into a homogeneous system we then have $\mu = gn_0$ and $V = 0$. Which means you can set the excitation are $u(x) = u_0 e^{\imath kx}$ and $v(x) = v_0 e^{\imath kx}$ by substituting these into the previous equation gives

$$\left(\frac{k^2}{2} + gn_0 - \omega\right)u_0 + (gn_0)v_0 = 0 \tag{4.18}$$

and

$$\left(\frac{k^2}{2} + gn_0 + \omega\right)v_0 + (gn_0)u_0 = 0\,. \tag{4.19}$$

These equations are known as the Bogoliubov Equations and when solved simultaneously give

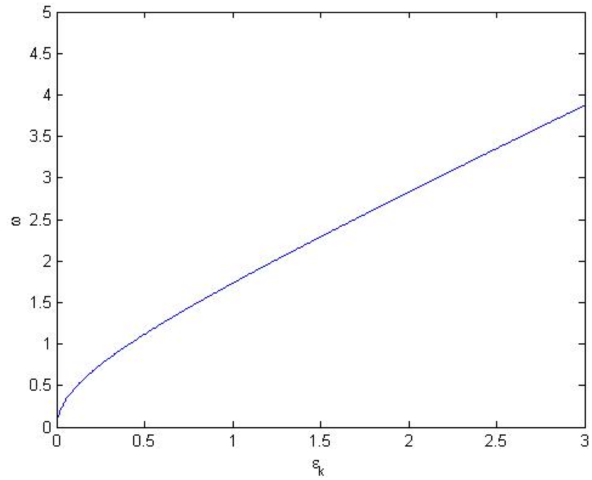
$$\left(\frac{k^2}{2} + gn_0 - \omega\right)\left(\frac{k^2}{2} + gn_0 + \omega\right) - (gn_0)^2 = 0\,, \tag{4.20}$$

$$\omega^2 = \frac{k^2}{2}\left(\frac{k^2}{2} + 2gn_0\right)\,. \tag{4.21}$$

Where $k^2/2 = \epsilon_k$ which gives the kinetic energy term of the equation. Substituting this in and simplify

$$\omega = \sqrt{\epsilon_k^2 + 2gn_0\epsilon_k}\,, \tag{4.22}$$

this is the Bogoliubov Excitation Spectrum. [11]



graph.jpg

Figure 4.1:

4.2.1 Results of Bogoliubov Equations

This graph shows that as ϵ_k increases the graph becomes more and more linear as the kinetic energy term $k^2/2 = \epsilon_k$ dominates the oscillation frequency. When $\epsilon_k \to 0$: Then, $\omega \propto \sqrt{\epsilon_k}$ with this limit, the interaction term dominates the superfluid and acts more collectively when the energy is low. When $\epsilon_k \to \infty$: Then, $\omega \propto \epsilon_k$ with this limit, the kinetic energy term dominates the superfluid and acts more like free particles.

4.3 Wave Propagation

To find the speed of a perturbation to a system in equilibrium, I firstly set $n = n_{EQ} + \delta n$ where n_{EQ} is the density of the system in equilibrium and the δn is the excitation on top of the equilibrium density. By linearizing the continuity equation we get

$$\frac{\partial \delta n}{\partial t} \approx -\frac{\partial}{\partial x}(n_{EQ} v)\,, \tag{4.23}$$

and when the Euler Equation is simplified to

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left(-\mu - \frac{1}{2}v^2\right)\,, \tag{4.24}$$

where μ is the generalised chemical potential.

$$\mu = \frac{1}{2}gn^2 + V - \frac{1}{2\sqrt{n}}\frac{\partial^2\sqrt{n}}{\partial x^2} \tag{4.25}$$

when there is a perturbation to the chemical potential and the Euler equation is then linearized it gives

$$\frac{\partial v}{\partial t} \approx -\frac{\partial}{\partial x}(\delta\mu) \tag{4.26}$$

it can then be shown that

$$\frac{\partial^2}{\partial t^2}(\delta n) = \frac{\partial}{\partial x}\left[n_{EQ}\left(\frac{\partial}{\partial x}(\delta\mu)\right)\right] \tag{4.27}$$

if we then take

$$\delta\mu = g(\delta n) \tag{4.28}$$

where the density is very slowly varying locally. I have ignored the gradient and the kinetic energy term as it negliable at point where it is measured. This gives

$$\frac{\partial^2}{\partial t^2}(\delta n) = \frac{\partial}{\partial x}\left[gn_{EQ}\left(\frac{\partial}{\partial x}(\delta n)\right)\right] = \frac{\partial}{\partial x}\left[c^2(x)\left(\frac{\partial}{\partial x}(\delta n)\right)\right] = c^2(x)\frac{\partial^2}{\partial x^2}(\delta n)\,,. \tag{4.29}$$

This is the wave equation with speed $c(x) = \sqrt{gn(x)}$ meaning the speed of sound in the BEC. An experiment was done to measure the speed of propagation in a BEC. This was done by using laser beams to induce a propagation within the system. The BEC is created by cooling 5x10^6 sodium in similar fashion as described in Chapter 1.

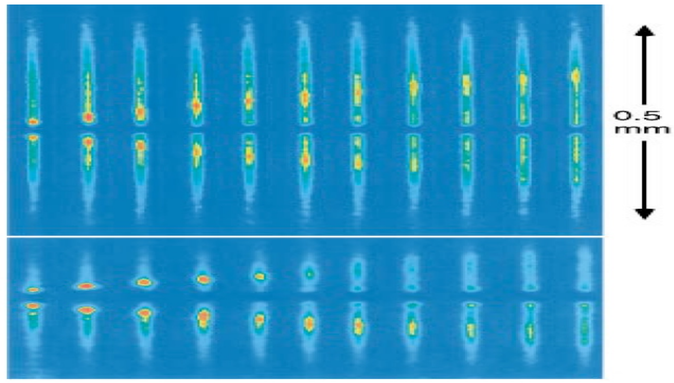
of sound image.jpg 

Figure 4.2: Pertubation in a BEC measured a 1.3ms time intervals [2]

This image is the propagation of sound in the BEC. Each image was taken every 1.3ms starting 1ms after the lase was turned on. The top image has a higher peak density meaing the speed of sound is higher in this image. This is clear from the image as the distance the wave propagates in the lower image is significantly less.[2]

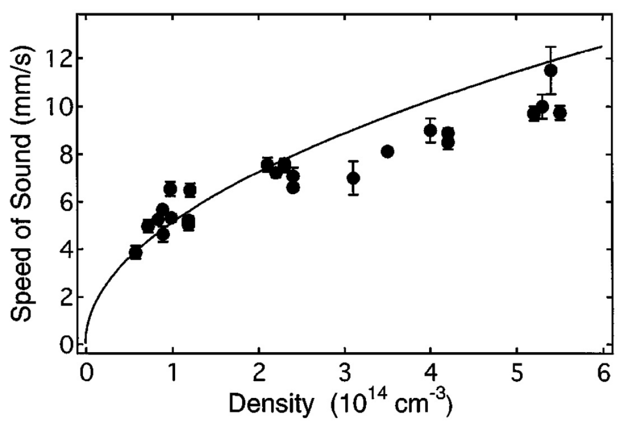
of sound.jpg 

Figure 4.3: Prediction of Speed of Sound vs. Expermental Data [2]

This graph shows the speed of sound versus the BEC peak density. Where the solid line shows $c(x) = \sqrt{gn_0}$ and the dots show the measure from the experiment will the statisical error bars. It shows that the analytics does match with the experimental data well. [2]

Chapter 5

Josephson Effects

In this chapter I shall discuss the history, analytics and then demonstate experiments to backup the analytics for Josephson effects. I also model the different Jospehson effects based on there initial conditions.

5.1 Introduction to Josephson Effect

In 1962 British physicist Brian David Josephson first predicted the Josephson Effects. It is a macroscopic quantum phenomenon which explores the oscillations between two weakly coupled Bose-Einstein Condensates (BEC). This system uses two perfect superfluids which are in harmonic osillators, which are weakly coupled, the separation is such that quantum tunnelling can take place between the two superfluids. I shall discuss how the system of two superfluids changes the oscillitary behaviour when there is change in phase and chemical potenial. I shall also simulate the initial conditions of a two weakly coupled BEC, to find what the limits for the variables satisfy the different Josephson Effects; these will be discussed later. [11]

The two Josephson effects we are looking at are Josephson Oscillations and Self-Trapping. Josephson Oscillations is when there the population oscillates from one side to the other when each of the superfluids are in two harmonic oscillators seperated by a weak coupling. Self-Trapping is when the population does not change from either side of the weak coupling and there is just a small amount of quantum tunneling that doesn't effect he population difference.

[13]

5.2 Obtaining of Josephson Equations

To model the two weakly coupled BECs on each side of the barrier we can treat each BEC as a separate wavefunctions. The quantum-mechanical wavefunction $\Psi(\boldsymbol{r}, t)$ is the sum of two individual wavefunctions on either side of the barrier, $\Psi_1(\boldsymbol{r}, t)$ and $\Psi_2(\boldsymbol{r}, t)$. [13]

$$\Psi(\boldsymbol{r}, t) = \Psi_1(\boldsymbol{r}, t) + \Psi_2(\boldsymbol{r}, t). \tag{5.1}$$

In the region of overlap of the two superfluid the wavefunctions can be shown to be

$$i\frac{\partial \Psi_1(\boldsymbol{r}, t)}{\partial t} = E_c N \Psi_1(\boldsymbol{r}, t) - E_J \Psi_2(\boldsymbol{r}, t)\,, \tag{5.2}$$

$$i\frac{\partial \Psi_2(\boldsymbol{r}, t)}{\partial t} = E_c N \Psi_2(\boldsymbol{r}, t) - E_J \Psi_1(\boldsymbol{r}, t)\,, \tag{5.3}$$

where $E_c N$ is the chemical potential in each BEC, where $N = N_1 - N_2$ is the population difference between the two BECs and the E_J represents the Josephson coupling energy. We then use the Madelung transformation on these equations by setting

$$\Psi_{1(2)}(\boldsymbol{r}, t) = \sqrt{N_{1(2)}}e^{i\theta_{1(2)}}\,, \tag{5.4}$$

$$e^{i\theta_{1(2)}} = \cos\theta_{1(2)} + i\sin\theta_{1(2)}\,, \tag{5.5}$$

Then by substitution of (5.4) and (5.5) into (5.2) and equating real and imaginary parts gives

<u>Real</u>

$$-\frac{d}{dt}(\sqrt{N_1}\sin\theta_1) = \frac{1}{2}(E_C N\sqrt{N_1}\cos\theta_1 - E_J\sqrt{N_2}\cos\theta_2)\,, \tag{5.6}$$

which simplifies to give

$$\left(\frac{dN_1}{dt}\right) = -2N_1\left(\frac{d\theta_1}{dt}\right)\frac{\cos\theta_1}{\sin\theta_1} - E_C N N_1\frac{\cos\theta_1}{\sin\theta_1} + E_J\sqrt{N_1 N_2}\frac{\cos\theta_2}{\sin\theta_1}. \tag{5.7}$$

<u>Imaginary</u>

$$\frac{d}{dt}(\sqrt{N_1}\cos\theta_1) = \frac{1}{2}(E_C N\sqrt{N_1}\sin\theta_1 - E_J\sqrt{N_2}\sin\theta_2)\,, \tag{5.8}$$

which simplifies to give

$$\left(\frac{dN_1}{dt}\right) = -2N_1\left(\frac{d\theta_1}{dt}\right)\tan\theta_1 + E_C N N_1\tan\theta_1 - E_J\sqrt{N_1 N_2}\frac{\sin\theta_2}{\cos\theta_1}\,, \tag{5.9}$$

and

$$\left(\frac{d\theta_1}{dt}\right) = \frac{1}{2N_1}\frac{\cos\theta_1}{\sin\theta_1}\left(\frac{dN_1}{dt}\right) - \frac{1}{2}E_C N + \frac{1}{2}E_J\sqrt{\frac{N_2}{N_1}}\frac{\sin\theta_2}{\sin\theta_1}. \tag{5.10}$$

Substituting (5.10) into (5.7) gives

$$\begin{aligned}(\sin^2\theta_1 + \cos^2\theta_1)\left(\frac{dN_1}{dt}\right) &= E_C N N_1 \cos\theta_1 \sin\theta_1 \\ &\quad - E_J\sqrt{N_1 N_2}\cos\theta_1\sin\theta_2 \\ -E_C N N_1 \cos\theta_1 \sin\theta_1 &+ E_J\sqrt{N_1 N_2}\cos\theta_2 \sin\theta_1,\end{aligned}$$

this simplifies to give

$$\frac{dN_1}{dt} = E_J\sqrt{N_1 N_2}\sin\theta. \tag{5.11}$$

It can then be shown

$$\frac{dN_2}{dt} = -\frac{dN_1}{dt}, \tag{5.12}$$

this gives

$$\frac{dN}{dt} = 2E_J\sqrt{N_1 N_2}\sin\theta, \tag{5.13}$$

where

$$\theta = \theta_1 - \theta_2\,. \tag{5.14}$$

To then find an equation for $d\theta_1/dt$ substitute (1.12) into (1.10). Therefore

$$\frac{d\theta_1}{dt} = -\frac{1}{2}E_C N + \frac{1}{2}E_J\sqrt{\frac{N_2}{N_1}}\cos\theta\,, \tag{5.15}$$

Similarily for $d\theta_2/dt$ gives

$$\frac{d\theta_2}{dt} = \frac{1}{2}E_C N + \frac{1}{2}E_J\sqrt{\frac{N_1}{N_2}}\cos\theta\,, \tag{5.16}$$

which gives

$$\frac{d\theta}{dt} = -E_C N + \frac{1}{2}E_J\left(\sqrt{\frac{N_2}{N_1}} - \sqrt{\frac{N_1}{N_2}}\right)\cos\theta\,. \tag{5.17}$$

[13]

5.2.1 Predictions for Results

With these expression(5.17) and with setting $n_1 \approx n_2 \approx n$ it can be shown

$$\frac{\partial}{\partial t}(\theta_1 - \theta_2) = -(\mu_1 - \mu_2)\,, \tag{5.18}$$

$$\frac{\partial}{\partial t}(n_1 - n_2) = -4\kappa\sqrt{n_1 n_2}\sin(\theta_1 - \theta 2)\,. \tag{5.19}$$

If we then set there to be no chemical difference $(\mu_1 - \mu_2) = 0$,

$$(\theta_1 - \theta_2) = constant\,, \tag{5.20}$$

so

$$\frac{\partial}{\partial t}(n_1 - n_2) = -4\kappa\sqrt{n_1 n_2}\sin(constant) = constant\,. \tag{5.21}$$

This shows that there is not a oscillation between the two BECs and it is a constant flow(DC Josephson) . However if a chemical difference is then put into the system

$$\frac{\partial}{\partial t}(\theta_1 - \theta_2) = -\Delta\mu\,, \tag{5.22}$$

so

$$(\theta_1 - \theta_2) = -\Delta\mu t + \phi_0\,, \tag{5.23}$$

$$\frac{\partial}{\partial t}(n_1 - n_2) = -4\kappa\sqrt{n_1 n_2}\sin(-\Delta\mu t + \phi_0)\,, \tag{5.24}$$

this shows that when there is a chemical potential difference there will be an oscillation between the two BEC this is an AC current. This next experiment uses Helium-3 and Helium-4 particles. They are seperated by a silicon nitride (SiN) membrane will 4225 holes. The variable is the pressure difference between the two sides of the superfluid to see how the oscillation frequency varies. Based on the analytics it can be shown:

$$\frac{\partial}{\partial t}\Delta\theta = -\frac{\Delta\mu}{\hbar}, \tag{5.25}$$

which then gives

$$\Delta\theta = -\frac{\Delta\mu}{\hbar} + \phi \tag{5.26}$$

The chemical potenial $\Delta\mu$ on the right side of the equation becomes

$$\Delta\theta = \frac{\Delta P m}{\rho}. \tag{5.27}$$

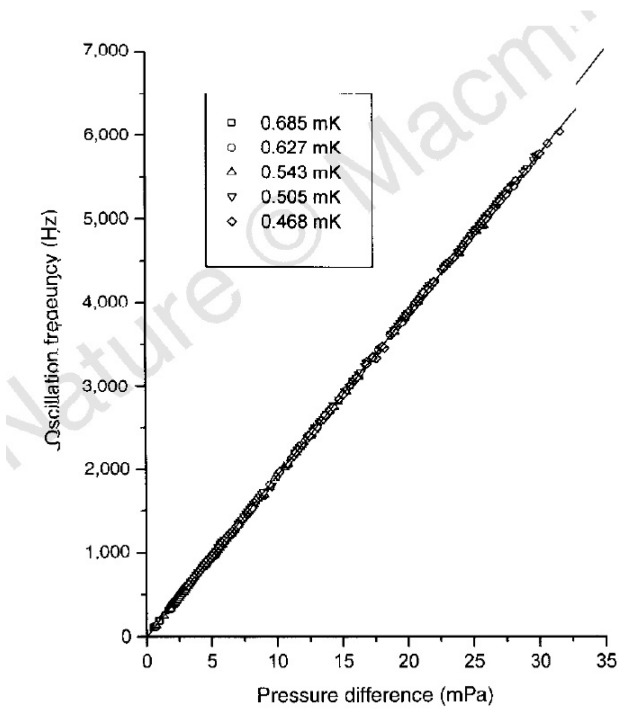
Where ΔP is the pressure difference across the weak link, ρ is the liquid density, and m is the mass of the Helium atoms. This then gives

$$\Delta\theta = \Delta P \frac{m}{\rho\hbar} t, \tag{5.28}$$

where the frequency of the oscillation is

$$f = \Delta P \frac{m}{\rho\hbar}. \tag{5.29}$$

Therefore the expected result is to see the frequency of the oscillation to be directly proportional to that of the pressure difference. [10] Figure(5.1)



PvF.jpg

Figure 5.1: Graph of the Frequency of the Quantum Oscillation [10]

shows the how the pressure of the system is directly proportional to the oscillation frequency which is what was expected from analytics as there is a linear increase in oscillation as the pressure increases.

5.2.2 Hamiltonian

Assume the normalization to be 1 to simpify these equations further

$$|\Psi_1|^2 + |\Psi_2|^2 = 1, \tag{5.30}$$

which means

$$N_1 N_2 = \frac{1 - N^2}{4}\,. \tag{5.31}$$

Substituting into (5.13) gives

$$\frac{dN}{dt} = E_J \sqrt{1 - N^2}\sin\theta\,, \tag{5.32}$$

and substituting into (5.16)

$$\frac{d\theta}{dt} = -E_C N - \frac{E_J N}{\sqrt{1-N^2}}\cos\theta. \tag{5.33}$$

By then taking

$$\frac{dN}{dt} = -\frac{\partial H}{\partial\theta}, \tag{5.34}$$

and

$$\frac{d\theta}{dt} = \frac{\partial H}{\partial N}. \tag{5.35}$$

This then goes to show

$$H(N,\theta) = -\frac{1}{2}E_C N^2 + E_J\sqrt{1-N^2}\cos\theta. \tag{5.36}$$

This is the hamiltonian for the Josephson system. Using equation (5.17)

The next step is to find an equation for θ this is done by first taking $d^2\theta/dt^2$

$$\frac{d^2\theta}{dt^2} = -E_C\frac{dN}{dt} - E_J\frac{d}{dt}\left(\frac{N}{\sqrt{1-N^2}}\cos\theta\right), \tag{5.37}$$

which then gives

$$\frac{d^2\theta}{dt^2} = -\frac{E_J E_C}{\sqrt{1-N^2}}\sin\theta - \frac{E_J^2}{2}\left(\frac{1+N^2}{1-N^2}\right)\sin 2\theta\,. \tag{5.38}$$

Then take $E_C = 0$ and $N \ll 1$ and take θ too be small.

$$\frac{d^2\theta}{dt^2} = -\frac{E_J^2}{2}\sin 2\theta \tag{5.39}$$

Set $\phi = 2\theta$

$$\frac{d^2\phi}{dt^2} = -E_J^2\phi\,, \tag{5.40}$$

which is a second order differential equation that can be solved to give

$$\phi = A\cos(E_J t + \phi_0)\,. \tag{5.41}$$

With the present of interactions in the superfluid gives

$$\phi = A\cos((\sqrt{E_J E_C + E_J^2})t + \phi_0)\,. \tag{5.42}$$

This expression gives tells us about the frequency oscillation when there a chemical potenial within each superfluid. However a similar equation cannot be given for the difference in the number of atoms N as this gives an expression with real and imaginary parts that only give a trivial solution.

5.2.3 Experimental Evidence

Several Experiments using the Josephson Junction which shows the Josephson Effects using a supercurrent between two superconductors. This occurs when there are two superconductors are separated by a thin insulator layer(which can be an oxide barrier). When the oxide barriers significantly overlap the Josephson Effects occurs. If there is no voltage between the superfluids then there is a DC current between the superfluid. However when there is a voltage across the junction then an AC Josephson effect occurs.

There have been several experiments showing these results to be true and also how changing different parameters have vary effects. This Experiment used Rubidium-87 atoms to create the BEC. The variable is the distance between the two BECs to change the E_J so Λ is above and below the critical value. [1] As we can see from this diagram of two weakly coupled BEC in a symmetric double-well potenial there are two different effects. Diagram A shows the distance between the two wave packets is 4.4 μm and there is an oscillation of the population between the two sides of the BEC this is the Josephson oscillation that is expected. In Diagram B the distance between the two wave packets has been increased to 6.7 μm and there is almost no change in population between the two BECs which his the self-trapping effect.

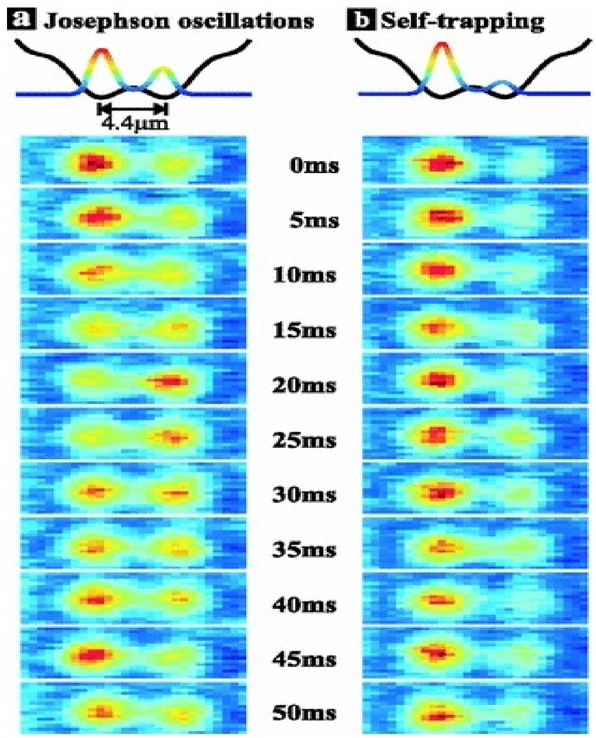


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Figure 5.2: Observation of tunneling dynamics of two weakly Linked BECs [1]

5.3 Josephson Oscillations vs Self-Trapping

The Hamiltonian for the Josephson effect is equivalent to that of a non-rigid pendulum. The length is dependent on the angular momentum. The pendulum oscillations describe the rate of change of the populations from one side to the other. However when the pendulum has enough energy to swing at a value greater than π it changes the system. This is analogous to the Josephson effects in a BEC as it goes from the Josephson oscillation where there is a population oscillation between each side of the barrier. Then when the system passes the critical value it changes to a self-trapping system which does not have an oscillation in population between each side. The hamiltonian for the system with a symmetric trap is [13]

$$H(N,\theta) = -\frac{1}{2}E_C N^2 + E_J\sqrt{1-N^2}\cos\theta\,, \tag{5.43}$$

as shown previously. By then setting ratio of the interactions within each of the superfluids separately, with the energy between the two superfluids as

$$\frac{E_C}{E_J} = \Lambda, \tag{5.44}$$

this gives a expression for the hamiltonian as

$$H(N, \theta) = \frac{\Lambda}{2} N^2 - \sqrt{1 - N^2} cos(\theta) \,. \tag{5.45}$$

By then setting $H(N, \theta) = 1$ [14] we then get the critical value for when the system changes from Self-trapping to Josephson oscillations. This can then be rearranged to give

$$\Lambda_c = \frac{1 + \sqrt{1 - N^2} cos(\phi)}{N^2/2} \,, \tag{5.46}$$

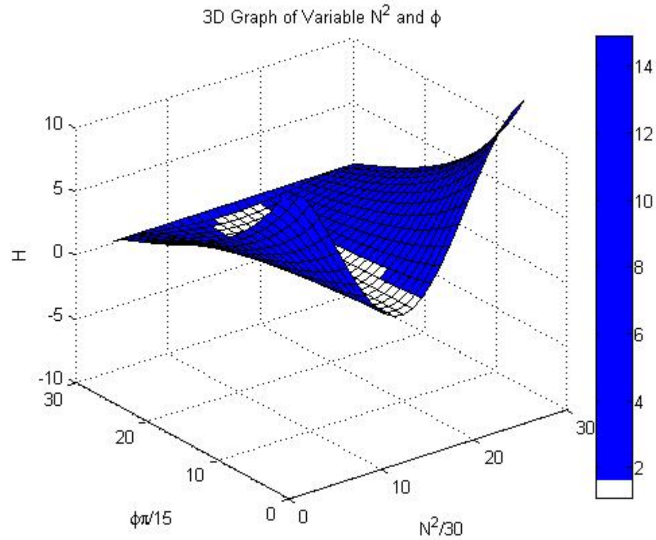
where Λ_c is the critical value. This can then be rearranged to give give a expression for N^2;

$$N^2 = \frac{2}{\Lambda^2} (\Lambda - cos^2(\theta)) \pm \frac{2}{\Lambda} \sqrt{\frac{1}{\Lambda^2} (\Lambda - cos^2(\theta))^2 - (1 - cos^2(\theta)} \,. \tag{5.47}$$

[14]

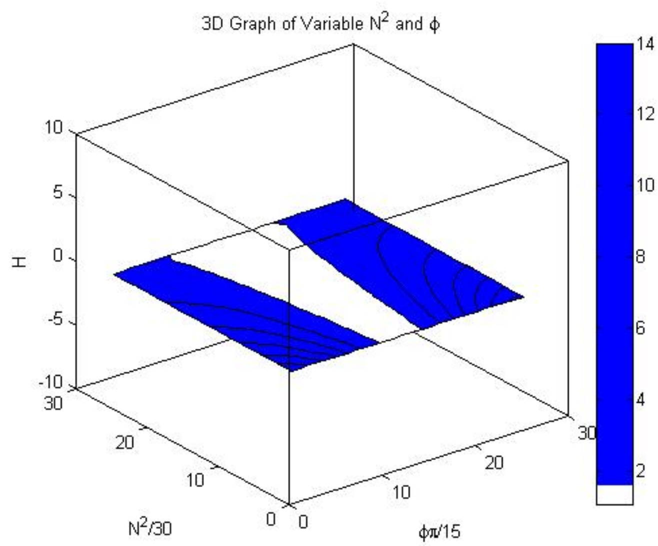
5.4 Graphically Representation and Analysis of the Critical Values of the Josephson Effects

In the models of the initial values I am taking values $0 \leq N^2 \leq 1$ this is the case as $-1 \leq N \leq 1$ this covers also population differences. I am taking values $0 \leq \theta \leq 2\pi$ as this takes all values for $\cos(\theta)$. I shall take a 3D representation of the system will both variables θ and N^2 against the value of Λ in order to see the overall pattern. Then in order to get the effect of each of the variables I will take fixed N^2 and variable θ and vice versa. Then taking the special cases of the systems in order see the other behaviours of the superfluid.



JE.jpg

Figure 5.3: 3D Graph Indicating Region of Josephson Effect



JE.jpg

Figure 5.4: 2D Graph Indicating Region of Josephson Effect

5.4.1 3D Representation

This graph is showing $0.5 \leq N^2 \leq 1$ and $0 \leq \theta \leq 2\pi$ with the Josephson regime shown in white and Self Trapping in blue. As white is where the

$H \leq 1$. This 2D plot shows a band of Josephson regime that broadens as N^2 is getting closer to 0 meaning that the smaller the population difference the more values the phase difference can be and still achieve a Josephson effect. This is for the initial value of the system.

5.4.2 Special Cases

I am now taking special cases for the θ with variable N^2.
The two special cases I am choosing are $\theta = 0/\pi/2$, which is when the superfluids are completely inphase ($\theta = 0$) and completely out of phase by $\pi/2$.
When a phase difference of $\phi = \pi/2$ it gives

$$\frac{2}{\Lambda} = N^2 \tag{5.48}$$

which means Josephson Effect is achieved when

$$\frac{2}{\Lambda} > N^2 \tag{5.49}$$

The other special case $\theta = \pi/2$ gives the expression

$$\frac{4}{\Lambda^2}(\Lambda - 1) = N^2. \tag{5.50}$$

This gives a Josephson Effect when

$$\frac{4}{\Lambda^2}(\Lambda - 1) > N^2. \tag{5.51}$$

I have then plotted this graph showing both special cases. It shows that the Josephson oscillation occurs inside the curves and Self-trapping outside. As we can see from this graph as $N \to 0$ then $\Lambda \to \infty$, this infers that if the population on either side of the barrier is the same as the critical value tends to infinity this infers it always will achieve Josephson oscillations however weak the coupling is. This is expected when there is no population difference then an interaction between each side is expected. However a very very large Λ is not a very interesting regime as the coupling energy is far too weak in relation to the chemical potenial energy within each superfluid to have meaningful tunnelling between each superfluid.

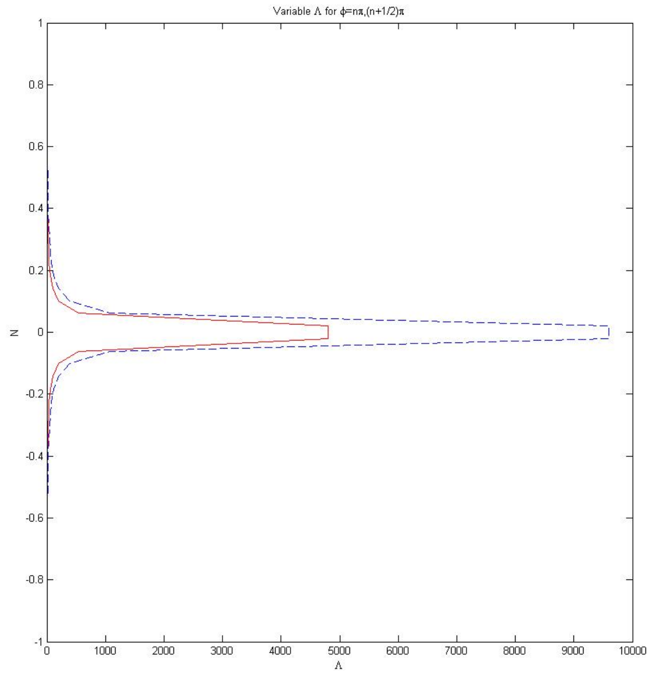


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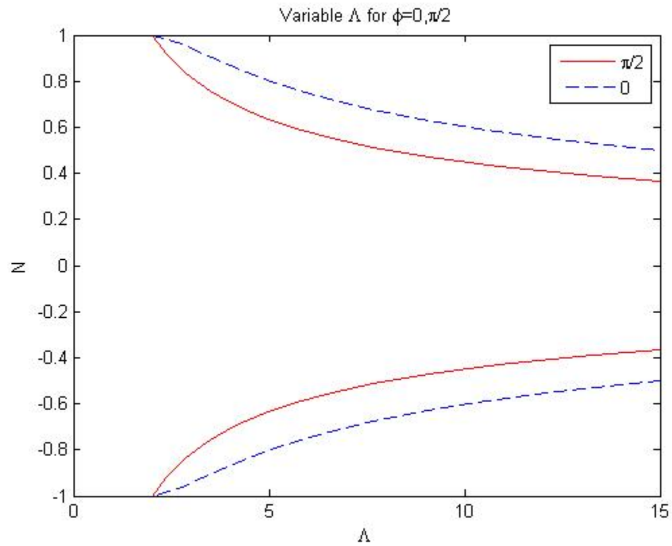
Figure 5.5: Graph of the Frequency of the Quantum Oscillation

This change of scale is to only include values of Λ that will have a meaningful interaction between the two BECs. With this change of scale of the axis in the graph we see that both special cases converge to 2 therefore the system will initially have a Josephson effect for the special cases when $\Lambda \leq 2$. This is due to the ratio of the coupling energy being so great that the superfluid have basically amalgomated into one and the system has changed into to one superfluid system.
It is also clear from this graph that the critical value rapidly increases as the population difference decreases, which is what is expected from the analytics. As the rate of change is given by $dN/dt = E_J\sqrt{1-N^2}\sin\theta$ therefore the smaller the population difference the greater rate of change of population which infers the presence of the Josephson Effect.

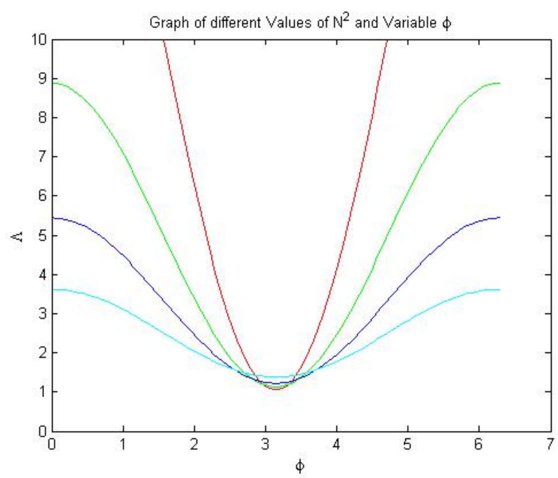


cases JE close.jpg

Figure 5.6: Graph of the Frequency of the Quantum Oscillation

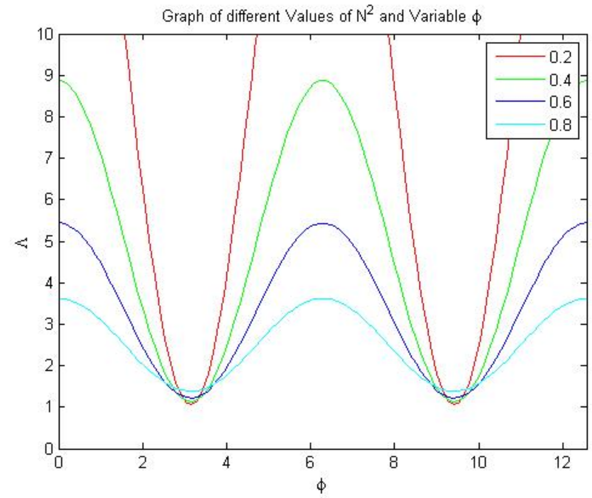
5.4.3 Variable Phase Difference

I am now going to see how the critical value varies when the intial phase difference is varied. I am choosing a range of $0 \leq \theta \leq 2\pi$ as this covers all possible values Λ. I have then chosen values of $N^2 = 0.2, 0.4, 0.6, 0.8$.



JE variable y.jpg

Figure 5.7: Graph of the Different Values of N and variable ϕ



JE variable N.jpg

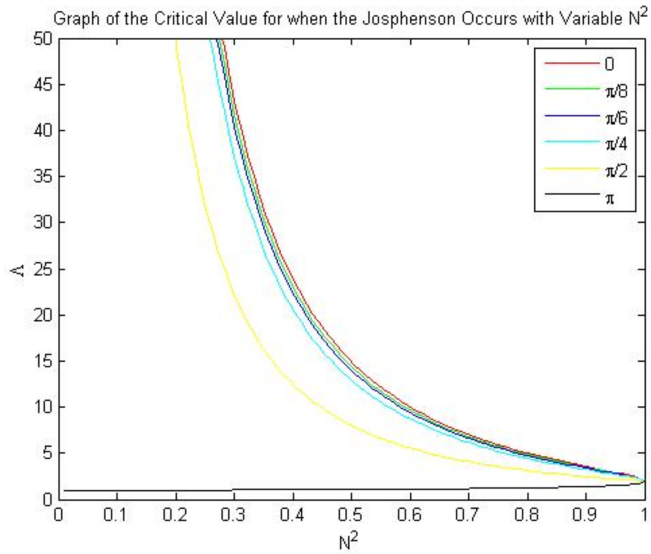
Figure 5.8: Graph of the Different Values of N and variable ϕ

In these graphs values under the curve signify Jospheson oscillations. From these graphs we see that the critical value follows a cosine curve, this infers the Josephson Effect is achieved will a higher values of Λ when closer to being in-phase. When the superfluids are completely out of phase $\theta = \pi/2$ Λ is at it's minimum. This is expected from the analytics as the only θ term is $cos\theta$. Thererfore when the two superfluids are more in phase the system require a lesser coupling energy in relation to the interaction within each superfluid.

We can also see from these graphs that the frequency of change for intial Λ does not change when N is varied because of the previously mentioned $cos\theta$ term. However the ampilitude of Λ increases as the population difference decreases. This is consistent will the results in the special cases.

5.4.4 Variable Population Difference

In this graph values of Λ under the curve are Josephson oscillations and those above are self trapping. We can see that for all values of θ there is a similar shape that tells us that the smaller the population difference the greater the value of Λ can be for a Jospheson oscillation to occur. Therefore a more even population will tunnel between the superfluids more easily. This is expected from the analytics for the same reason as descibed in the special cases. The



variable N2.jpg

Figure 5.9: Graph of the Different Values of N and variable ϕ

other key feature of this graph is as the phase difference becomes larger the range of values of Λ become larger and a Josephson effect is achieved will a weaker coupling between the two superfluids.

5.4.5 Conclusion on Josephson Effect

It is clear from the graphs that the Josephson oscillation occurs more easily when the coupling energy is higher which is expected as it will have a greater interaction between the two superfluids. When the coupling energy is stronger than the chemical potenial within each superfluid then the each side of the barrier would just mix and become one superfluid and thus destroy the system. It is also clear from these results that the initial phase difference between the two superfluids is a significant factor in that it oscillates the critical value in a cosine curve. Which is expected as when the two sides are out of phase by π the coupling energy must be very high to achieve a Josephson oscillation. However when the phase different is very small then it requires a much lower coupling energy in order for the two superfluid on either side to interact with Josephson oscillations. The final variable was the population difference which also had a large effect, when the population difference was very small then the critical value would become very large and the Jospheson oscillation would happened far more easily. However when there is a complete imbalance then there are no criteria for Josephson effect as there are no atoms on one side of the barrier and therefore no interaction. If there are the same amount on either side then the superfluid will interact with each other very easily.

Chapter 6

Conclusion

This report shows dynamic and static behaviours of Bose-Einstein condensates(BEC).
Firstly I have taken an approximation of Gross-Pitaevskii Equation (GPE) to find the stability of the system depending on the strength of the interation between the atoms in the BEC and found the critical parameter for this approximation. I found that the shape of the BEC in a harmonic oscillator becomes taller and narrower as interactions between atoms become more attractive until the system becomes unstable and collaspes. I then wanted to look at a dynamic solution of the BEC. I first showed the relation with the GPE and the fluid equations. Then showed how a pertubation can show different properties of the superfluid: showing the different behaviour of the BEC to see at what energies the system acts as free particles or collectively and the speed of a propagation within the BEC.
The next stage was to see the behaviour of two BECs with each other and I modelled and simulated the Josephson Effects between two superfluids that where weakly coupled to see how they would interact with either a Josephson oscillation or Self-Trapping Effect. I did this by first obtaining the hamiltonian for the Josephson Effects and ploting when the system follows a self-trapping effect and when it has Josephson oscillations. I showed how the population effects the Josephson Effects, which showed that the the greater the population difference the less lower the critical parameter for Josephson oscillations became. I then showed how the phase difference changes the critical parameter for the Josephson Effects. In order to further my study of BECs I would like to see how BEC behave at a finite temperature and model how the thermal cloud changes will the presence of another thermal

cloud damps out a BEC and looking into modelling and the analytics this situation.

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