MAS8091: MMath Dissertation

School of Mathematics & Statistics

Investigating How Particles Will Orbit Schwarzschild and Kerr Black Holes

Author:
Stuart Brown

Supervisor:
Dr. David Toms

May 1, 2015

Abstract

We have studied Einstein's field equations of general relativity and produced orbit paths for test particles and photons, time-like and null geodesics respectively, orbiting Schwarzschild and Kerr black holes. A Schwarzschild black hole is a stationary spherically symmetric black hole. A Kerr black hole is a rotating black hole that is axis-symmetric about the axis of rotation. We observed that there are similarities between time-like and null geodesics. We have also examined phenomena such as the precession of perihelion and light bending. This was to investigate whether the cosmological constant affects these phenomena. We showed that the cosmological constant does affect the precession of perihelion, but not the bending of light. We shall present a detailed conclusion of how we would expect test particles to move around these types of black holes.

Contents

1 Introduction **3**

2 Schwarzschild Black Holes **5**

2.1 Timelike Geodesics 8

2.1.1 Bound orbits of the first-kind 9

2.1.2 Bound orbits of the second-kind 16

2.1.3 Unbound orbits of the first and second-kind 19

2.2 Null Geodesics 21

2.2.1 Bound Orbits of the first-kind 22

2.2.2 Bound Orbits of the second-kind 24

2.2.3 Unbound Orbits of the first-kind 24

2.2.4 Unbound Orbits of the second-kind 27

3 Kerr Black Holes **28**

3.1 Timelike Geodesics 30

3.2 Null Geodesics 33

4 Effects of the Cosmological Constant **37**

4.1 Precession of Perihelion 37

4.1.1 Direct integration of perturbation force 38

4.1.2 Standard perturbation 39

4.1.3 Comments on Precession of Perihelion 41

4.2 Light Bending 41

5 Conclusion **44**

References **47**

1 Introduction

The aim of my dissertation is to examine Einstein's Theory of General Relativity field equations to describe how a test particle (an uncharged, non-rotating, very small particle that does not affect space-time) moves around a black hole. There are different types of black holes. Therefore, we will examine, in depth, how the test particle is affected around a Schwarzschild and Kerr black hole and see if the test particle's orbital path differs between each black hole.

Before we get into the Einstein's theory of general relativity field equations we will begin by understanding basic concepts of space-time and black holes. Einstein's theory of general relativity fundamentally changed how we understood our universe. The basic idea behind Einstein's Theory of general relativity is that there is no such thing as a gravitational force between two bodies. Rather, what we view as a force is a manifestation of the fact that motion takes placed on a curved space [1]. There was two parts to Albert Einstein's theory and they were split into special relativity and general relativity. His theory of special relativity determined that the laws of physics are the same for all non-accelerating observers, and he showed that the speed of light within a vacuum is the same no matter the speed at which an observer travels. As a result, he found that space and time were interwoven into a single continuum known as space-time. Events that occur at the same time for one observer could occur at different times for another. It introduced a new framework for all of physics and proposed new concepts of space and time.

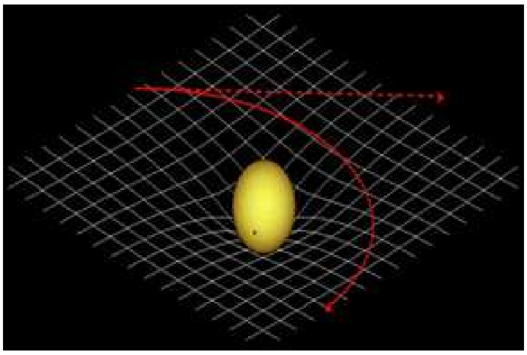

Figure 1: A test particle passing a mass in spacetime [2].

Gravity is not a force but a curvature in space-time. Space-time is analogous to a very large sheet and objects such a planets or galaxies weigh down on the sheet causing the sheet to dimple. For example if you look at figure 1, the mass of the entity is proportional to the dimple it causes in the sheet, i.e. space-time. Objects with mass cause a curvature in space and time, a larger mass will cause a greater dip in the sheet. Einstein realized that massive objects caused a distortion in space-time. So if a smaller mass object is within the distortion it will fall into the dip created by the larger mass. This is an intuitive description of general relativity.

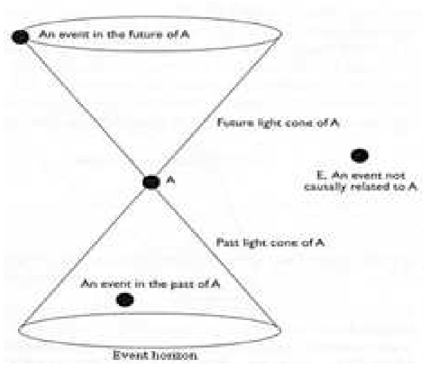

Figure 2: A past and future time-cone [4].

This new founded idea means that a test particle moving through these distortions in space does not move in straight lines. We refer to the path line of a test particle moving in space-time as its geodesics [3]. We can see this in Figure 1. The dotted line shows the path our test particle would follow if not for general relativity however the full line shows the actual path line of the particle. There are three types of geodesics dependent on the direction and position of the geodesic in comparison to an event A. We will explain this by examining Figure 2. This shows a future and past light cone. A future light cone shows every position light can reach from an event A. The past light cone is all the positions that can send a light signal to the event. Hence, a timelike geodesic describes the motion of a test particle moving within the light cone at a speed less than the speed of light [3]. A null geodesic describes the motion of a test particle moving along the edge of the light cone at the speed of light [3]. A spacelike geodesic describes the motion of a test particle moving outside the light cone and hence moving at a speed greater than that of the speed of light. We will not discuss any work on spacelike geodesics but focus on timelike and null geodesics of a test particle around certain types of black holes.

A test particle orbiting a black hole has two sets of possibilities. It can either follow a bound or an unbound orbit. A bound orbit are those that describe the motion of a fixed orbit around the black hole analogous to the way the planets in our system orbit the sun. An unbound orbit describes the motion of the test particle approaching the black hole but never entering a stable orbit and eventually exiting the gravitational field of the black hole. Bound and unbound orbits also have two possibilities. Either they can remain in their described motion around the exterior of the black hole or enter the event horizon and plunge into the singularity. We will go into more details for the event horizon and the singularity of a black hole later on in the dissertation.

So we will examine the bound timelike orbit paths for both Schwarzschild and Kerr geometries. We will vary the distances from the black hole that the test particles orbit the black hole at and see how this changes the orbit path. It will be engaging to see how gravity affects a massless particle such as light, as well as entities with different masses. For the Schwarzschild geometry we will also discuss the effect of a cosmological constant in light bending and precession of perihelion. The cosmological constant was what Einstein added to his field equations to achieve a static universe. Perihelion

distance means the distance of closest approach so perihelion precession describes how a particle would advance by a specific angle, in perihelion distance, every time the particle passed this distance from the black hole [1].

There are a couple of items to distinguish before we start. Firstly, I will be consistent in using the metric signature $(-+++)$ throughout the report. Secondly, since the cosmological constant, Λ, is so small it will be negligible in the mathematics applied in the majority of the work done in this dissertation and, hence, I will set $\Lambda = 0$ until the final section.

2 Schwarzschild Black Holes

A Schwarzschild black is the simplest type of black hole due to the fact that it has a non-rotating core and the make-up of the black hole is essentially an event horizon and singularity [1]. The event horizon is not a physical surface, it is the point at which the gravitational pull becomes so great as to make escape impossible, even light can't escape. The singularity is an infinitesimal point in space where the pull of gravity is infinitely strong and space-time infinitely curved. We don't actually know what happens in the singularity because the densities are so great, the laws of physics break down as we know them. The distance between the singularity and the event horizon is known as the Schwarzschild radius. The Schwarzschild solution which describes the space-time outside of a spherically symmetric mass distribution was the first exact solution to the Einstein field equations of general relativity and was founded by Karl Schwarzschild in 1916 [1]. Our solar system is also symmetrically spherical so the Schwarzschild solution can give a very accurate description of the space-time outside the sun and the earth; hence we can use the Schwarzschild solution to accurately describe the behaviour of the space-time within our own solar system [7].

As pre-detailed in the introduction we shall be giving a detailed description of how tests particles with different types of mass behave orbiting a Schwarzschild black hole. Before we delve into our investigation of how a test particle reacts around a black hole we must consider the line element of the Schwarzschild solution that describes the space-time outside any spherically symmetric mass distribution which is given below in spherical coordinates (t, r, θ, ϕ) [3]

$$\mathrm{d}s^2 = -c^2\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2\right). \tag{1}$$

This metric is made up of t = time, r = radius, M = mass of the spherically symmetric mass distribution, c = speed of light and G = Newtons gravitational constant. We can easily see that the metric coefficients tend to infinity as $r \to 0$ and $r \to 2GM/c^2$, where $r = 0$ is the singularity at the centre of the black hole and $r = 2GM/c^2$ is the Schwarzschild radius. The standard convention is to let $c = G = 1$ which are called geometrized units and enables us to make the metric far simpler.

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + sin^2(\theta)d\phi^2). \tag{2}$$

We can see that this is not too far off the line element for Minkowski space-time. For example if we let $M \to 0$ we obtain the line element for Minkowski space-time (i.e. $ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)$). We can also see metric tends to Minkowski metric as $r \to \infty$; this is a property known as asymptotic flatness [5]. Now we have found the line element for the Schwarzschild solution we can derive the geodesic equations for this specific line element using the Lagrangian equation for curved space-time. The Lagrangian equation is defined as [3]

$$L(x^\mu(\tau), x^\mu(\tau)) = \frac{1}{2}\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}x(\tau)\dot{x}^\mu\dot{x}^\nu, \tag{3}$$

where $g_{\mu\nu}$ is a metric tensor and is also referred to as the Schwarzschild metric. So to derive the geodesic equation for this line element by dividing equation (2) by $d\tau^2$ which will give the following

$$2L = -\left(1 - \frac{2M}{r}\right) \dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2) \tag{4}$$

where $(ds/d\tau)^2 = 2L$ and the dots denote differentiation with respect to τ (i.e. $\dot{t} = dt/d\tau$). The corresponding canonical momenta are

$$\begin{aligned} p_t &= -\frac{\partial L}{\partial \dot{t}} = \left(1 - \frac{2M}{r}\right)\dot{t}, &(5)\\ p_r &= \frac{\partial L}{\partial \dot{r}} = \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}, &(6)\\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = r^2\dot{\theta}, &(7)\\ p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = (r^2\sin^2(\theta))\dot{\phi}. &(8)\end{aligned}$$

and the corresponding Hamiltonian is [5]

$$H = -p_t\dot{t} + p_r\dot{r} + p_\theta\dot{\theta} + p_\phi\dot{\phi} - L = L. \tag{9}$$

The equality of the Hamiltonian and the Lagrangian shows that all the energy in the problem derives solely from kinetic energy. It is also important to note that by rescaling τ we can show that [6]

$$2L = \begin{cases} 0 & \text{for null geodesic ,} \\ -1 & \text{for timelike geodesic .} \end{cases} \tag{10}$$

and spacelike geodesics are outside the scope of this dissertation. Now we will consider each space-time coordinate through the Euler-Lagrange equation which has the following form [3]:

$$0 = \frac{\partial L}{\partial x^\mu} - \frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^\mu}. \tag{11}$$

So if we start by looking at the t-coordinate we get

$$0 = \frac{\partial L}{\partial t} - \frac{d}{d\tau}\frac{\partial L}{\partial \dot{t}}\ , \text{where}\ \frac{\partial L}{\partial t} = 0. \tag{12}$$

Hence,

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{t}} = \frac{dp_t}{d\tau} = 0. \tag{13}$$

Integrating equation (13) out we can show

$$p_t = \left(1 - \frac{2M}{r}\right)\dot{t} = \text{constant} = \beta. \tag{14}$$

Now if we look into the ϕ coordinate

$$0 = \frac{\partial L}{\partial \phi} - \frac{d}{d\tau}\frac{\partial L}{\partial \dot{\phi}}\ , \text{where}\ \frac{\partial L}{\partial \phi} = 0. \tag{15}$$

Thus,

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{\phi}} = \frac{dp_\phi}{d\tau} = 0. \tag{16}$$

Integrating equation (16) out, we can show

$$p_\phi = (r^2\sin^2(\theta))\dot{t} = \text{constant}. \tag{17}$$

We will denote this constant with a letter once we have looked at the θ component of the Euler-Lagrange, which takes the form,

$$\frac{\partial}{\partial \tau}(r^2\dot{\theta}) - r^2\sin(\theta)\cos(\theta)\dot{\phi}^2 = 0. \tag{18}$$

Using the chain rule on the first term in (18) it expands too

$$2r\dot{r}\dot{\theta} + r^2\ddot{\theta} - r^2\sin(\theta)\cos(\theta)\dot{\phi}^2 = 0. \tag{19}$$

We can assign the value of $\pi/2$ to θ which implies $\dot{\theta} = 0$ and also $\ddot{\theta} = 0$. Hence, the geodesic described is invariant in an invariant plane which we may distinguish by $\theta = \pi/2$. Thus,

$$p_\phi = (r^2\sin^2(\theta))\dot{t} = r^2\dot{\phi} = \text{constant} = \alpha^{-1}. \tag{20}$$

Now we have got values for the components of the Euler-Lagrange equation the constancy of the Lagrangian gives

$$2L = -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2 = -\eta. \tag{21}$$

Where $\eta = 1$ for a timelike geodesic and $\eta = 0$ for a null geodesic [6]. Now we are going to derive an orbital equation which just involves r. Let us examine and manipulate equations (14) and (17) we get the following:

$$d\tau = \alpha r^2 d\phi \qquad \text{and} \qquad \left(1 - \frac{2M}{r}\right)dt = \beta d\tau = \beta\alpha r^2 d\phi. \tag{22}$$

These newly formed equations can transform our equation (21) into

$$dr^2 + \left[r^2\left(1 - \frac{2M}{r}\right) - \alpha^2\beta^2 r^4 + \eta\alpha^2 r^4\left(1 - \frac{2M}{r}\right)\right]d\phi^2 = 0. \tag{23}$$

Let $r = 1/u$ and divide equation (23) by $r^4 d\phi^2$ to obtain the equation that determines the geometry of the geodesics in the invariant plane.

$$\left(\frac{du}{d\phi}\right)^2 = \alpha^2\beta^2 - (u^2 + \eta\alpha^2)(1 - 2Mu) = f(u). \tag{24}$$

2.1 Timelike Geodesics

A timelike geodesic describes the motion of a test particle moving within the light cone at a speed less than the speed of light. The governing equations for the timelike geodesic orbits is the following

$$\left(\frac{du}{d\phi}\right)^2 = f(u), \tag{25}$$

where,

$$\begin{aligned} f(u) &= \alpha^2\beta^2 - (u^2 + \eta\alpha^2)(1 - 2Mu), & (26)\\ &= 2M(u - u_1)(u - u_2)(u - u_3), & (27)\\ &= 2M(u^3 - u^2(u_1 + u_2 + u_3) + u(u_1u_3 + u_2u_3 + u_1u_2) \\ &\quad - u_1u_2u_3). & (28) \end{aligned}$$

For $f(u) = 0$ we have chosen the roots to be u_1, u_2 and u_3. Since we are looking at timelike geodesics that means $\eta = 1$. Hence,

$$\begin{aligned} u_1 + u_2 + u_3 &= \frac{1}{2M}, & (29)\\ u_1u_3 + u_2u_3 + u_1u_2 &= \alpha^2, & (30)\\ u_1u_2u_3 &= \frac{\alpha^2}{2M}(1 - \beta^2). & (31) \end{aligned}$$

We are looking at bound orbits so we can put restrictions on some of the parameters such as $\beta^2 < 1$, $\alpha^2 > 0$ and $M > 0$. This shows the right hand side of equation (31) must be positive which means $u_1 u_2 u_3 > 0$ also. This must mean one of the roots is positive and real and leaves three other possibilities for the other two which are:

- Both positive and real,
- Both negative and real,
- A complex pair (i.e. a complex number and its conjugate).

However since $(1 - \beta^2) > 0$, the equation $f(u) = 0$ must have a positive real root. We can also see that $f(u) > 0$ for $u = 0$ and that $f(u) \to \pm\infty$ for $u \to \pm\infty$. This means we must consider the following cases:

1. Three distinct roots which are in the range $0 < u_1 < u_2 < u_3$,
2. First two roots are equal such that $0 < u_1 = u_2 < u_3$,
3. The last two roots are equal such that $0 < u_1 < u_2 = u_3$,
4. All three roots are equal such that $0 < u_1 = u_2 = u_3$.

Each of these cases has two different possibilities either stay in its described orbit which we call orbits of the first-kind or plunge into the singularity of the black which will now be referred to as orbits of the second-kind.

2.1.1 Bound orbits of the first-kind

As described in the previous section orbits of the first kind are orbits where a test particle stays in its described orbit without entering the event horizon and plunging into the singularity. We have 4 different cases where the roots are different so we shall look at each of these in turn and see how the orbit changes with the roots of $f(u) = 0$ changing.

Case 1

So we shall begin with first described case where we have three distinct roots of $f(u) = 0$ such that $0 < u_1 < u_2 < u_3$. We have started with this case because we will be able to derive the equations that describe the motion of a test particle for this case and adapt it for the following cases. We shall denote the roots of $f(u) = 0$ by the following[5]:

$$\begin{aligned} u_1 &= \frac{1}{l}(1 - e), && (32) \\ u_2 &= \frac{1}{l}(1 + e), && (33) \\ u_3 &= \frac{1}{2M} - \frac{2}{l}. && (34) \end{aligned}$$

Where $e =$ eccentricity of orbit which has the range $0 \leq e < 1$ for $\beta^2 < 1$ and $l =$ the lactus rectum[5]. Some important properties to note for the value of the eccentricity of the orbit are:

- $e = 0$ produces a circular orbit,,
- $0 < e < 1$ produces an elliptical orbit,
- $e = 1$ produces a parabolic orbit,
- $e > 1$ produces hyperbolic trajectories.

Since $0 < u_1 \leq u_2 \leq u_3$ we can note that

$$\frac{1}{2M} - \frac{2}{l} \geq \frac{1}{l}(1+e) \Rightarrow l \geq 2M(3+e). \tag{35}$$

If we let μ denote M/l then we can derive an important inequality

$$\mu \leq \frac{1}{2}(3+e) \Rightarrow 1 - 6\mu - 2\mu e \geq 0. \tag{36}$$

Now if we input our newly defined roots into f(u) we get the condition that

$$f(u) = 2M\left(u - \frac{1}{l}(1-e)\right)\left(u - \frac{1}{l}(1+e)\right)\left(u - \left(\frac{1}{2M} - \frac{2}{l}\right)\right) \tag{37}$$

If we now substitute our roots into equations (30) and (31) we can get the following:

$$u_1u_3 + u_2u_3 + u_1u_2 = \alpha^2 \Rightarrow \frac{1}{l^2}\left[l - M\left(3+e^2\right)\right] = M\alpha^2, \tag{38}$$

and

$$u_1u_2u_3 = \frac{\alpha^2}{2m}(1-\beta^2) \Rightarrow \frac{1}{l^3}(1-4M)(1-e^2) = \alpha^2(1-\beta^2). \tag{39}$$

Using our new parameter μ we can simplify equations (38) and (39) to

$$\frac{1}{l^2}[l - M(3+e^2)] = M\alpha^2 \Rightarrow \frac{1}{lM}[1 - \mu(3+e^2)] = \alpha^2 \tag{40}$$

and

$$\frac{1}{l^3}(1-4M)(1-e^2) = \alpha^2(1-\beta^2) \Rightarrow \frac{1}{l^2}(1-4\mu)(1-e^2) = \alpha^2(1-\beta^2). \tag{41}$$

From (40) and (41) we can easily show that

$$\mu < \frac{1}{(3+e^2)} \Rightarrow \mu < \frac{1}{4}. \tag{42}$$

We can confirm that these inequalities are correct from the inequalities we worked out earlier and that $e < 1$. If we now return to our expression for $f(u)$, we will make the substitution $u = 1/l - (e/l)\cos(\delta)$ [5], where δ denotes the relativistic anomaly. Looking at this substitution we can produce the following statements:

1. When $\delta = \pi$ then our test particle is at aphelion distance in the orbit (distance of furthest approach). This is equivalent to $u = u_1$, and

2. When $\delta = 0$ then our test particle is at perihelion distance in the orbit (distance of closest approach). This is equivalent to $u = u_2$.

Now we have made these inferences with our newly formed roots $f(u) = 0$ lets take another look at equation (25). If we do some rearrangements, expansions and use the chain rule we can reformat equation (25) too

$$\left(\frac{du}{d\delta}\right)^2\left(\frac{d\delta}{d\phi}\right)^2 = 2M\left(u^3 - \frac{1}{2M}u^2 + \frac{1}{lM}u - \frac{4}{l^2}u + \frac{(1-e^2)}{l^2}u - \frac{(1-e^2)}{2l^2M}\right. \\ \left. + \frac{2(1-e^2)}{l^3}\right) \tag{43}$$

We can easily calculate $(du/d\delta)^2 = [(d/d\delta)(1/l - (e/l)\cos(\delta))]^2 = (e/l)^2\sin^2(\delta)$. Using this and using the substitution $u = 1/l - (e/l)\cos(\delta)$ in (43) we get

$$\sin^2(\delta)\left(\frac{d\delta}{d\phi}\right)^2 = 1 - 6\mu - 2\mu e\cos(\delta) - \cos^2(\delta) + 6\mu\cos^2(\delta). \tag{44}$$

Hence,

$$\left(\frac{d\delta}{d\phi}\right)^2 = 1 - \frac{2}{Mu}(3 + e\cos(\delta)). \tag{45}$$

This is from factorizing and knowing $\cos^2(\delta) + \sin^2(\delta) = 1$. We can adapt the form of equation (45) if we use the identity $\cos(2\delta) = 2\cos^2(\delta) - 1$. Thus,

$$\left(\frac{d\delta}{d\phi}\right)^2 = \left[(1 - 6\mu + 2\mu e) - 4\mu e\cos^2\left(\frac{\delta}{2}\right)\right]. \tag{46}$$

Hence,

$$\frac{d\delta}{d\phi} = \pm(1 - 6\mu + 2\mu e)^{\frac{1}{2}}\left(1 - K^2\cos^2\left(\frac{\delta}{2}\right)\right)^{\frac{1}{2}}. \tag{47}$$

The letter K is a new parameter to simplify the differential equation (47) and takes the form

$$K^2 = \frac{4\mu e}{1 - 6\mu + 2\mu e} \tag{48}$$

If we look back at the inequalities (36) that we derived at the beginning of this section we can deduce

$$1 - 6\mu + 2\mu e > 0 \Rightarrow K^2 \geq 1 \tag{49}$$

We shall only look at the positive value of the (47) so that ϕ increases strictly positively and we can produce solution in terms of Jacobian elliptical functions.

$$\int \frac{1}{\sqrt{(1-6\mu+2\mu e)\left(1-K^2\cos^2(\frac{\delta}{2}\right)}}\mathrm{d}\delta = \int \mathrm{d}\phi \tag{50}$$

Integrating equation (50), gives

$$\phi = \frac{2}{\sqrt{1-6\mu+2\mu e}}\int_0^{\psi}\frac{1}{\sqrt{1-K^2\sin^2(\tau)}}\mathrm{d}\tau, \tag{51}$$

where $\psi = (\pi - \delta)/2$. Our expression for ϕ in (51) can be simplified by letting

$$F\left(\frac{1}{2}(\pi-\delta), K\right) = F(\psi, K) = \int_0^{\psi}(1-K^2\sin^2(\tau))^{-\frac{1}{2}}\mathrm{d}\tau. \tag{52}$$

Thus,

$$\phi = \frac{2}{\sqrt{1-6\mu+2\mu e}}F(\psi, K). \tag{53}$$

Using the substitution $\psi = (\pi - \delta)/2$ means our statements about aphelion and perihelion distances change slightly.

1. When $\psi = 0$ then our test particle is at aphelion distance in the orbit, and
2. When $\psi = \frac{\pi}{2}$ then our test particle is at perihelion distance in the orbit.

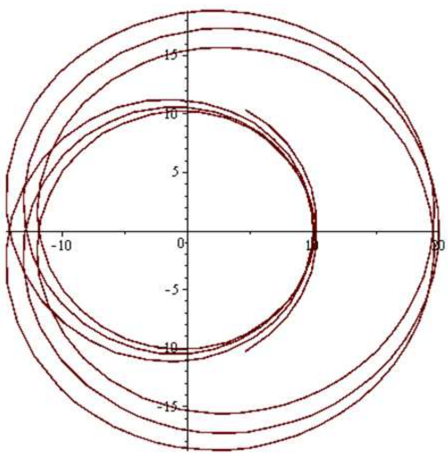

Figure 3: Orbital path of a test particle in Scwarzschild geometry with three distinct roots with M=1.

Figure 3 shows the bound orbit path of a test particle with three distinct roots orbiting a black hole with mass equal to one. It can be seen that the aphelion distance $r_1 = 20$ and the perihelion distance $r_3 = 10$.

Case 2

The second case we discussed was when the first two roots of $f(u) = 0$ are equal, such that $0 < u_1 = u_2 < u_3$. Hence

$$u_1 = \frac{1}{l}(1-e) = \frac{1}{l}(1+e) = u_2 \Rightarrow \begin{cases} e = 0, \\ r_1 = r_2 = l. \end{cases} \tag{54}$$

Since we know r_1 and r_2 refer to the aphelion and perihelion distance respectively. Then if $r_1 = r_2$ would imply that the test particle follows a circular orbit with the singularity of the black hole at its centre. Therefore we have a constant radius which we will call $r_c = l$ and this means we can change our value of μ.

$$\mu = \frac{M}{l} = \frac{M}{r_c} \tag{55}$$

We can substitute (55) into equation (40) to give

$$\frac{1}{lM}[1 - \mu(3+e^2)] = \alpha^2 \Rightarrow \frac{1}{r_c M}\left[1 - 3\frac{M}{r_c}\right] = \alpha^2. \tag{56}$$

Re-arranging, we get

$$r_c^2 - \frac{1}{\alpha^2 M} r_c + \frac{3}{\alpha^2} = 0. \tag{57}$$

We can also substitute (55) into equation (41) to give

$$\frac{1}{l^2}(1-4\mu)(1-e^2) = \alpha^2(1-\beta^2) \Rightarrow \frac{1}{r_c^2}\left(1 - 4\frac{M}{r_c}\right) = \alpha^2(1-\beta^2). \tag{58}$$

From equation (57) we can deduce that

$$r_c = \frac{\frac{1}{\alpha^2 M} \pm \sqrt{\left(\frac{1}{\alpha^2 M}\right)^2 - \frac{12}{\alpha^2}}}{2} = \frac{1}{2\alpha^2 M}[1 \pm \sqrt{1 - 12\alpha^2 M^2}]. \tag{59}$$

Since we are dealing with real roots then we have the restriction $1/(\alpha^2 M^2) \leq 1$. This is to stop us using complex numbers which we will discuss later.

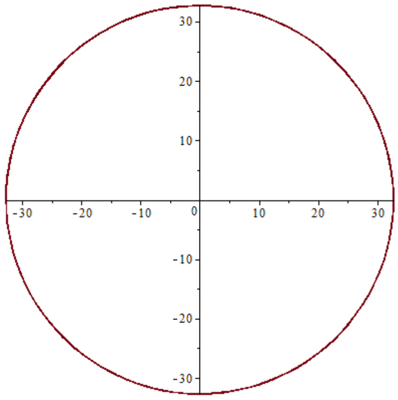

Figure 4: Orbital path of a test particle in Scwarzschild geometry with the first two roots equal.

For Figure 4 we used the conditions $M = 1$ and $\alpha^2 = 1/36$ which gives a circular orbit with radius, $r = 18 + 6\sqrt{6}$. The orbit will be circular and look very similar to figure 4 for any values in the range $1/(\alpha^2 M^2) \leq 1$.

Case 3

The third case we discussed was when the last two roots of $f(u) = 0$ are equal such that $0 < u_1 < u_2 = u_3$. When the roots form in this way we know that aphelion distance of the orbit will be equal to u^{-1} and then fall into a circular orbit with radius r_2 or u_2^{-1}. Since the last two roots equal we can adapt equation (36) to give

$$1 - 6\mu + 2\mu e = 0 \Rightarrow l = 2M(3 + e). \tag{60}$$

If we substitute (60) into the roots of the equation $f(u) = 0$ we find the following.

$$u_1 = \frac{1}{l}(1 - e) \Rightarrow r_1 = \frac{2M(3 + e)}{1 - e}, \tag{61}$$

$$u_2 = \frac{1}{l}(1 + e) \Rightarrow r_2 = \frac{2M(3 + e)}{1 + e}. \tag{62}$$

We can also adapt equation (46) with our knowledge that $1 - 6\mu + 2\mu e = 0$ and using the trigonometric identity $\cos^2(\delta) + \sin^2(\delta) = 1$.

$$\left(\frac{d\delta}{d\phi}\right)^2 = \left[(1 - 6\mu + 2\mu e) - 4\mu e \cos^2\left(\frac{\delta}{2}\right)\right] \Rightarrow \left(\frac{d\delta}{d\phi}\right)^2 = 4\mu e \sin^2\left(\frac{\delta}{2}\right) \tag{63}$$

This simplifies too

$$\frac{d\delta}{d\phi} = \pm 2\sqrt{\mu e} \sin\left(\frac{\delta}{2}\right) \tag{64}$$

From here we will only take the negative value in (64) to ensure that ϕ is always increasing as the test particle moves inwards from its aphelion distance. Now we will integrate this to get an equation with respect to ϕ.

$$d\phi = -\frac{1}{\sqrt{\mu e}}\int \frac{1}{2}\csc\left(\frac{\delta}{2}\right)d\delta = -\frac{1}{\sqrt{\mu e}}\ln\left[\frac{1}{\sin\left(\frac{\delta}{2}\right)} - \frac{\cos\left(\frac{\delta}{2}\right)}{\sin\left(\frac{\delta}{2}\right)}\right] \tag{65}$$

From here we can use the two trigonometric identities $\sin(2\delta) = 2\sin(\delta)\cos(\delta)$ **and** $\cos(2\delta) = 1 - \sin^2(\delta)$ **to reduce equation (65) too**

$$\phi = -\frac{1}{\sqrt{\mu e}}\ln\left[\tan\left(\frac{\delta}{4}\right)\right] \tag{66}$$

We will now rearrange equation (66) to get δ as the subject of the equation and substitute this into our substitution of u to produce the solution for the orbital equation when the final two roots of $f(u) = 0$ are equal.

$$u = \frac{1}{l}(1 - e\cos(\delta)) \Rightarrow u = \frac{1}{l}[1 - e\cos(4\tan^{-1}(e^{-\phi\sqrt{\mu e}}))] \tag{67}$$

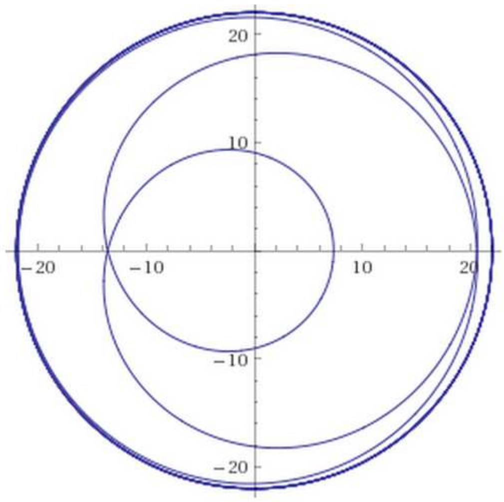

Figure 5: The path of a test particle with a bound orbit of the first-kind with the last two roots of $f(u) = 0$ equal. The orbit has the initial conditions M=11/7 and l=7.

We can use equation (67) to produce the orbital path of our test particle when the last two roots of f(u)=0 are equal which we can see in Figure 5. This orbit it as an aphelion distance of 22 and an perihelion distance of 22/3.

Case 4

The fourth and final case we discussed was when all the roots of $f(u) = 0$ are equal such that $0 < u_1 = u_2 = u_3$. This type of orbit describes a test particle in an unstable circular orbit which will eventually end by plunging into the singularity [5]. Hence no orbit of the first kind can be found for three equal roots.

2.1.2 Bound orbits of the second-kind

As described in the introduction orbits of the second kind are orbits where a test particle orbits a black hole and enters the event horizon and plunges into the singularity. This point to distinguish derives from knowing that $u_1+u_2+u_3 = 1/2M$ and also knowing that orbits of the second kind of aphelion distance of u^{-3} which implies $u_3 < 1/2M$. This means that $u_1 + u_2 > 0$ and, hence, we know that all orbits start outside of the event horizon. In section 2.1.1 we formed a substitution for u. Since we are dealing with a different type of orbit we must alter this substitution slightly [5].

$$u = \left(\frac{1}{2M} - \frac{2}{l}\right) + \left(\frac{1}{2M} - \frac{3+e}{l}\right) \tan^2\left(\frac{1}{2}\varepsilon\right). \tag{68}$$

If we substitute this into equation (25) and use the identities $\cos^2(\varepsilon) + \sin^2(\varepsilon) = 1$ and $1 + \tan^2(\varepsilon/2) = \sec^2(\varepsilon/2)$ we get the following.

$$\left(\frac{du}{d\phi}\right)^2 = f(u) \Rightarrow \left(\frac{d\varepsilon}{d\phi}\right)^2 = (1 - 6\mu + 2\mu e)\left(1 - K^2 \sin^2\left(\frac{1}{2}\varepsilon\right)\right). \tag{69}$$

Our value for K still stands the same as in equation (50). This looks very similar to the solution in 2.1.1 and we can actually express ϕ in terms of the same elliptical integral (54). Hence,

$$\phi = \frac{2}{\sqrt{1 - 6\mu + 2\mu e}} F\left(\frac{1}{2}\varepsilon, K\right). \tag{70}$$

Looking at equation (70) and examining our substitution (68) we can make a couple of points. When the test particle is at aphelion when $u = u_3$ and this only happens when $\varepsilon = 0$ which sets $\phi = 0$. The second point is to note is when we approach the singularity $u \to \infty$ which happens when $\varepsilon \to \pi$ and we can show that

$$\phi_0 = \frac{2}{\sqrt{1 - 6\mu + 2\mu e}} K(k), \tag{71}$$

where, we have let K(k) describe a complete elliptical integral.

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - K^2 \sin^2(\tau)}} \mathrm{d}\tau. \tag{72}$$

Now we have formed a complete equation for ϕ in (70) we will investigate two different cases. The first being when the eccentricity of the orbit is equal to 0 and the second case when we have $2\mu(3+e)=1$.

Case 1 (e = 0)

When the eccentricity is equal to zero we know that this only happens when the first two roots of $f(u)=0$ are equal. We can also notice that $K^2=0$. Therefore we can re-adjust our substitution in (68) and integrate to give

$$\varepsilon = (1-6\mu)^{\frac{1}{2}}(\phi-\phi_0), \text{ where } \phi_0 = \text{constant}. \tag{73}$$

We can now find the solution easily by substituting (73) into (68) with $e=0$ and use the trigonometric identity $1+\tan^2(\varepsilon)=\sec^2(\varepsilon)$ to simplify it. Giving the solution,

$$u = \frac{1}{l} + \left(\frac{1}{2M} - \frac{3}{l}\right)\sec^2\left[\frac{1}{2}(1-6\mu)^{\frac{1}{2}}(\phi-\phi_0)\right]. \tag{74}$$

We know that $l>0$ and $M>0$. So we can make the following observations about equation (74):

- The test particle is at its aphelion distance in the orbit, u_3^{-1} when $\phi=\phi_0$, and
- The test particle reaches the singularity, i.e. $r=o$, when $\phi-\phi_0=\frac{\pi}{(1-6\mu)^{\frac{1}{2}}}$.

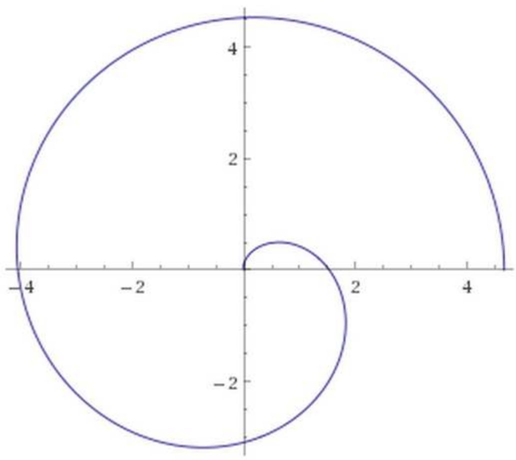

Figure 6: The path of a test particle with a bound orbit of the second-kind with $e=0$.

Depending on how close μ is too a sixth will describe how many times the test particle orbits the black hole before plunging into the singularity. The reason I specifically looked at what happens at $e=0$, and hence $\mu=1/6$, is because it means all roots of $f(u)=0$ must equal. Therefore, since

$u_1 = u_2 = 1/6M$ then we have a circular orbit of $6M$. However, $6M$ is at the very lowest radius of the stable orbiting distance from the equations of motion. Therefore, when $e = 0$ we have the general solution

$$\left(\frac{du}{d\phi}\right)^2 = 2M\left(u - \frac{1}{6M}\right)^3. \tag{75}$$

Hence,

$$u = \frac{1}{6M} + \frac{2}{M(\phi - \phi_0)^2}. \tag{76}$$

We have now found the specific orbital equation for bound orbits when $e = 0$ and we can use it to plot the orbital path of our test particle. We can see an example of an orbit path in Figure 6. The test particle spirals inwards until plunging into the singularity.

Case 2 - 2$\mu(3+e) = 1$

This case is also known when the last to roots of $f(u) = 0$ are equal. With this case we are immediately faced with a complication. This being that we cannot simply input $\mu = 1/(6+2e)$ because it removes the coefficient from the $\tan^2$ in our substitution in (68). Therefore we must examine this case from the beginning without assuming anything from past sections. So to start with we can easily see that when this case arises the roots of $f(u) = 0$ take the form:

$$u_1 = \frac{1-e}{l} \qquad \text{and} \qquad u_2 = u_3 = \frac{1}{4M} - \frac{1-e}{2l} = \frac{1+e}{l}. \tag{77}$$

This means we need to change our substitution slightly and the suggested substitution, taken from [5], is

$$u = \frac{1}{l}\left(1 + e + 2e\tan^2\left(\frac{1}{2}\varepsilon\right)\right). \tag{78}$$

If we examine the substitution (78) we can see that

$$\begin{cases} u = u_2 = u_3 = \frac{1+e}{l} & \text{when } \varepsilon = 0, \\ u \to \infty & \text{when } \varepsilon = \pi. \end{cases} \tag{79}$$

Using the substitution (78) we can see that our general solution (25) transforms too

$$\left(\frac{du}{d\phi}\right)^2 = f(u) \Rightarrow \left(\frac{d\varepsilon}{d\phi}\right)^2 = 4\mu e\sin^2\left(\frac{1}{2}\varepsilon\right) \tag{80}$$

We can simplify this too,

$$\frac{d\varepsilon}{d\phi} = \pm 2\sqrt{\mu e}\sin\left(\frac{1}{2}\varepsilon\right). \tag{81}$$

From here we will only take the negative value in (81) to ensure that ϕ is always increasing as the test particle moves inwards from its aphelion distance. So we will integrate (81) to get an equation with respect to ϕ and get

$$\phi = -\frac{1}{\sqrt{\mu e}} \ln \left[\tan\left(\frac{\varepsilon}{4}\right)\right]. \tag{82}$$

You will see that (82) is exactly the same general solution in case 3 of bound orbits of the first-kind (66). However, we have different roots here so the main difference between these two cases is that the perihelion distance for the orbital equation (66) is the aphelion distance in this case. Observing equation (82) we can show the following

$$\begin{cases} r \to \infty & \text{when}\, \varepsilon = \pi \Rightarrow \phi = 0, \\ r = \frac{l}{1+e} & \text{when}\, \varepsilon \to \infty \Rightarrow \phi \to \infty. \end{cases} \tag{83}$$

Hence, the aphelion distance of the orbit is at $r = l/(1+e)$. We will now rearrange equation (82) to get ε as the subject of the equation and substitute this into equation (78) to produce the solution for the orbital equation when $\mu = 1/(6+2e)$.

$$u = \frac{1}{l}\left(1 + e + 2e\tan^2\left[2\tan^{-1}\left(\exp^{-\phi\sqrt{\mu e}}\right)\right]\right) \tag{84}$$

We now have the specific orbital equation for the $\mu = 1/(6+2e)$ case which we can use to plot and describe its orbital path.

Imaginary roots

We will not be covering imaginary roots but if you are interested in finding out more about what happens to an orbit with imaginary roots then have a look at [5] (pages 111-113).

2.1.3 Unbound orbits of the first and second-kind

An unbound orbit describes the motion of a test particle approaching the black hole but never entering a stable orbit and either eventually exits the gravitational field of the black hole or plunges into the singularity. They occur when $\beta^2 > 1$. Since $u_1u_2u_3 = \alpha^2(1-\beta^2)/2M$ then we must allow for one negative root. Since we know $u_1 < u_2 < u_3$ we know u_1 must be the negative root and leaves three possibilities for the roots:

1. Three distinct roots with the restrictions $u_1 < 0 < u_2 < u_3$, or
2. The last two roots are equal with the restrictions $u_1 < 0 < u_2 = u_3$, or
3. The last two roots are complex-conjugate roots.

We must also consider both types of orbits. For orbits of the first-kind we must have the restriction $0 < u \leq u_2$ which will lead to hyperbolic trajectories. For orbits of the second kind we must have the restriction $u \geq u_3$ which will hold similar features to that of bound orbits of the second-kind. Our second possibility when the final two roots are equal is a special case. When the final two roots are equivalent then the trajectories they describe are the same for the first and second-kind orbits.

Case 1 - All three roots are real

Since we are dealing with unbound orbits then our restrictions upon the eccentricity of the orbit changes to $e \geq 1$. Therefore our new roots of $f(u) = 0$ take the form [5]:

$$u_1 = -\frac{1}{l}(e-1), \tag{85}$$

$$u_2 = \frac{1}{l}(e+1), \tag{86}$$

$$u_3 = \frac{1}{2M} - \frac{2}{l}. \tag{87}$$

Our inequality in (36) still holds at $1-6\mu-2\mu e \geq 0$ since we have $u_1 < u_2 \leq u_3$. However since $e \geq 1$ we need to change one of our relations. The changes don't affect (40) but they do affect (41) so we need to re-write this as

$$\alpha^2(\beta^2 - 1) = \frac{1}{l^2}(1-4\mu)(e^2-1). \tag{88}$$

We know that $\alpha^2 > 0$ and $\beta^2 - 1 \geq 0$. This means if we examine (40) and (88) we can show

$$1 - \mu(3+e^2) > 0 \Rightarrow \mu \leq \frac{1}{4}. \tag{89}$$

We can use the same substitution as in section 2.1.1 which is $u = 1/l + e\cos(\delta)/l$ but since we have $e \geq 1$ in this section then we have different values for the aphelion and perihelion distances of the orbit.

$$\begin{cases} u = 0 & \text{when } \delta = \cos^{-1}(-e^{-1}), \\ u = \frac{1+e}{l} & \text{when } \delta = 0. \end{cases} \tag{90}$$

Hence, an unbound orbit approaches from infinity then reaches an aphelion distance of $u = (1+e)/l$. We can now find the solution for ϕ using a similar technique as in section 2.1.1 and get a similar result as (53) with some slight differences.

$$\phi = \frac{2}{\sqrt{1-6\mu+2\mu e}}\left[K(k) - F\left(\frac{1}{2}(\pi-\delta), K\right)\right]. \tag{91}$$

For orbits of the first kind the test particle spirals inwards towards the aphelion distance then the test particles trajectory goes off to infinity along the direction, ϕ_∞.

$$\phi_\infty = \frac{2}{\sqrt{1-6\mu+2\mu e}}\left[K(k) - F\left(\frac{1}{2}\cos^{-1}(e^{-1}), K\right)\right]. \tag{92}$$

Orbits of the second kind with an unbound orbit follow exactly the same procedure as orbits with bound orbits. So the discussion we had in section 2.1.2 (pages 12-16) applies here but we change the eccentricity of the orbit to a value larger than or equal to one for unbound orbits.

Imaginary roots

We will not be covering imaginary roots but if you are interested in finding out more about what happens to an orbit with imaginary roots then have a look at [5] (pages 115-122).

2.2 Null Geodesics

A null geodesic, as described in the introduction, describes the motion of a test particle moving along the edge of the light cone at the speed of light. Einsteins Theory of Relativity states that light is affected by gravitational fields of an entity such as a star or a black hole. Depending on the initial trajectory of the photon we can have two different types of orbits. We could have a bound orbit which can either remain in their described motion around the exterior of the black hole or enter the event horizon and plunge into the singularity. Or it could follow an unbound orbit which describes the motion of the test particle approaching the black hole but never entering a stable orbit and eventually exit the gravitational field of the black hole. From our working in section 2.1 we know the equation that determines the geometry of the geodesics in the invariant plane.

$$\left(\frac{\partial u}{\partial \phi}\right)^2 = \alpha^2\beta^2 - (u^2+\eta\alpha^2)(1-2Mu) \tag{93}$$

For null geodesics we know that the Lagrangian is equal to zero and $\eta = 0$, which gives

$$\begin{aligned}\left(\frac{du}{d\phi}\right)^2 = f(u) &= \alpha^2\beta^2 - u^2(1-2Mu) = 2Mu^3 - u^2 + \alpha^2\beta^2 &(94)\\ &= 2M(u-u_1)(u-u_2)(u-u_3). &(95)\end{aligned}$$

This means we have different roots for $f(u) = 0$:

$$\begin{aligned}u_1+u_2+u_3 &= \frac{1}{2M}, &(96)\\ u_1u_3+u_2u_3+u_1u_2 &= 0, &(97)\\ u_1u_2u_3 &= -\frac{\alpha^2\beta^2}{2M} = -\frac{\lambda^2}{2M}. &(98)\end{aligned}$$

We shall let $\alpha\beta = \lambda$, where $\lambda =$ the impact parameter. The impact parameter is the perpendicular distance between the tangent to the test particle as it enters the gravitational pull of the mass and a line running parallel to it which intersects the mass. Varying this will increase or decrease the size of the gravitational pull on the test particle.

Introducing the impact parameter means we can re-write the general orbital equation (94) too

$$\left(\frac{du}{d\phi}\right)^2 = f(u) = 2Mu^3 - u^2 + \lambda^2. \tag{99}$$

2.2.1 Bound Orbits of the first-kind

To solve these simultaneous equations for $f(u) = 0$ we must consider all values u_1,u_2 and u_3. Since $u_1u_2u_3 < 0$ we must allow one negative real root. We know $u_1 \leq u_2 \leq u_3$ then u_1 must be the negative root. So for the remaining two roots we have the following two possibilities: two real roots (either distinct or equivalent) or a complex conjugate pair. For the two real roots we can derive the roots by equating $f'(u) = 0$.

$$f'(u) = 6Mu^2 - 2u = 2u(3Mu - 1) = 0. \tag{100}$$

We can see that equation (100) gives roots of either $u = 0$ or $u = M/3$. For a non-zero impact parameter we can see that $u = M/3$ will be a root of (99). If we put $u = M/3$ into equation (99) we can see it is actually a double root

$$f(u) = 2M\left(\frac{1}{3M}\right)^3 - \left(\frac{1}{3M}\right)^2 + \lambda^2 = 0. \tag{101}$$

This implies,

$$\lambda^2 = \frac{1}{27M^2} \qquad \text{or} \qquad \lambda = \frac{1}{3\sqrt{3}M}. \tag{102}$$

So now we have the value for our impact parameter we can easily derive the roots, which are:

$$u_1 = -\frac{1}{6}M, \qquad u_2 = \frac{1}{3}M \qquad \text{and} \qquad u_3 = \frac{1}{3}M. \tag{103}$$

It can be noted that due to these roots a circular orbit can be achieved, however the orbit would be very unstable. Now let us rewrite the orbital equation with our new roots

$$\left(\frac{du}{d\phi}\right)^2 = 2M(u-u_1)(u-u_2)(u-u_3) \tag{104}$$

$$= 2m\left(u + \frac{1}{6}M\right)\left(u - \frac{1}{3}M\right)^2 \tag{105}$$

We now have the base equation to describe the motion in bound orbits within null geodesics. With certain substitutions we may adapt this equation for those of the first and second kind of orbit.

First Case - two real roots

For this case we can just use a straight substitution for u which satisfies equation (105) [5].

$$u = -\frac{1}{6M} + \frac{1}{2M}\tanh^2\left[\frac{1}{2}(\phi - \phi_0)\right], \text{where}\, \phi_0 = \text{constant}. \tag{106}$$

From the substitution (106) we can show the following two points.

$$\begin{cases} u = 0 \Rightarrow r \to \infty & \text{when}\, \phi = 0 \,\text{and}\, \tanh^2(\phi_0/2) = \frac{1}{3}, \\ u = \frac{1}{3M} \Rightarrow r = 3M & \text{when}\, \phi \to \infty \,\text{and}\, \tanh^2((\phi - \phi_0)/2) = 1. \end{cases} \tag{107}$$

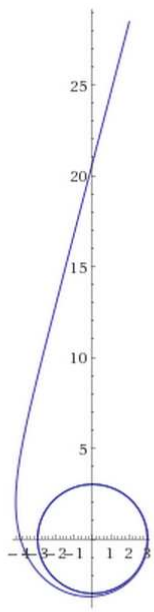

Figure 7: The path of a test particle with a bound orbit of the first-kind with real roots.

So we can take the assumption that as ϕ increases the test particle approaches its stable circular orbit of $r = 3M$ which we can see in Figure 7.

Imaginary roots

We will not be covering imaginary roots but if you are interested in finding out more about what happens to an orbit with imaginary roots then have a look at [5] (pages 133-134).

2.2.2 Bound Orbits of the second-kind

For orbits of the second kind we can use a straight substitution [5] for u which satisfies equation (105).

$$u = \frac{1}{3M} + \frac{1}{2M}\tan^2\left(\frac{1}{2}\varepsilon\right). \tag{108}$$

If we substitute (108) into equation (105) and use the trigonometric identity $1 + \tan^2(\varepsilon/2) = \sec^2(\varepsilon/2)$ **we get**

$$\frac{d\varepsilon}{d\phi} = \sqrt{\sin^2\left(\frac{\varepsilon}{2}\right)}. \tag{109}$$

Hence,

$$\phi = 2\ln\left[\tan\left(\frac{\varepsilon}{4}\right)\right]. \tag{110}$$

We found ϕ **by using standard trigonometric identities and rearranging. We can rearrange equation (110) to put it in terms of** ε**, such that** $\varepsilon = 4\tan^{-1}(\exp(\phi/2))$ **and then we can substitute this back into our substitution (108) of** u **to obtain the full solution for bound orbits of the second-kind.**

$$u = \frac{1}{3M} + \frac{1}{2M}tan^2\left[2\tan^{-1}\left(e^{\frac{\phi}{2}}\right)\right] \tag{111}$$

We can re-write u **in terms of exponentials instead, such that**

$$u = \frac{1}{3M} + \frac{2e^{\phi}}{M(e^{\phi}-1)^2} \tag{112}$$

Now we have the solution we can show the following attributes of the orbit.

$$\begin{cases} u \to \infty \Rightarrow r \to 0 & \text{when}\,\phi \to 0, \\ u \to \frac{1}{3M} \Rightarrow r \to 3M & \text{when}\,\phi \to \infty. \end{cases} \tag{113}$$

2.2.3 Unbound Orbits of the first-kind

In this section we will be using the perihelion distance of the orbit so for ease of notation we shall now refer to it as P**. If we have three distinct roots for** $f(u) = 0$ **then they take the form [5]:**

$$u_1 = \frac{P-2M-C}{4MP}, \qquad u_2 = \frac{1}{P} \qquad \text{and} \qquad u_3 = \frac{P-2M+C}{4MP}. \tag{114}$$

where C equals a constant. Since our roots take the form $u_1 < u_2 < u_3$ then we can show that we have the following restriction.

$$\frac{1}{P} < \frac{P - 2M + C}{4MP}. \tag{115}$$

Re-arranging, we get

$$0 < P - 6M + C. \tag{116}$$

If we square equation (116) and re-arrange it so we have C as the subject of the equation we get

$$C^2 < (P - 6M)^2. \tag{117}$$

We will need to use this inequality shortly. Now we have defined the roots of $f(u) = 0$ then we shall input them into our relation equations (97) and (98). We shall begin by substituting our roots into equation (97).

$$\frac{P - 2M - C}{4MP^2} + \frac{P - 2M + C}{4MP^2} + \frac{P^2 + 4M^2 - C^2 - 4MP}{16M^2P^2} = 0. \tag{118}$$

Re-arranging equation (118) for C, we get

$$C^2 = (P + 6M)(P - 2M). \tag{119}$$

Now we have got a value for C^2 we can input this into our inequality (117) and rearrange to find the restrictions on our perihelion distance.

$$(P + 6M)(P - 2M) < (P - 6M)^2. \tag{120}$$

This implies that

$$P > 3M. \tag{121}$$

Now we shall look at what happens when we put our roots (114) into the relation (98).

$$\left(\frac{P - 2M - C}{4MP}\right)\left(\frac{1}{P}\right)\left(\frac{P - 2M + C}{4MP}\right) = -\frac{\lambda^2}{2M}. \tag{122}$$

Re-arranging equation (122), we get

$$\lambda^2 = \frac{1}{8MP^3}\left[C^2 - (P - 2M)^2\right]. \tag{123}$$

If we input equation (119) into equation (123) we can remove the constant term, C.

$$\lambda^2 = \frac{1}{8MP^3}[(P + 6M)(P - 2M) - (P - 2M)^2] = \frac{P - 2M}{P^3}. \tag{124}$$

If we put our restriction of the perihelion distance (121) into equation (124) we can see what restrictions we have on the impact parameter.

$$\lambda^2 > \frac{3M - 2M}{27M^3} \Rightarrow \lambda^2 > \frac{1}{27M^2}. \tag{125}$$

Therefore we have the following restrictions upon our orbit. That $r > 3M$ and $\lambda^2 > 1/27M^2$ which means we are considering a completely different orbit path to those in section 2.2.1. Therefore, we need a different substitution for u. The suggested substitution, taken from [5] is

$$u = \frac{P - 2M - C}{4MP} + \frac{C - P + 6M}{8MP}[1 - \cos(\delta)]. \tag{126}$$

So if we substitute (126) into our general orbital equation (105) and use the trigonometric identity $\cos(2\delta) = 1 - 2\sin^2(\delta)$ we arrive at

$$\left(\frac{d\delta}{d\phi}\right)^2 = \frac{C}{P}\left[1 - B^2 \sin^2\left(\frac{1}{2}\delta\right)\right] \tag{127}$$

where,

$$B^2 = \frac{1}{2C}[C - P + 6M]. \tag{128}$$

We can write the solution to (127) in terms of Jacobian elliptic integrals. Therefore our solution for ϕ is

$$\phi = 2\sqrt{\frac{P}{C}}\left[K(k) - F\left(\frac{1}{2}\delta, B\right)\right]. \tag{129}$$

The values $K(k)$ is defined in equation (72) and $F(\frac{1}{2}\delta, B)$ holds similar form to that of equation (52). For orbits of the first kind the test particle spirals inwards towards the aphelion distance, which is $u = 1/P$, then the test particles trajectory goes off to infinity along the direction, ϕ_∞.

$$\phi_\infty = 2\sqrt{\frac{P}{C}}\left[K(k) - F\left(\frac{1}{2}\delta_\infty, B\right)\right]. \tag{130}$$

where,

$$\frac{1}{2}\delta_\infty = \arcsin\left(\sqrt{\frac{C - P + 2M}{C - P + 6M}}\right)$$

If we use equation (129) and our substitution (126) we can plot the orbital path of our test particle.

2.2.4 Unbound Orbits of the second-kind

When we have unbound orbits of the second-kind we know that we have the range $u_3 \leq u < \infty$. Therefore we need to adopt a slightly different substitution [5] to that of (126) which is

$$u = \frac{1}{P} + \frac{C + P - 6M}{4MP} \sec^2 (\frac{1}{2}\delta). \tag{131}$$

Examining this substitution we can make the following two observations

$$\begin{cases} u = u_3 = \frac{C+P-2M}{4MP} & \text{when } \delta = 0 \text{ (aphelion distance)}, \\ u \to \infty \Rightarrow r \to 0 & \text{when } \delta = \pi. \end{cases} \tag{132}$$

So if we substitute (131) into equation (105) and integrate out we get the solution

$$\phi = 2\sqrt{\frac{P}{C}} F \left[\frac{1}{2}\delta, B\right], \tag{133}$$

where B is still defined as (128). Therefore if we examine equation (133) and our observations it is possible to plot the orbital path of unbound orbits of the second-kind in Schwarzschild geometry.

3 Kerr Black Holes

A Kerr black hole is far more complex than a Schwarzschild black hole mainly due to the fact that a Kerr black hole has a rotating core. Since it has a rotating body it does not have spherical symmetry so we cannot use the Schwarzschild solution. It took a long time to discover the Kerr solution, nearly 50 years after Einstein published his paper on relativity. This solution has been derived by looking at the axial symmetries around the axis of rotation [8]. Therefore the Kerr solution has a different metric which we will look into for both timelike and null geodesics.

In this section we will determine a radial equation for Kerr geometry and adapt for null and timelike geodesics. We will also delve into how a test particle trajectory acts in a retrograde orbit (where the test particle moves in the opposite direction to that of the black hole) and a direct orbit (where the test particle moves in the same direction to that of the black hole) [8]. The line element for the space-time around a Kerr black hole is given by equation (134) with angular momentum a and letting $c = G = 1$.

$$\begin{aligned} ds^2 = &-\left(1-\frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mar\sin^2(\theta)}{\rho^2}d\phi dt + \frac{\rho^2}{\Delta(r)}dr^2 + \rho^2 d\theta^2 \\ &+ \left(r^2+a^2+\frac{2Ma^2r\sin^2(\theta)}{\rho^2}\right)\sin^2(\theta)d\phi^2, \end{aligned} \tag{134}$$

where,

$$\Delta(r) = r^2 - 2Mr + a^2 \text{ and } \rho^2 = r^2 + a^2cos(\theta). \tag{135}$$

There are two principle features to this metric which we will examine. The first relates to the nature of the null surfaces which we know occur at $\Delta(r) = 0$ which gives

$$r = \frac{2M \pm \sqrt{4M^2-4a^2}}{2} = M \pm \sqrt{M^2-a^2}. \tag{136}$$

This gives us $r_\pm = M \pm \sqrt{M^2-a^2}$ which are both positive as long as $a^2 < M^2$. We can note that r_- covers the singularity, which is inside r_+ so we will use base our calculations on the value of r_+. The second principle we need to examine is what happens at $r = 0$ and $\theta = \pi/2$ which corresponds to $\rho^2 = 0$. If we put these values into the metric we can see the coefficients become undefined and hence we are looking at the singularity of the Kerr black hole.

In this section we are only going to consider the orbits in the equatorial plane with $\dot{\theta} = 0$ and $\theta = \pi/2$ as it makes the metric much easier to deal with. It reduces ρ^2 to r^2 and the metric to

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 - \frac{4Ma}{r}d\phi dt + \frac{r^2}{\Delta(r)}dr^2 + \left(r^2+a^2+\frac{2Ma^2}{r}\right)d\phi^2. \tag{137}$$

Now if we follow a similar mathematical approach to the Kerr metric we should obtain an radial equation for Kerr geometry. So if we divide (137) by $d\tau^2$ we obtain (i.e $(ds/d\tau)^2 = 2L$) the Lagrangian for Kerr geometry.

$$2L = -\left(1 - \frac{2M}{r}\right)\dot{t}^2 - \frac{4Ma}{r}\dot{\phi}^2\dot{t}^2 + \frac{r^2}{\Delta(r)}\dot{r}^2 + \left(r^2 + a^2 + \frac{2Ma^2}{r}\right)\dot{\phi}^2. \tag{138}$$

The dots denote differentiation with respect to τ (i.e. $\dot{t} = dt/d\tau$). The corresponding canonical momenta are

$$\begin{aligned} p_t &= \frac{\partial L}{\partial \dot{t}} = -\left(1 - \frac{2M}{r}\right)\dot{t} - \frac{2Ma}{r}\dot{\phi}, & (139)\\ p_r &= \frac{\partial L}{\partial \dot{r}} = \frac{r^2}{\Delta(r)}\dot{r}, & (140)\\ p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = -\frac{2Ma}{r}\dot{t} + \left(r^2 + a^2 + \frac{2Ma^2}{r}\right)\dot{\phi}. & (141)\end{aligned}$$

The corresponding Hamiltonian is,

$$H = p_t\dot{t} + p_r\dot{r} + p_\phi\dot{\phi} - L = L. \tag{142}$$

Now we will consider each space-time coordinate through the Euler-Lagrange equation (which has the form $0 = \partial L/\partial x^\mu - (d/d\tau)(\partial L/\partial x^\mu)$). So if we start by looking at the t-coordinate we get

$$0 = \frac{\partial L}{\partial t} - \frac{d}{d\tau}\frac{\partial L}{\partial \dot{t}}\ , \text{where } \frac{\partial L}{\partial t} = 0. \tag{143}$$

Thus,

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{t}} = \frac{dp_t}{d\tau} = 0. \tag{144}$$

Integrating out, we can show

$$p_t = -\left(1 - \frac{2M}{r}\right)\dot{t} - \frac{2Ma}{r}\dot{\phi} = \text{constant} = -\beta. \tag{145}$$

Now if we look into the ϕ coordinate,

$$0 = \frac{\partial L}{\partial \phi} - \frac{d}{d\tau}\frac{\partial L}{\partial \dot{\phi}}\ , \text{where } \frac{\partial L}{\partial \phi} = 0. \tag{146}$$

Thus,

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{\phi}} = \frac{dp_\phi}{d\tau} = 0. \tag{147}$$

Integrating this out, we can show

$$p_\phi = -\frac{2Ma}{r}\dot{t} + \left(r^2 + a^2 + \frac{2Ma^2}{r}\right)\dot{\phi} = \text{constant} = \alpha. \tag{148}$$

We can note here that α is equal to the angular momentum about an axis normal to the equatorial plane. The equality of the Hamiltonian and the Lagrangian shows that all the energy in the problem derives solely from kinetic energy. Now we have got values for the components of the Euler-Lagrange equation the constancy of the Lagrangian gives

$$2H = 2L = -\beta \dot{t} - \alpha \dot{\phi} + \frac{r^2}{\Delta(r)} \dot{r}^2 = \delta_g \tag{149}$$

where,

$$2L = \delta_g = \begin{cases} 0 = & \text{null geodesic ,} \\ 1 = & \text{timelike geodesic.} \end{cases} \tag{150}$$

From here we need to do some rearrangements and substitutions to get an equation with respect to r. Lets begin rearranging equation (145)

$$-\left(1 - \frac{2M}{r}\right) \dot{t} - \frac{2Ma}{r} \dot{\phi} = -\beta \Rightarrow \dot{t} = \left(\beta - \frac{2Ma}{r} \dot{\phi}\right) \left(1 - \frac{2M}{r}\right)^{-1}. \tag{151}$$

Now substitute this into equation (148) and rearrange it

$$-\frac{2Ma}{r} \dot{t} + \left(r^2 + a^2 + \frac{2Ma^2}{r}\right) \dot{\phi} = \alpha \Rightarrow \dot{\phi} = \frac{1}{\Delta} \left[\left(1 - \frac{2M}{r}\right) \alpha + \frac{2Ma}{r} \beta\right]. \tag{152}$$

Now if we substitute (152) into (151) we get

$$\dot{t} = \frac{1}{\Delta} \left[\left(r^2 + a^2 + \frac{2Ma^2}{r}\right) \beta - \frac{2Ma}{r} \alpha\right]. \tag{153}$$

If we substitute these newly formed equations into equation (149) we obtain the radial equation for Kerr geometry, which takes the form,

$$r^2 \dot{r}^2 = r^2 \beta^2 + \frac{2M}{r} (\alpha - a\beta)^2 - (\alpha^2 - a^2 \beta^2) - \delta_g \Delta. \tag{154}$$

3.1 Timelike Geodesics

So for a timelike geodesic in Kerr geometry $\delta_g = 1$ which transforms our radial equation(154) to

$$\dot{r}^2 = \beta^2 + \frac{2M}{r^3} (\alpha - a\beta)^2 - \frac{1}{r^2} (\alpha^2 - a^2 \beta^2) - \frac{1}{r^2} \Delta \tag{155}$$

where, β denotes the energy per unit mass of our test particle describing the trajectory. From section 2 we know that $u = 1/r$ so we transform our radial equation (155) to

$$u^{-4} \dot{u}^2 = \beta^2 + 2Mu^3 (\alpha - a\beta)^2 - u^2 (\alpha^2 - a^2 \beta^2) - (1 - 2Mu + a^2 u^2). \tag{156}$$

We will fix r and, hence, u too which will mean we will be describing a circular orbit. So our initial objective is to attain values of α and β which we can find when $\dot{u} = 0$.

$$0 = \beta^2 + 2Mu^3(\alpha - a\beta)^2 - u^2(\alpha^2 - a^2\beta^2) - (1 - 2Mu + a^2u^2). \tag{157}$$

We also know that $\ddot{u} = 0$ so if we differentiate equation (157) with respect u gives

$$0 = 3Mu^2(\alpha - a\beta)^2 - u(\alpha^2 - a^2\beta^2) - (-Mu + a^2u). \tag{158}$$

For ease of notation we will let $\xi = \alpha - a\beta$ which transforms our equations (157) and (158), respectively, too

$$\begin{aligned} 0 &= 2M\xi^2u^3 - u^2(\xi^2 - 2a\beta\xi + a^2) + 2Mu + (\beta^2 - 1), & (159)\\ 0 &= 3M\xi^2u^2 - u(\xi^2 - 2a\beta\xi + a^2) + M. & (160)\end{aligned}$$

If we multiply equation (160) by u and subtract it from equation (159) we find

$$\beta^2 = M\xi^2u^3 - Mu + 1. \tag{161}$$

We can now substitute this into equation (160) to give

$$2a\xi\beta u = \xi^2(3Mu - 1)u - (a^2u - M). \tag{162}$$

If we square both sides of equation (162) and substitute into equation (161) we can eliminate β, to give

$$\begin{aligned} 0 = \xi^4u^2[(3Mu-1)^2 - 4a^2Mu^3] - 2\xi^2u[(3Mu-1)(a^2u - M) \\ - 2a^2u(Mu-1)] + (a^2u - M)^2. \end{aligned} \tag{163}$$

We can see straight away that this is a quadratic in ξ^2 so we can easily find a solution. However due to the length of the equation we shall break it down slight by initially working out the discriminant.

$$\begin{aligned} \frac{1}{4}(b^2 - 4ac) &= u^2[(3Mu-1)(a^2u - M) - 2a^2u(Mu-1)]^2 \\ &\quad - u^2[(3Mu-1)^2 - 4a^2Mu^3](a^2u - M)^2 & (164)\\ &= 4a^2M\Delta^2(u)u^3 & (165)\end{aligned}$$

where, $\Delta(u) = 1 - 2Mu + a^2u^2$. So the we can now find the solution to equation (163).

$$\begin{aligned} \xi^2u^2 &= \frac{[(3Mu-1)(a^2u - M) - 2a^2u(Mu-1)]^2 \pm \sqrt{4a^2M\Delta^2(u)u^3}}{(3Mu-1)^2 - 4a^2Mu^3}, & (166)\\ &= \frac{\left[1 - 3Mu \pm 2a\sqrt{Mu^3}\right]\Delta(u) - [(3Mu-1)^2 - 4a^2Mu^3]}{(3Mu-1)^2 - 4a^2Mu^3}. & (167)\end{aligned}$$

To simplify this equation we shall use the following substitution [5]:

$$\chi_{\pm} = 1 - 3Mu \pm 2a\sqrt{Mu^3} \Rightarrow \chi_{+}\chi_{-} = (3Mu-1)^2 - 4a^2Mu^3. \tag{168}$$

Therefore, we can rewrite equation (167) with this substitution, such that

$$\xi^2u^2 = \frac{\chi_{\pm}\Delta(u) - \chi_{+}\chi_{-}}{\chi_{+}\chi_{-}} = \frac{\chi_{\pm}\Delta(u)}{\chi_{+}\chi_{-}} - 1 = \frac{\chi_{\pm}\chi_{\mp}}{\chi_{+}\chi_{-}}\frac{\Delta(u)}{\chi_{\mp}} - 1 = \frac{1}{\chi_{\mp}}(\Delta(u) - \chi_{\mp}). \tag{169}$$

Since $\Delta(u) - \chi_{\mp} = u(a\sqrt{u} \pm \sqrt{M})^2$ we can show the following for ξ.

$$\xi = -\frac{a\sqrt{u} \pm \sqrt{M}}{\sqrt{u\chi_{\mp}}} \Rightarrow \xi = \begin{cases} -\frac{a\sqrt{u}+\sqrt{M}}{\sqrt{u\chi_{\mp}}} = & \text{retrograde orbit,} \\ -\frac{a\sqrt{u}-\sqrt{M}}{\sqrt{u\chi_{\mp}}} = & \text{direct orbit.} \end{cases} \tag{170}$$

If we now substitute equation (170) into equation (161) we can obtain a value for β.

$$\beta^2 = \frac{1}{\chi_{\mp}}\left(1 + 4M^2u^2 + Ma^2u^3 - 4mu \mp 2a\sqrt{Mu^3} \pm 4aMu\sqrt{Mu^3}\right). \tag{171}$$

Square-rooting, we can show

$$\beta = \frac{1}{\sqrt{\chi_{\mp}}}\left(1 - 2Mu \mp a\sqrt{Mu^3}\right). \tag{172}$$

We can now easily find a value for α by substituting (172) into the substitution $\xi = \alpha - a\beta$.

$$\alpha = \mp\sqrt{\frac{M}{u\chi_{\mp}}}\left[1 + a^2u^2 \pm 2a\sqrt{Mu^3}\right]. \tag{173}$$

So we have found the energy and angular momentum for our test particle and the reason for this was too make the following calculations far easier. We shall now select certain values of β and α from equations (172) and (173), respectively, which we will assign β_c and α_c for the reciprocal radius u_c. Assigning them this way means equation (156) will allow a double root at $u = u_c$ and, hence, reduces equation (156) down too

$$u^{-4}\dot{u}^2 = 2M(\alpha_c - a\beta_c)^2(u-u_c)^2\left[u + 2u_c - \frac{\alpha_c^2 - a^2\beta_c^2 + a^2}{2M(\alpha_c - a\beta_c)^2}\right]. \tag{174}$$

Let us examine the final term on the right-hand side of equation (174) and simplify it down.

$$\frac{\alpha_c^2 - a^2\beta_c^2 + a^2}{2M(\alpha_c - a\beta_c)^2} = \frac{\frac{M}{u_c\chi_{\mp}}\left[1 + 3a^2u_c^2 \pm 4a\sqrt{Mu_c^3}\right]}{\frac{M}{u_c\chi_{\mp}}(a\sqrt{u_c} \pm \sqrt{M})^2} = \frac{1 + 3a^2u_c^2 \pm 4a\sqrt{Mu_c^3}}{(a\sqrt{u_c} \pm \sqrt{M})^2}. \tag{175}$$

Using equation (175) we can show,

$$u_c - \frac{\alpha_c^2 - a^2\beta_c^2 + a^2}{2M(\alpha_c - a\beta_c)^2} = \frac{2Mu_c - a^2u_c^2 - 1}{(a\sqrt{u_c} \pm \sqrt{M})^2} = -\frac{\Delta_{u_c}}{2(a\sqrt{u_c} \pm \sqrt{M})^2} \tag{176}$$

where,

$$\Delta_{u_c} = a^2u_c^2 - 2Mu_c + 1. \tag{177}$$

If we set $u_* = -u_c + \frac{\Delta_{u_c}}{2(a\sqrt{u_c}\pm\sqrt{M})^2}$ **we can re-write equation (174) as**

$$u^{-4}\dot{u}^2 = 2M(\alpha_c - a\beta_c)^2(u - u_c)^2(u - u_*). \tag{178}$$

If we multiply equation (178) through by u^4 **and use our substitution** ξ**, we get**

$$\left(\frac{d\phi}{d\tau}\right)^2\left(\frac{du}{d\phi}\right)^2 = 2M\xi_c^2u^4(u - u_c)^2(u - u_*). \tag{179}$$

We can re-arrange equation (152) and use our critical values we get,

$$\frac{d\phi}{d\tau} = \frac{u^2}{\Delta_u}(\alpha_c - 2M\xi_c u). \tag{180}$$

Substituting this into equation (179) we get,

$$\frac{du}{d\phi} = \frac{\sqrt{2M}\xi(u - u_c)\sqrt{u - u_*}}{\alpha_c - 2M\xi_c u}. \tag{181}$$

Thus,

$$\phi = \frac{1}{\xi_c a^2\sqrt{2M}}\int \frac{\alpha_c - 2M\xi_c u}{(u - u_+)(u - u_-)(u - u_c)\sqrt{u - u_*}}\,du, \tag{182}$$

where,

$$u_\pm = \frac{1}{a^2}\left[M \pm \sqrt{M^2 - a^2}\right]. \tag{183}$$

Hence, we have found the orbital equation for time-like geodesics in direct and retrograde orbits. Due to space and time constraints we will not be able to show you the plots but you can do this by putting equation (182) in maple.

3.2 Null Geodesics

So for null geodesics in Kerr geometry $\delta_g = 0$ **which transforms our radial equation (154) to**

$$\dot{r}^2 = \beta^2 + \frac{2M}{r^3}(\alpha - a\beta)^2 - \frac{1}{r^2}(\alpha^2 - a^2\beta^2). \tag{184}$$

Again we will follow a similar path to adapt the radial equation in Kerr geometry for null geodesics as we did for the orbital equation in Schwarzschild geometry. So to begin we need to define the impact parameter which we will set as $\gamma = \alpha/\beta$. As discussed in section 2.2 different values of the impact parameter will produce different types of orbit so we will consider the critical value of the impact parameter, γ_c, which corresponds to an unstable circular orbit of radius r_c. We can make two conclusions from this. The first being that orbits of the first kind arriving from infinity will enter a stable orbit with perihelion distance larger than that of r_c. The second being that orbits of the second kind arriving from infinity will have an aphelion distance less than that of r_c and, hence, will plunge into the singularity at $r = 0$ (and $\theta = \pi/2$). For any geodesic with a value of $\gamma < \gamma_c$ then they will describe orbits of the second kind.

So we will find the value for the critical radius by letting $r = r_c$ and $\dot{r} = 0$. Equation (184) becomes,

$$0 = \beta^2 + \frac{2M}{r^3}(\alpha - a\beta)^2 - \frac{1}{r^2}(\alpha^2 - a^2\beta^2). \tag{185}$$

Now we shall differentiate (185) with respect to r. Since $\dot{r} = 0$, then we know $\ddot{r} = 0$. Hence,

$$0 = -\frac{6M}{r^4}(\alpha - a\beta)^2 + \frac{2}{r^3}(\alpha^2 - a^2\beta^2). \tag{186}$$

If we know rearrange this equation we can form an equation which gives the critical radius.

$$r_c = 3M\frac{\alpha - a\beta}{\alpha + a\beta} = 3M\frac{\gamma_c - a}{\gamma_c + a}. \tag{187}$$

Now if we substitute this into equation (185) we get,

$$\beta^2 = \frac{1}{27M^2}\frac{(\alpha - a\beta)^3}{\alpha + a\beta} = \frac{\beta^2}{27M^2}\frac{(\gamma_c - a)^3}{\gamma_c + a} \Rightarrow 27M^2(\gamma_c + a) = (\gamma_c - a)^3. \tag{188}$$

If we know let $y = \gamma_c + a$ [5] we can reduce this equation into a simple cubic equation.

$$y^3 - 27M^2 y + 54aM^2 = 0 \Rightarrow \begin{cases} a > 0 = & \text{direct orbit,} \\ a < 0 = & \text{retrograde orbit.} \end{cases} \tag{189}$$

We can also input the substitution of y into equation (187) to get,

$$r_c = 3M\frac{\gamma_c - a}{\gamma_c + a} \Rightarrow r_c = 3M[1 - \frac{2a}{y}]. \tag{190}$$

We will now use the substitution $y = A\cos(\theta + B)$ where A and B are constants. To find the constants we will put this substitution into equation (189) to give

$$A^3\cos^3(\theta + B) - 27AM^2\cos(\theta + B) + 54aM^2 = 0. \tag{191}$$

We can slightly adapt equation (191) using the identity $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$.

$$A^3[\frac{1}{4}\cos[3(\theta + B)] + \frac{3}{4}\cos(\theta + B)] - 27AM^2\cos(\theta + B) + 54aM^2 = 0. \tag{192}$$

Since we know $A \neq 0$, $a > 0$ and $M > 0$ so we know the terms in equation (192) must cancel each other out. Hence, we can write equation (192) into 2 separate equations (for explanation on this see [9]). Lets examine the first of the separated equations and find the value for A.

$$\frac{3A^3}{4}\cos(\theta + B) - 27AM^2\cos(\theta + B) = 0. \tag{193}$$

Hence,

$$\frac{3A}{4}\cos(\theta + B)[A^2 - 36M^2] = 0. \tag{194}$$

Therefore, we can easily see $A = 0, \pm 6M$. Since $A \neq 0$ we know that $A = \pm 6M$. So we can update our substitution to $y = 6M\cos(\theta + B)$. Now we can write the second part of the separated equation which is the following.

$$\pm 54M^3\cos(3[\theta + B]) + 54aM^2 = 0. \tag{195}$$

Hence,

$$|a| = M\cos(3[\theta + B]). \tag{196}$$

We can now show some results for direct and retrograde orbits with the aid of [5]. Firstly for direct orbits ($a > 0$) we can show that $y = -6M\cos(\theta + 120^o)$. Finding this means we can define the value for the impact parameter and the critical radius, which are γ_c and r_c respectively.

$$\gamma_c = y - a \qquad \text{and} \qquad r_c = 3M\left(1 - \frac{2a}{y}\right). \tag{197}$$

Now for retrograde orbits ($a < 0$) we can derive $y = 6M\cos(\theta)$. Finding this means we can now define the values for the impact parameter and the critical radius, which are γ_c and r_c respectively.

$$\gamma_c = y + |a| \qquad \text{and} \qquad r_c = 3M\left(1 + \frac{2|a|}{y}\right). \tag{198}$$

Now we have derived the conditions for direct and retrograde orbits we need to get the orbital equation (184) in terms of u and we can also use our value of the impact parameter to simplify it. Hence,

$$\left(\frac{du}{d\tau}\right)^2 = M\beta^2(\gamma_c - a)^2u^4\left[\frac{1}{(\gamma_c - a)^2} - \frac{\gamma_c + a}{M(\gamma_c - a)}u^2 + 2u^3\right]. \tag{199}$$

If re-arrange equation (188) and substitute it into equation (199) we can show,

$$\left(\frac{du}{d\tau}\right)^2 = M\beta^2(\gamma_c - a)^2u^4\left[\left(\frac{\gamma_c + a}{3M(\gamma_c - a)}\right)^3 - 3\left(\frac{\gamma_c + a}{3M(\gamma_c - a)}\right)u^2 + 2u^3\right]. \tag{200}$$

The suggested substitution, taken from [5], is $u_c = \frac{\gamma_c + a}{3M(\gamma_c - a)}$. So if we substitute this into equation (200) and do some more re-arrangements we can show

$$\left(\frac{d\phi}{d\tau}\right)^2\left(\frac{du}{d\phi}\right)^2 = M\beta^2(\gamma_c - a)^2u^4(u - u_c)^2(2u + u_c). \tag{201}$$

Now, if we look back at our equation (152) and substitute our critical impact parameter into it we get

$$\dot{\phi} = \frac{\beta u^2}{a^2u^2 - 2Mu + 1}[\gamma_c - 2Mu(\gamma_c - a)], \tag{202}$$

$$= \frac{\beta u^2}{3u_c(a^2u^2 - 2Mu + 1)}[3u_c\gamma_c - 2u(\gamma_c + a)]. \tag{203}$$

Inputting this into equation (201) and re-arranging we can get

$$\frac{du}{d\phi} = \frac{(\gamma_c + a)(a^2u^2 - 2Mu + 1)}{\sqrt{M}}\frac{(u - u_c)\sqrt{2u + u_c}}{3\gamma_c u_c - 2(\gamma_c + a)u}. \tag{204}$$

Hence, we can find the solution for the orbital equation for direct and retrograde orbits.

$$\phi = \pm\frac{\sqrt{M}}{a^2(\gamma_c + a)}\int\frac{3\gamma_c u_c - 2(\gamma_c + a)u}{(u - u_+)(u - u_-)(u - u_c)\sqrt{2u + u_c}}\,du, \tag{205}$$

where,

$$u_\pm = \frac{1}{a^2}\left[M \pm \sqrt{M^2 - a^2}\right]. \tag{206}$$

We have found the orbital equation for null geodesics in direct and retrograde orbits. Due to space and time constraints we will not be able to show you the plots but you can do this by putting equation (205) in maple or matlab.

4 Effects of the Cosmological Constant

In this section we will be looking at how the cosmological constant effects the precession of perihelion and light bending. The cosmological constant, which is denoted by Λ, is an arbitrary constant that Einstein added to his field equations to achieve a static universe. So this term was used to counteract the gravitational pull of gravity. At the time there was observational justification to create this constant since observations were limited to stars within our own galaxy. However, the Hubble telescope proved that the universe was expanding in 1929 and hence the cosmological constant was disregarded. So from 1929 to the early 1900s scientists dealing with the cosmological constant equalized the cosmological constant to zero. However, with recent findings the cosmological constant, or dark matter, is now believed to explain the acceleration of the expansion of the universe [10]. If this research is indeed true we need to see what affects the cosmological constant has when we re-introduce it into Einstein's field equations. To do this we will examine what affects it has upon the precession of perihelion and the bending of light to see the impact it has upon fundamental properties of orbits.

The precession of perihelion and light bending are two extremely important ideas behind working out the path of an orbit. Perihelion is the distance of closest approach so the precession of perihelion is how the perihelion distance changes with each orbit. This is due to the fact that a full period of an elliptical orbit is not equal to 2π but a value that far exceeds this. An elliptical orbit advances by an angle $\Delta\phi$ with each rotation [1]. Light bending, or gravitational lensing, is the process of light been bent by the gravitational force of a planet or star. Gravitational lensing is an effect of Einstein's theory of general relativity. The gravitational field of a massive object will extend far into space, and cause light rays passing close to that object (and thus through its gravitational field) to be bent and refocused somewhere else [1]. The more massive the object, the stronger its gravitational field and hence the greater the bending of light rays - just like using denser materials to make optical lenses results in a greater amount of refraction. Thus through this process an observer of gravitational lensing can see multiple images.

4.1 Precession of Perihelion

So precession of perihelion describes how the perihelion distance changes with each orbit but does the cosmological constant affect the angle the elliptical orbit advance by. To examine the effects perihelion due to the cosmological constant we will assume the space-time around the mass is spherically symmetric (i.e. such as the space-time around a Schwarzschild black hole). The introduction of the cosmological constant transforms our

metric too

$$ds^2 = -\left(1 - \frac{2GM}{r} - \frac{1}{3}\Lambda r^2\right) dt^2 + \left(1 - \frac{2GM}{r} - \frac{1}{3}\Lambda r^2\right)^{-1} dr^2$$
$$+ r^2 \left(d\theta^2 + \sin^2(\theta) d\phi^2\right). \tag{207}$$

We can see if we reduce Λ to zero the metric reduces to the original Schwarzschild metric. We can show that the angular momentum within spherically symmetric distribution is the following.

$$A_\Lambda = \frac{1}{3}\Lambda c^2 r. \tag{208}$$

Hence, the angular momentum is directly proportional to Λ. The general formula for orbital eccentricity is [10],

$$\Delta\omega_\Lambda = \frac{1}{2}\frac{\Lambda c^2}{n}\sqrt{1 - e^2}. \tag{209}$$

where $n = \sqrt{GM/a^3}$ which is referred to as the Keplerian mean motion of the planet moving around a central body of mass M. We shall use two different approaches to assess the affect the cosmological constant has on the advances of perihelion. The first approach shall be a analytical approach by direct integration of perturbation force and the second approach shall be by a standard perturbation method.

4.1.1 Direct integration of perturbation force

We will examine the effects on the precession of perihelion due to Λ using direct integration in a space-time that is spherically symmetric. So will use the metric (207) and for the ease of notation we will let $g(r) = -\frac{1}{3}\Lambda r^2$. We can show that the geodesic equation of the metric (207) for a test particle is [10],

$$\left(\frac{dr}{d\tau}\right)^2 = \left(\frac{\beta}{Mc}\right) - U(r), \tag{210}$$

where,

$$U(r) = c^2\left(1 + \frac{\alpha^{-2}}{c^2 M^2 r^2}\right)\left(1 - \frac{2GM}{r} - g(r)\right). \tag{211}$$

We have been consistent with denoting β and α^{-1} as the total energy and angular momentum of the test particle, respectively. We have also denoted τ as the value for proper time. It is helpful here to work out the Newtonian potential, $\Phi(r)$, which is the following [?],

$$\Phi(r) \equiv \frac{1}{2}\lim_{c^2\to\infty}\left[U(r) - c^2\right] = \frac{\alpha^{-2}}{2M^2r^2} - \frac{GM}{r} - \frac{1}{2}g(r)c^2. \tag{212}$$

If we $L = 1/(\alpha M)$ and to for the ease of notation lets set $\varepsilon = (\beta/Mc)^2/2$ which means our equation (210) transforms too

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \Phi(r) = \varepsilon \tag{213}$$

If we now set $h(r) = -g(r)c^2/2(= -\Lambda c^2 r^2/6)$ and approximate proper time to our time coordinate, t, we get an equation in the same form as in [11]. So we can follow there steps and arrive at the integral for the advance of perihelion $\Delta\Theta_p$ due to V(r) and with respect to the Newtonian potential is the following [11],

$$\Delta\Theta_p \equiv -\frac{2p}{GMe^2}\int_{-1}^{1}\frac{z}{\sqrt{1-z^2}}dz, \tag{214}$$

where,

$$r = \frac{p}{1+ez}, \qquad p = \frac{L^2}{GM} \qquad \text{and} \qquad z = \frac{\cos(\xi) - e}{1 - e\cos(\xi)}. \tag{215}$$

In [11] $V(r)$ has the mass component included, whereas we didn't include it. So for calculations henceforth, we will disregard the M term. We are interested only in the power-law perturbating potentials of the type $V(r) = \gamma_n r^n$. So we can now find the precession formula [10].

$$\Delta_p[-(n+1)] = -\frac{2\gamma_{-(n+1)}(n+1)}{GMp^n e}\int_{-1}^{1}\frac{z(1+ez)^n}{\sqrt{1-z^2}}dz \tag{216}$$

Using the substitution for z and that $n = -3$, which we got from [10], we can show,

$$\Delta_p[2] = \frac{2\Lambda c^2 r^2}{3GMp^{-3}e}\frac{3\pi e\sqrt{1-e^2}}{2(1-e^2)^3}. \tag{217}$$

We can also show that $p = a(1-e^2)$, where we denote a as the semi-major axis of the orbit. So we can simplify equation (217) to give the formula for the advance of perihelion due to the cosmological constant, via direct integration, as

$$\Delta\omega_\Lambda = \frac{\pi c^2 \Lambda a^3}{GM}\sqrt{1-e^2}. \tag{218}$$

4.1.2 Standard perturbation

We will know have a look at a standard perturbation approach to assess the effects of the cosmological constant upon the precession of perihelion. We again we will use a space-time that is spherically symmetric. So we will use the metric (207) and examine the second-order geodesic equation for the metric (207). We can derive this by differentiating the orbital equation for

timelike geodesics for the metric (207) (follow similar steps from section 2 for the metric (1) to get the orbital equation).

$$\frac{d^2u}{d\phi^2} + u - GM\alpha_z^{-2} = \frac{3GM}{c^2}u^2 - \frac{\Lambda c^2 \alpha_z^{-2}}{3u^3}. \tag{219}$$

where, α_z^{-1} denotes the angular momentum of the orbit in the z-direction. The right hand side of equation (219) describes the Keplerian motion of the orbit. Knowing this we know that the left-hand side of equation (219) is equal to zero. From reference [10] we are advised to use the substitution $u = GM\alpha^{-2}[1 + ef(\phi)]$. Hence, we can derive a solution for u, (follow the steps in [1]) which we know is one over the radius.

$$u = GM\alpha_z^{-2}(1 + e\cos(\phi)). \tag{220}$$

Since, equation (219) is a linear differential equation we can concentrate on the perturbation due to the cosmological constant (i.e. only look at the second term of equation (219)).

$$\frac{d^2u}{d\phi^2} + u - GM\alpha_z^{-2} = -\frac{\Lambda c^2 \alpha_z^{-2}}{3u^3}. \tag{221}$$

If we use our substitution (220) in the right-hand side of equation (221) we get,

$$-\frac{\Lambda c^2 \alpha_z^{-2}}{3u^3} = -\frac{\Lambda c^2 \alpha_z^4}{3G^3M^3}\frac{1}{(1 + e\cos(\phi))^3}. \tag{222}$$

Lets take a closer look at the second fraction on the right-hand side of equation (222) so when we integrate equation (222) with respect to ϕ we can make it simpler.

$$\frac{1}{(1 + e\cos(\phi))^3} = 1 - 3e\cos(\phi) + 6e^2\cos^2(\phi) + O(e^3) \approx 1 - 3e\cos(\phi). \tag{223}$$

This is since $e\cos(\phi) \ll 1$ [1] so any order of $e\cos(\phi)$ higher than one is negligible and, hence, we can ignore. Using similar derivation techniques to those used in section 4.1.1 we can derive,

$$\Delta\omega_\Lambda \approx \frac{\pi\Lambda c^2 \alpha_z^6}{G^4M^4}. \tag{224}$$

The orbital angular momentum, α_z, is given by [10],

$$\alpha_z = \sqrt{GMp} = \sqrt{GMa(1 - e^2)}. \tag{225}$$

If we insert our value for the angular momentum into equation (224) we can show our solution for the precession of perihelion via standard perturbation techniques.

$$\Delta\omega_\Lambda = \frac{\pi\Lambda c^2 a^3}{GM}(1 - e^2)^3. \tag{226}$$

4.1.3 Comments on Precession of Perihelion

So we have derived formulas for the precession of perihelion due to the cosmological constant via two techniques. These techniques both provided a solution which clearly shows that they are directly affected by the cosmological constant. However, a point to highlight is that the two solutions differ slightly. The orbit path then is effected by the eccentricity term and if you look at Figure 8 you can see that the different solution give very different orbits, especially as the eccentricity of the orbit increases.

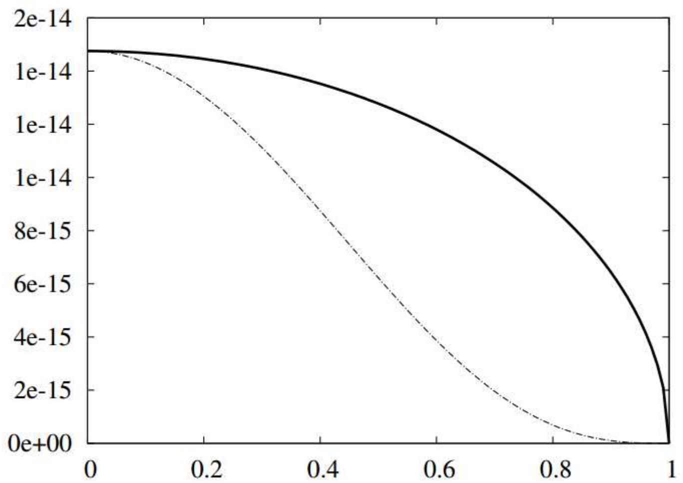

Figure 8: A plot of $\Delta\omega_\Lambda$ against the eccentricity of the orbit [10].

Examining Figure 8, we can easily see that the different solutions give very different orbits. Unfortunately, due to the space and time constraints we cant investigate further to give a precise solution to the precession of perihelion due to the cosmological constant but this an area which you can do further research into. However, we can conclude that the precession of perihelion is directly affected by the cosmological constant.

4.2 Light Bending

We know gravity acts on light due to phenomenon known as gravitational redshift. We also know that since we are dealing with light then our test particle, whatever path it takes, must follow a null geodesic. Before we go any further we will examine an example of light bending in figure to better understand the situation.

If we examine Figure 9, the point of closest approach is denoted by A and the distance by R_0. As the light ray approaches A the gravitational pull increases and, hence, the bending of light increases. As the light ray moves away from A the gravitational pull decreases and, hence, the bending of light decreases until the light ray is moving straight.

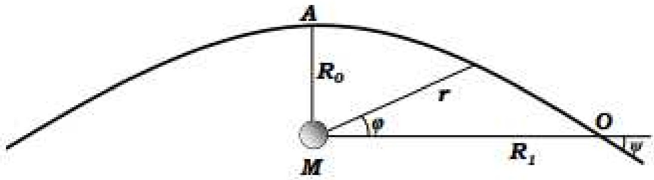

Figure 9: This diagram represents a photon coming in from infinite distance, passing a minimum distance $r = R_0$, and then escaping to infinity. It is photon undergoes a change of direction due to the gravitational pull of M [12].

To examine the effects of the cosmological constant upon the bending of light we shall be assuming a spherical symmetric space-time. Hence, we shall look at the Schwarzschild-de Sitter metric (207) and let $f(r) = (1 - 2GM/r - \Lambda r^3/3)$ for the ease of notation. We can easily extract the lagrangian in the equatorial plane $\theta = \pi/2$ similarly to how we did in section 2.1.

$$L = -f(r)\dot{t}^2 + f^{-1}(r)\dot{r}^2 + r^2\dot{\phi}^2 \tag{227}$$

If we examine the Euler-Lagrange equations (11) we get the same values as (14) and (20) which were,

$$f(r)\dot{t} = \beta \qquad \text{and} \qquad r^2\dot{\phi} = \alpha^{-1}. \tag{228}$$

Since we are dealing with null geodesics we can set the Lagrangian equal to zero. If we use the equations we found in (228) we can show that,

$$\left(\frac{dr}{d\phi}\right)^2 = r^4\left(\frac{1}{b^2} - \frac{f(r)}{r^2}\right), \text{ where } b = \text{impact parameter.} \tag{229}$$

The impact parameter still denotes $b = \alpha\beta$. If we set the radius to R_0 (distance of closest approach) then the left-hand side of equation (229) equates to zero. Knowing this, we can show that $b^2 = R_0^2/f_0$. If we now substitute this back into equation (230) we get,

$$\left(\frac{dr}{d\phi}\right)^2 = r^4\left(\frac{f_0}{R_0^2} - \frac{f}{r^2}\right). \tag{230}$$

Now we have our orbital equation we can now derive the bending angle which can be found by the bending angle formula [3],

$$\phi_B = 2\int_{R_0}^{\infty} \frac{d\phi}{dr} dr - \pi, \tag{231}$$

where $\phi_B =$ bending angle. So if we re-arrange equation (230) we can show that,

$$\frac{R_0}{\sqrt{f_0}}\frac{d\phi}{dr} = \frac{1}{r^2\sqrt{1 - \frac{R_0^2}{r^2}\frac{f}{f_0}}}. \tag{232}$$

Hence, the bending angle can be found to be,

$$\frac{\phi_B + \pi}{2} = \frac{R_0}{f_0} \int_{R_0}^{\infty} \frac{dr}{r^2 \sqrt{1 - \frac{R_0^2}{r^2} \frac{f}{f_0}}}. \tag{233}$$

To begin with lets simplify the square-root in the integral.

$$r^2 \sqrt{1 - \frac{R_0^2}{r^2} \frac{f}{f_0}} = \frac{r}{f_0} \sqrt{r^2 f_0 - R_0^2 f}. \tag{234}$$

This simplifies our bending angle formula (234) too,

$$\frac{\phi_B + \pi}{2} = R_0 \int_{R_0}^{\infty} \frac{dr}{r \sqrt{r^2 f_0 - R_0^2 f}}. \tag{235}$$

Lets have a closer look at the equation inside the square-root,

$$\begin{aligned} r^2 f_0 - R_0^2 f &= r^2 \left(1 - \frac{2GM}{R_0} - \frac{\Lambda}{3} R_0^2\right) - R_0^2 \left(1 - \frac{2GM}{r} - \frac{\Lambda}{3} r^2\right), & (236) \\ &= (r^2 - R_0^2) - 2GM \left(\frac{r^2}{R_0} - \frac{R_0^2}{r}\right). & (237) \end{aligned}$$

Therefore, we can see that the cosmological constant cancels out and, hence, does not intervene in the bending angle of light.

5 Conclusion

Through this report we have investigated the orbital path of a test particle orbiting Schwarzschild and Kerr black holes. We then examined what effects the constant had upon the precession of perihelion and the bending of light.

Due to the nature of a black hole we realised we would have two types of orbit paths. Those were orbits of the first-kind which remain in their described motion around the exterior of the black hole or orbits of the second-kind which enter the event horizon and plunge into the singularity. Both these types of orbits could have also followed a bound orbit path or an unbound orbit path.

We began by deriving an orbital equation for geodesics in Schwarzschild geometry and set certain values for null and time-like geodesics (space-like geodesics were outside the scope of the report). We began examining the orbital equation for time-like geodesics.

We investigated bound orbits of the first-kind and found there was four different cases for the orbital equation in this instance. Therefore, we derived specific orbital equations for each case and plotted the orbital path. The orbital equation for three distinct roots gave an elliptical orbit. When the first two roots were equal we achieved a circular orbit. When the final two roots were equal gave produced an orbit path of a test particle approaching its perihelion distance then stabilises itself in a circular orbit. The final case we discussed was when all the roots were equal. We realised the orbit would be an unstable circular orbit that eventually plunged into the singularity. Hence, no bound orbit of the first-kind with all roots equal exists.

After, thoroughly, investigating orbits of the first-kind we then examined bound orbits of the second-kind. To begin with we altered the orbital equation for those of the second-kind and looked into two different cases. The first case was when the first two roots are equal (i.e. eccentricity equals zero). We discovered that orbits of this kind orbit the black hole at a certain aphelion distance and enter the black hole and plunge into the singularity. The second case was when we had $2\mu(3+e)=1$**. Due to space constraints we couldn't plot this orbital path but we could see from the equation it had an aphelion distance of** $r = l/(1+e)$ **and orbits the black hole until it plunges into the singularity. This case actually just an extension to that of the third case of bound orbits of the first-kind and, hence, why we decided to leave this plot out. There was further cases we could have discussed, such as when all roots of the orbital equation were the same, but due to space and time constraints we couldn't delve any further.**

Once we had discussed bound orbits we moved onto unbound orbits. We made the distinction that the first root of the orbital equation must be negative, leaving three possibilities. The first case we looked at was when all roots were real and distinct. We adapted our roots slightly and found that that orbit path spiraled inwards towards its aphelion distance and then went off to infinity on the trajectory (92). We didn't discuss the second case when the last two roots of the orbital equation were the same that much due to the space restrictions of the report but we did make the distinction that the trajectories they describe are the same for first and second-kind orbit paths. The third case was to do with imaginary eccentricities which is outside the scope of the report so was not discussed.

Finally for the Schwarzschild geometry we looked at unbound orbits of the second-kind. We immediately discovered that these followed exactly the same procedure as bound orbits of the second-kind.

We now moved onto null geodesics (i.e. the motion of light) which meant adapting our orbital equation since the Lagrangian is equal to zero for null geodesics and introduced an impact parameter. We quickly realised that light acts very similar to that of a test particle. We began by looking at bound orbits of the first-kind. We realised that the first root must be negative so we therefore looked at the case when the final two real roots were real. After getting the specific orbital equation and plotting we discovered the light enters a stable, circular orbit of radius $3M$. For bound orbits of the second-kind we found that the light had an aphelion distance of radius of $3M$ then orbited the black hole and eventually plunge into the singularity.

We then looked at unbound orbits in null geodesics. They acted very similar to those of null geodesics. With the orbits of the first kind spiralling inwards towards an aphelion distance then shooting off on the trajectory (130). Orbits of the second-kind began orbiting at an aphelion distance $(C + p - 2M)/4MP$ and ended by plunging into the singularity.

We had now discussed Schwarzschild geometry in huge depth so we moved onto looking at Kerr black holes. We were faced with a much tougher challenge here due to the fact a Kerr black hole has a rotating core and, hence, is not symmetric. We decided to only look at orbits in the equatorial plane (i.e. $\theta = \pi/2$) which made the Kerr metric far more easier to deal with. Due to the fact that a Kerr black hole and, hence, a rotating core we had two types of orbit, direct and retrograde orbits (i.e. orbits that follow the rotation of the black and those that oppose it). We began our examination of the Kerr geometry by deriving an orbital equation (154) which could be adapted for time-like and null geodesics.

From here, we examined the time-like geodesics of direct and retrograde orbits. We began by adapting the orbital equation for that of time-like

geodesics. Then we derived formulas for the energy and angular momentum of our test particle orbiting the black hole. This enabled us to derive a solution for the orbital equation for the orbit path of a test particle for either direct or retrograde orbits. Unfortunately, we could not plot the graph due to space restrictions.

Next, we examined the null geodesics (i.e. the motion of light) in Kerr geometry and again saw similarities between the motion of light and that of a test particle. We began by adapting the general orbital equation for that of null geodesics and restricted the radius to a critical value. Using a substitution allowed us to find the critical radius and impact parameter for direct and retrograde orbits. This enabled us to derive a specific orbital equation which we intrigued to see held very similar qualities to that of the specific orbital equation for time-like geodesics. Again we could, unfortunately, not plot the graph due to space restrictions.

Finally we examined two fundamental orbiting ideas and how the introduction of the cosmological constant would affect them. Firstly, we examined the precession of perihelion and took two different approaches to deriving the solution for the precession of perihelion due to the cosmological constant. We used direct integration and standard perturbations. Both the solutions showed that the cosmological constant directly affected the precession of perihelion but gave slightly different results. We, also, looked at how the cosmological constant affected the bending of light. We found that the cosmological constant cancels itself out of the light bending equation and, hence, does not affect the bending of light.

Through this report we have examined, in very large depth, the orbital paths around black holes and discovered some very interesting results. However, due to space and time restriction unfortunately I couldn't research or include everything I wanted. If I had no time or space restrictions we could have delved into more detail on certain points. I could have examined and plotted the orbital path of all the cases for time-like and null geodesics in Schwarzschild and Kerr geometry instead of looking at specific cases. I could have also examined what happened when we have imaginary eccentricities. There are other black holes as well, such as the Reissner-Nordstrom black hole, which I could have examined. We didn't even touch upon what happens when we have a charged test particles. For instance, what happens when we have a charged and rotating test particle orbiting a charged black hole with a rotating core? I could also have looked further into what other fundamental orbiting ideas that the cosmological constant affects. I could have also looked further into solutions we got for the precession of perihelion due to the cosmological constant and get a specific solution.

References

[1] D.J. Toms, *MAS314: Relativity,* Newcastle University, 2006.

[2] Time Travel Research Center:
http://www.zamandayolculuk.com/cetinbal/HTMLdosya1/RelativityFile.html,
Date of access: 29th April 2015

[3] D.J. Toms, *MAS8113: Differential Geometry and General Relativity,* Newcastle University, 2015.

[4] Encyclopedia of Science: Light Cone
http://www.daviddarling.info/encyclopedia/L/light_cone.html, Date of access: 29th April 2015

[5] S. Chandrasekhar, *The Mathematical Theory of Black Holes,* Oxford Oxfordshire: Clarendon Press; New York: Oxford University Press; 1983.

[6] J.L. Synge, *Relativity: The general theory,* Amsterdam: North-Holland Pub. Co.;1960.

[7] S.M. Carroll, *Lecture Notes on General Relativity,*
http://preposterousuniverse.com/grnotes/grnotes-seven.pdf, 1997.

[8] J.B. Hartle, *Gravity: An Introduction to Einstein's General Relativity,* 1301 Sansome St., San Francisco: Addison Wesley; 2003.

[9] R.S. Johnson, *The Notebook Series: The solution of cubic and quartic equations,,*
http://www.ncl.ac.uk/maths/students/teaching/notebooks/EquationNotebook.pdf,
2006.

[10] H. Arakida, *Note on the Perihelion/Periastron Advance due to Cosmological Constant,* International Journal of Theoretical Physics, Vol.52(5), pp.1408-1414, 2013.

[11] O.I. Chashchina and Z. K. Silagadze, *Remark on orbital precession due to central-force perturbations,* Physical Review D, Vol.77(10),pp. 107502, 2008.

[12] F. Hammond, *A note on the effect of the cosmological constant on the bending of light,* Modern Physics Letters A, 28, pp.1-7, 2013.