Chaotic Motion in a Central Force and Magnetic Field

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Abstract

When subjected to a magnetic field, a particle can display interesting motion. If the particle is also in a central force, then the motion becomes a lot more complicated and chaos occurs. In this project I will be observing and analysing this motion in both two and three dimensions. I will start by looking at 2 dimensional motion in a central force, then adding in the magnetic field and observing how the motion has changed. I will then move into three dimensions and plot the motion. This is where the particle begins to display chaotic behaviour so I will analyse this motion and produce Poincaré maps to analyse the chaos. I will also examine how the size of the magnetic field affects how chaotic the motion is and try to determine how the strength of the magnetic field affects the onset of chaos.

Contents

Chapter 1

Introduction

Particles in a central force display fairly simple orbits, we know this from observing the orbits of planets around the sun. However, if we add in a magnetic field, it is less well known what the motion will look like. In fact the motion becomes chaotic and this is what I will be looking at throughout this project.

Throughout the project I will be assuming that the particle is an electron in a Rydberg atom. A Rydberg atom is an atom which has been excited and whose furthest electron has a high radial quantum number n. The radial quantum number is one of 4 quantum numbers used to describe an electron of an atom, these are: the radial quantum number n, the angular quantum number l_q, the magnetic quantum number m and the spin quantum number s. As n increases, the average position of an electron becomes further from the nucleus. This means that the length scales between the outer electrons and the nucleus become so large, that the motion can be represented classically, rather than using quantum mechanics. Rydberg atoms behave in a similar way to hydrogen atoms as the outer electron is shielded from the nucleus by the core electrons. The outer electron sees the nucleus and all other electrons as one body and therefore the atom can be treat as a hydrogen atom with only one electron.

Another case where the motion I will be studying may be observed is the orbit around a black hole. Matter which orbits a black hole with a magnetic field can display particle trajectories similar to those I will be plotting in this project.

I will begin by focusing on the motion in two dimensions. Starting with the motion in only a central force, I will derive some equations describing the motion. I will use these to plot the orbits of the particle, as well as looking at the energy and momentum of the system. Staying in two dimensions, I will then add in a constant magnetic field. I will modify the equations derived for the central force and plot the motion in the magnetic field. I will briefly take a look at the case where the magnetic field is not constant, but depends upon the radial position of the particle. The next thing to do is to move into three dimensions. I will derive the equations and then make them dimensionless. Plotting them will allow me to see how the three dimensional trajectories compare to the two dimensional motion previously observed. Finally I will analyse the chaotic motion by looking at Poincaré maps of the bound orbits. By reducing the strength of the magnetic field I will find fixed points and periodic solutions of the system. Increasing the field slightly will allow me to observe the transition into chaos. I will also use the dimensionless parameters to determine the size of the magnetic field in Tesla at each stage.

Analysis of a single Poincaré section has been carried out in previous work on this system. [1] [2] However I will be using different Poincaré sections which have a better structure and are more informative.

Chapter 2

Central Force Motion

In this chapter we will be studying the motion of a particle under a central force in 2 dimensions. We will start by examining the properties of a central force. We will then find the momentum and energy of the particle. Finally we are going to plot the motion of the particle.

2.1 Central Force

A central force acting on a particle is one which depends only on the position of the particle relative to the origin. The force acts on the particle by moving it towards or away from the origin.

An example of a central force is the gravitational field around the earth. Gravity pulls objects towards the centre of the earth and gets stronger the closer the object is to the origin.

The central force is given by

$$\boldsymbol{F} = f\left(\boldsymbol{r}\right) = f\left(r\right)\hat{\boldsymbol{r}},$$

where $\boldsymbol{F}$ is the force and $\boldsymbol{r}$ is the position vector of the particle.

We can write the position of a particle as

$$\boldsymbol{r}\left(t\right) = r\hat{\boldsymbol{r}}, \tag{2.1}$$

and use this to find the velocity and acceleration. However, first we need to know how to differentiate $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ with respect to time. It can be shown that

$$\begin{aligned} \dot{\hat{\boldsymbol{r}}} &= \dot{\theta}\hat{\boldsymbol{\theta}}, \\ \dot{\hat{\boldsymbol{\theta}}} &= -\dot{\theta}\hat{\boldsymbol{r}}. \end{aligned}$$

Now these can be used to differentiate (2.1) to get the velocity

$$\dot{\boldsymbol{r}}\left(t\right) = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}, \tag{2.2}$$

and again to get the acceleration

$$\ddot{\boldsymbol{r}}(t) = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\boldsymbol{r}} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\boldsymbol{\theta}}. \tag{2.3}$$

Newton's 2nd Law of motion states that

$$\boldsymbol{F} = m\boldsymbol{a} = m\ddot{\boldsymbol{r}},$$

where m is the mass and $\boldsymbol{a}$ is the acceleration. Combining this with the central force gives

$$m\ddot{\boldsymbol{r}} = f(r)\,\hat{\boldsymbol{r}}. \tag{2.4}$$

So now we can substitute equations (2.1), (2.2) and (2.3) into this equation to get

$$m\left(\ddot{r} - r\dot{\theta}^2\right)\hat{\boldsymbol{r}} + m\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\boldsymbol{\theta}} = f(r)\,\hat{\boldsymbol{r}},$$

splitting this into separate components gives the radial component as

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = f(r), \tag{2.5}$$

and the angular component as

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0. \tag{2.6}$$

There are two conserved quantities in central force motion, which will be needed to simplify these equations and allow us to plot the orbits. These quantities are the angular momentum L and the total energy E. Since they are conserved, they are called constants of motion.

2.2 Angular Momentum

Angular momentum is the measure of rotation of an object or particle. In classical mechanics it is defined as

$$\boldsymbol{L} = \boldsymbol{r} \times m\dot{\boldsymbol{r}}.$$

This gives

$$\boldsymbol{L} = r\hat{\boldsymbol{r}} \times m\left(\dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}\right),$$

and calculating this cross product gives

$$L = mr^2\dot{\theta}. \tag{2.7}$$

This is the angular momentum of the particle, and is one of the constants of motion of the particle. Also, note that the time derivative of this is the angular component of the force.

We can now show that angular momentum is conserved by differentiating this to give

$$\dot{L} = mr\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right),$$

and if we compare this to equation (2.6) it can easily be seen that

$$\dot{L} = 0.$$

So the angular momentum is constant and therefore is a conserved quantity.

Another way to show that the angular momentum is conserved is to consider the Torque. Torque is the force which causes rotation and is defined as

$$\boldsymbol{\tau} = \dot{\boldsymbol{L}} = \boldsymbol{r} \times \boldsymbol{F}.$$

Calculating this gives

$$\begin{aligned}\dot{L} &= \boldsymbol{r} \times m\ddot{\boldsymbol{r}}. \\ &= mr\left(\ddot{r} - r\dot{\theta}^2\right)\end{aligned}$$

As before, if we compare this to (2.6) we can see that $\dot{L} = 0$ and L is constant so the angular momentum is conserved.

2.3 Energy

The second constant of motion to look at is the energy. The total energy of a system is made up of the kinetic energy and the potential energy.

Kinetic energy is the energy generated by motion, and is defined as

$$K = \frac{1}{2}m\dot{\boldsymbol{r}}^2.$$

The potential energy of an object is determined by its spatial position. Potential energy is defined using the force as

$$\boldsymbol{F} = -\nabla \boldsymbol{V}.$$

As we are working in two dimensions and we have a central force, this can be written as

$$f\left(\boldsymbol{r}\right) = -\frac{\partial V}{\partial r}. \tag{2.8}$$

Now, the total energy of our system is given by

$$E = \frac{1}{2}m\dot{\boldsymbol{r}}^2 + V. \tag{2.9}$$

Calculating $\dot{\boldsymbol{r}}^2 = \dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}}$ and substituting in gives

$$E = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) + V.$$

We can now show that the energy is a conserved quantity. Differentiating equation (2.9) gives

$$\dot{E} = m \left(\dot{r}\ddot{r} + r\dot{r}\dot{\theta}^2 + r^2 \dot{\theta}\ddot{\theta} \right) + \dot{V}.$$

Now note that

$$\dot{\boldsymbol{r}} \cdot \ddot{\boldsymbol{r}} = \dot{r}\ddot{r} + r\dot{r}\dot{\theta}^2 + r^2 \dot{\theta}\ddot{\theta},$$

and equation (2.8) implies that

$$\dot{V} = \frac{\partial V}{\partial t} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial t} = -f\frac{\partial r}{\partial t} = -f\dot{r}.$$

Substituting both of these into $\dot{E}$ leaves

$$\begin{aligned} \dot{E} &= m\dot{\boldsymbol{r}} \cdot \ddot{\boldsymbol{r}} - f\dot{r}, \\ &= \dot{\boldsymbol{r}} \cdot F - f\dot{r}, \\ &= \dot{r}f - f\dot{r}, \\ &= 0. \end{aligned}$$

Therefore, integrating this implies that E is constant, so the energy is conserved.

2.4 Orbits

2.4.1 Predicting the Motion

We are now going to use the energy to think about what we expect the motion of the particle to look like and to make some predictions about what forms the orbits might take.

We can now take the total energy in equation (2.9), and rearrange it using equation (2.7) to get

$$E = \frac{1}{2} m \left(\dot{r}^2 + W \right),$$

where

$$W = W(r) = \frac{L^2}{m^2 r^2} + \frac{2V}{m}.$$

We can now plot W. This will enable us to predict what the orbits will look like.

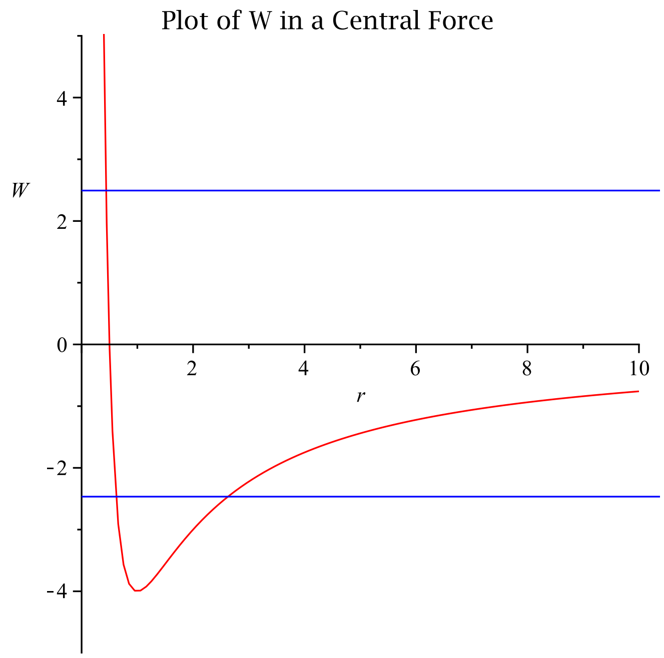

Figure 2.1: Plot of W which can be used to predict the form of the orbits. The blue lines represent positive and negative energy.

In figure 2.1 we have plotted W and then added in two horizontal lines, one at positive W and one at negative W. These correspond to the particle having positive and negative energy respectively.

From the plot, you can see that the line corresponding to negative energy cuts the line for W twice. This shows that the orbits will be closed. Therefore we expect the orbits to be either circular or elliptical. Now looking at the line corresponding to positive energy, we can see that it cuts W only once. This means that the orbits will not be closed, but instead be open. Hence we expect that they will be hyperbolic curves.

2.4.2 Orbits

I am now going to plot the trajectories of the particle, but to do this we need an equation for the position of the particle. We can start by taking the angular momentum and rearranging it to get $\dot{\theta}$. We can then substitute into equation (2.5) to get

$$m\ddot{r} - \frac{L^2}{mr^3} = f(r)\,. \tag{2.10}$$

We can now use this to plot some orbits under different conditions.

We have first chosen the energy to be positive and plotted the trajectory of the particle. Looking at figure 2.2, you can see that the particle comes

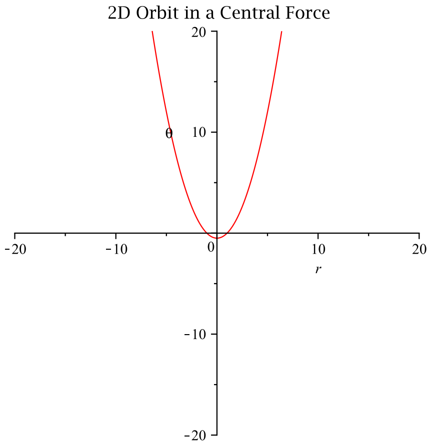

Figure 2.2: A hyperbolic orbit occurs when the energy is positive.

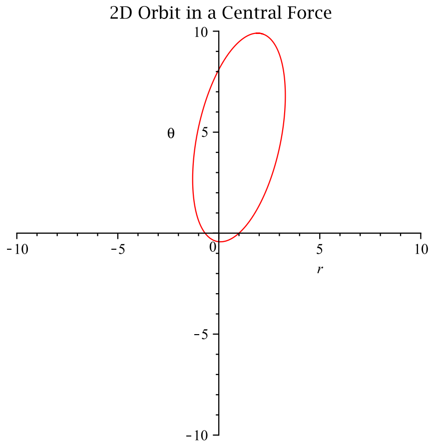

Figure 2.3: Elliptical orbit in negative energy.

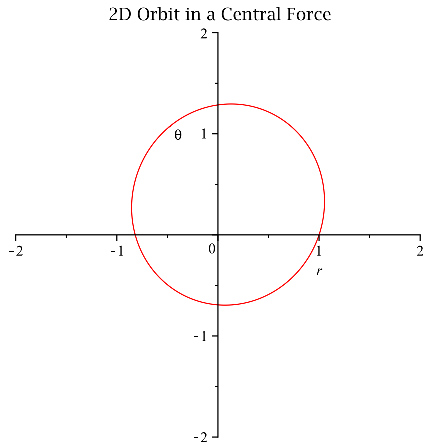

Figure 2.4: Circular orbits can also appear in negative energy.

down from far away. As it enters the central force, it changes direction and finally leaves again, going back the way it came in. This is indeed a hyperbolic curve, just as we predicted.

Next, we made the energy negative and plotted the trajectory again, which can be seen in figure 2.3. This is an elliptical orbit which stays inside the central force forever.

Finally, we adjusted the boundary conditions and plotted one more orbit, figure 2.4. This time the trajectory is circular.

Once again, our prediction was correct that orbits would be circular or elliptical when the energy is negative.

Chapter 3

Central Force Motion in a Magnetic Field

We can now add in a magnetic field and observe how this affects the motion of the particle.

3.1 Magnetic Field

Introduce a magnetic field in the z direction

$$\boldsymbol{B} = B\hat{\boldsymbol{z}}.$$

The force on the particle is now made up of the central force as before and the Lorentz force of the magnetic field. It is given by

$$\boldsymbol{F} = f(r)\,\hat{\boldsymbol{r}} + e\dot{\boldsymbol{r}} \times \boldsymbol{B}, \tag{3.1}$$

where e is the charge of the particle. Using this with Newton's 2nd Law gives

$$f(r)\,\hat{\boldsymbol{r}} + e\dot{\boldsymbol{r}} \times \boldsymbol{B} = m\ddot{\boldsymbol{r}}. \tag{3.2}$$

Since the magnetic field is in the z direction, note that

$$\begin{aligned} \hat{\boldsymbol{r}} \times \boldsymbol{B} &= -B\hat{\boldsymbol{\theta}}, \\ \hat{\boldsymbol{\theta}} \times \boldsymbol{B} &= B\hat{\boldsymbol{r}}. \end{aligned}$$

Substituting equations (2.2), (2.3) and (2.4) into equation (3.2) gives

$$\begin{aligned} \boldsymbol{F} &= f(r)\,\hat{\boldsymbol{r}} - e\dot{r}B\hat{\boldsymbol{\theta}} + er\dot{\theta}B\hat{\boldsymbol{r}}, \\ &= m\left(\ddot{r} - r\dot{\theta}^2\right)\hat{\boldsymbol{r}} + m\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\boldsymbol{\theta}}. \end{aligned}$$

Now we can split this into separate components to get

$$f(r) + er\dot{\theta}B = m\left(\ddot{r} - r\dot{\theta}^2\right), \tag{3.3}$$

$$-e\dot{r}B = m\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right). \tag{3.4}$$

3.2 Angular Momentum

We can now calculate the angular momentum by finding something that differentiates to give the angular component of the force in equation (3.4). This is

$$L = mr^2\dot{\theta} + \frac{1}{2}eBr^2. \tag{3.5}$$

Differentiating this gives

$$\dot{L} = mr\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right) + eBr\dot{r},$$

and using (3.4) we can see that

$$\dot{L} = 0.$$

so the angular momentum is conserved.

3.3 Energy

Equation (3.5) can be rearranged to get $\dot{\theta}$

$$\dot{\theta} = \frac{L - \frac{1}{2}eBr^2}{mr^2}. \tag{3.6}$$

We know that the total energy can be rearranged to give

$$E = \frac{1}{2}m\left(\dot{r}^2 + W\right),$$

where this time

$$W = \frac{\left(L - \frac{1}{2}eBr^2\right)^2}{m^2r^2} + \frac{2V}{m}.$$

Substituting W back into the energy gives

$$E = \text{constant},$$

so the energy is conserved.

3.4 Orbits

3.4.1 Predicting the Motion

Once again we can use W to predict what form the orbits will take.

Figure 3.1 shows a plot of W and, like in section 2.4.1, some lines have been added in corresponding to positive and negative energy. This plot shows that the orbits are always closed, whether the energy is positive or negative. However, as the central force and the magnetic field are interacting with each other, we cannot predict the exact form of the orbits.

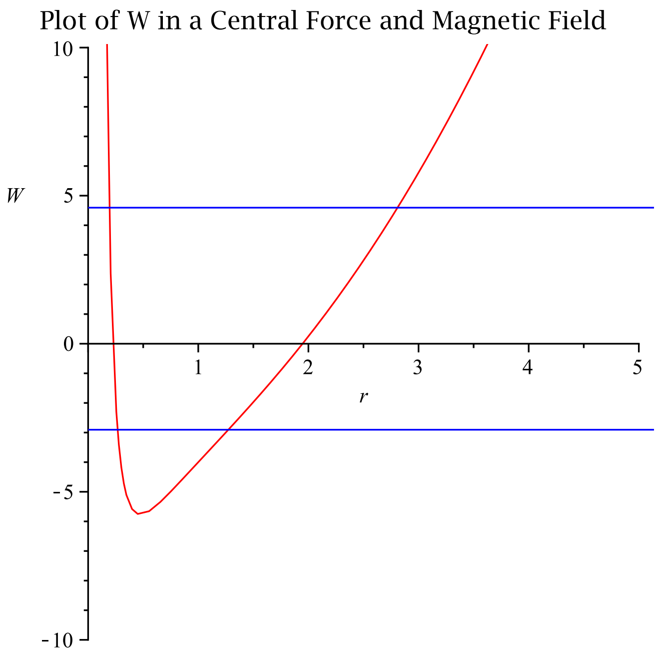

Figure 3.1: Plot of W with magnetic field to try to predict the form of the motion. Blue lines correspond to positive and negative energy.

3.4.2 Orbits

We now want to plot the trajectories of the particle under different conditions. Before we do this, we will first introduce some parameterizations to simplify the equations

$$l = \frac{L}{m}, \tag{3.7}$$

$$b = \frac{eB}{2m}. \tag{3.8}$$

Also, the force can be written as

$$f(r) = \frac{Ze^2}{4\pi\epsilon_0 r^2},$$

which can be parameterized as

$$f(r) = \frac{gm}{r^2}, \tag{3.9}$$

to make the equations simpler. Here, Z is the atomic number of the atom and e is the charge of the electron.

We can substitute these parameterisations and equation (3.6) into equation (3.3) to give

$$\ddot{r} - \frac{l^2}{r^3} + b^2 r - \frac{g}{r^2} = 0. \tag{3.10}$$

We can now use this equation to plot some orbits.

3.4.3 Magnetic Field Only

We can first try getting rid of the central force, i.e. setting $F = 0$, to see what the motion will look like under only a magnetic field. This corresponds to setting $g = 0$ in equation (3.10).

Before plotting this orbit, we will derive an equation for the form the motion will take. We will begin with equation (3.2) and set $f(r) = 0$, leaving

$$m\ddot{\boldsymbol{r}} = e\dot{\boldsymbol{r}} \times \boldsymbol{B}.$$

Integrating this leaves

$$m\dot{\boldsymbol{r}} = e\boldsymbol{r} \times \boldsymbol{B} + \text{constant}. \tag{3.11}$$

If we assume that $\dot{r}$ is perpendicular to $\boldsymbol{B}$, then this means that the constant on the right hand side is also perpendicular to $\boldsymbol{B}$. Therefore we can write

$$\text{constant} = -e\boldsymbol{r}_0 \times \boldsymbol{B},$$

where r_0 is constant. We can substitute this into equation (3.11) and rearrange to get

$$m\left(\dot{\boldsymbol{r}} - \dot{\boldsymbol{r}}_0\right) = e\left(\boldsymbol{r} - \boldsymbol{r}_0\right) \times \boldsymbol{B}.$$

Multiplying through by $(\boldsymbol{r} - \boldsymbol{r}_0)$ and rearranging gives

$$\frac{1}{2}m\frac{\partial}{\partial t}\left(\boldsymbol{r} - \boldsymbol{r}_0\right)^2 = 0.$$

Integrating this equation leaves

$$\left(\boldsymbol{r} - \boldsymbol{r}_0\right)^2 = \text{constant}. \tag{3.12}$$

This is clearly the equation of a circle with the centre depending on the value of the constant. Different initial conditions will position the circle in different places.

We can now plot the motion using equation (3.10) to see if we are correct about the shape of the orbit.

You can see from figure 3.2 that motion in only a magnetic field is indeed circular as we predicted from equation (3.12). This is very similar to the motion under only a central force, as seen in chapter 2. This is because, in both cases, there is only one force acting on the particle.

We can now combine these two cases and observe the motion under both a central force and a magnetic field.

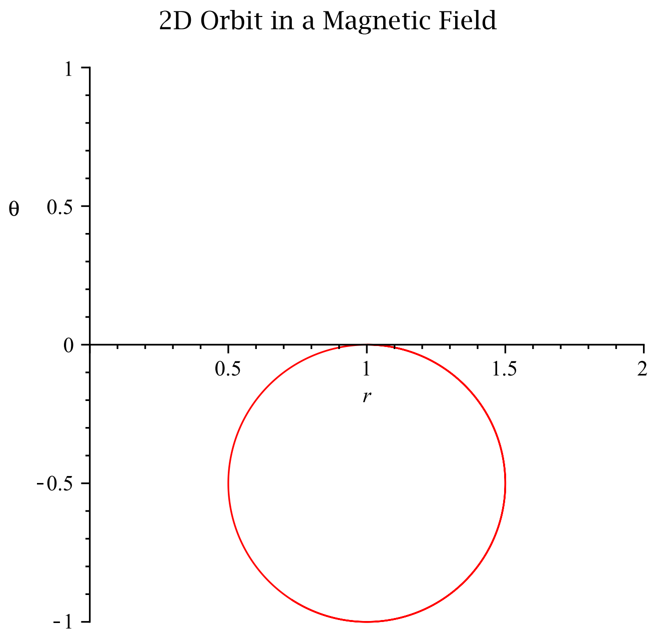

Figure 3.2: Motion in a magnetic field without a central force is just circular.

3.4.4 Central Force and Magnetic Field

Now we can look at the orbits in both a central force and magnetic field.

Looking at figures 3.3, 3.4 and 3.5, you can see that the motion of the particle is more complicated when there are two forces acting on the particle. The particle seems to loop around itself while orbiting around the origin.

3.5 Magnetic Field depending on r

Previously, we have only looked at the case where the magnetic field is constant. However, this is not the only possibility. We are now going to briefly look at the case where the magnetic field is not constant, in particular we will look at the case where it is a function of the position of the particle.

Before looking at this case, we will first need to consider the vector potential.

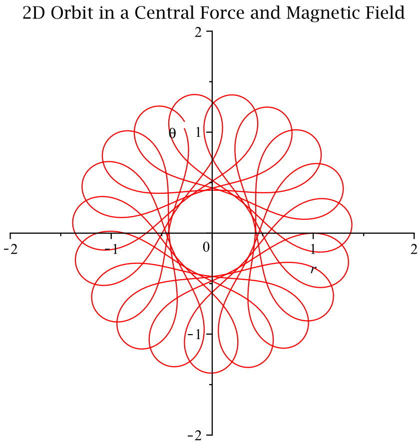

Figure 3.3: Motion in a central force and magnetic field.

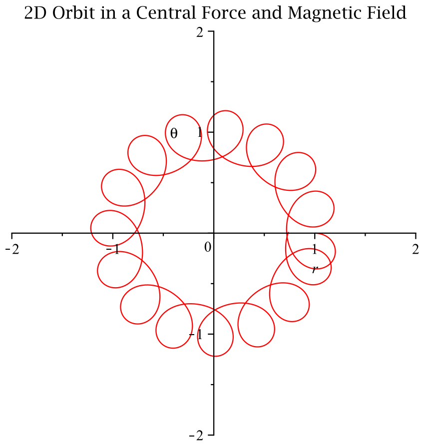

Figure 3.4: Motion in a central force and magnetic field.

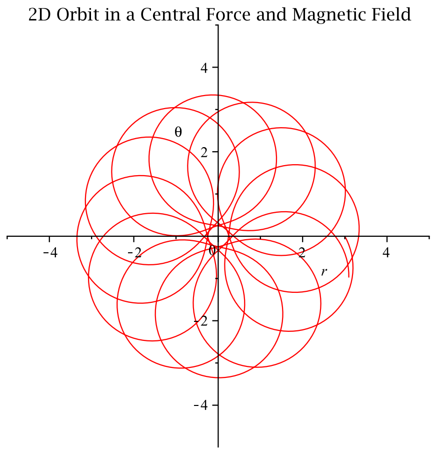

Figure 3.5: Motion in a central force and magnetic field.

3.5.1 Vector Potentials

The vector potential is a vector field A and is defined as

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \tag{3.13}$$

where $\boldsymbol{B}$ is the magnetic field.

Note that taking the divergence of this gives

$$\nabla \cdot \boldsymbol{B} = \nabla \cdot (\nabla \times \boldsymbol{A}) = 0.$$

Which leads to the Maxwell equation

$$\nabla \cdot \boldsymbol{B} = 0.$$

The definition of the vector potential is not uniquely defined, since we can add terms to A which still satisfy equation (3.13). These must be terms such that when we take the curl, it still gives 0, meaning that the magnetic field remains unchanged. A gradient term is an example of one that would work. For example

$$\boldsymbol{A} + \nabla\phi,$$

is also a vector potential, for some scalar function ϕ. This property is called gauge invariance.

3.5.2 The Magnetic Field

We can now introduce the magnetic field which depends on position.

$$\boldsymbol{B} = B\left(\boldsymbol{r}\right).$$

We know that the magnetic field is in the z direction, which means that the components of $\boldsymbol{B}$ are

$$\boldsymbol{B} = \left(0, 0, B\right).$$

We can make a guess at the vector potential. We will try using

$$\boldsymbol{A} = \left(0, A, 0\right).$$

Other vector potentials will also work as long as they have r or θ components. Using equation (3.13) to combine these we get

$$B = r^{-1}\frac{\partial}{\partial r}\left(rA\right). \tag{3.14}$$

We can also calculate the angular momentum of a particle in this magnetic field. This is done by starting with equation (3.4) as before and substituting in equation (3.14). Then rearranging leaves

$$\frac{\partial}{\partial t}\left(mr^2\dot{\theta} + erA\right) = 0.$$

We know that the angular momentum is conserved so this means that the angular momentum is given by

$$L = mr^2\dot{\theta} + erA. \tag{3.15}$$

We can clearly see that the angular momentum contains a component of the vector potential.

When given a magnetic field B as a function of position, it can be substituted into equation (3.14) and solved to obtain the vector potential A. From there, we can substitute A into equation (3.15) to obtain the angular momentum of the system. All other analysis of the motion can follow the same way as previously demonstrated

Chapter 4

Motion in 3 Dimensions

We have looked in detail at the motion in two dimensions. We can now generalise these ideas to 3 dimensions and analyse the motion.

4.1 The Equations

We are going to begin with the force in three dimensions and use this to derive three equations that will be used to plot some particle trajectories In this chapter we will be using the notation

$$\boldsymbol{R} = (r, \theta, z)\,,$$

where $\boldsymbol{R}$ is the position in 3 dimensional space.

If we now let the force be

$$\boldsymbol{F} = f\left(\hat{\boldsymbol{R}}\right) = \frac{-Ze^2}{4\pi\epsilon R^2}\hat{\boldsymbol{R}}, \tag{4.1}$$

where Z is the atomic number of the atom and e is the charge of an electron. Now we can rearrange this force to get

$$\boldsymbol{F} = f\,(r, z)\,\hat{\boldsymbol{r}} + G\,(r, z)\,\hat{\boldsymbol{z}},$$

where

$$f\,(r, z) = \frac{-Ze^2 r}{4\pi\epsilon\left(r^2 + z^2\right)^{\frac{3}{2}}}, \tag{4.2}$$

$$G\,(r, z) = \frac{-Ze^2 z}{4\pi\epsilon\left(r^2 + z^2\right)^{\frac{3}{2}}}. \tag{4.3}$$

If we follow the same method as we have previously done in chapters 2 and

3 we can split the force into three components

$$\begin{aligned} m\left(\ddot{r}-r\dot{\theta}^2\right) &= f\left(r,z\right)+er\dot{\theta}B, & (4.4)\\ m\left(2\dot{r}\dot{\theta}+r\ddot{\theta}\right) &= -e\dot{r}B, & (4.5)\\ m\ddot{z} &= G\left(r,z\right). & (4.6) \end{aligned}$$

Since the angular component, equation (4.5), is exactly the same is it was in chapter 3, then the angular momentum will also be exactly the same.

$$L = mr^2\dot{\theta}+\frac{1}{2}eBr^2.$$

We have already shown that this quantity is constant and hence the angular momentum is conserved.

If we now rearrange the angular momentum to get $\dot{\theta}$ we get

$$\dot{\theta}=\frac{L-\frac{1}{2}eBr^2}{mr^2}. \tag{4.7}$$

Using the parameterizations defined in equations (3.7), (3.8) and (3.9) this becomes

$$\dot{\theta}=\frac{l}{r^2}-b. \tag{4.8}$$

Now we can substitute equation (4.7) into equation (4.4) and use the parameterizations again to obtain a differential equation in r.

$$\ddot{r}-\frac{l^2}{r^3}+b^2r-\frac{gr}{\left(r^2+z^2\right)^{\frac{3}{2}}}=0. \tag{4.9}$$

Notice that here I have substituted the form of $f\left(r,z\right)$ from (4.2) back in and then simplified. Here g is the same parameterization of the force as was used in equation (3.9).

Finally we can substitute $G\left(r,z\right)$ into (4.6) which gives

$$\ddot{z}=\frac{gz}{\left(r^2+z^2\right)^{\frac{3}{2}}}. \tag{4.10}$$

These three differential equations describe the motion of the particle through space.

4.2 Dimensionless Equations

Now that we have the equations to describe the motion, we can make them dimensionless before we plot them.

First consider equation (4.9). Ignoring the z term, this leaves

$$\ddot{r} - \frac{l^2}{r^3} + b^2 r - \frac{g}{r^2} = 0.$$

If we let

$$r' = \frac{r}{R}, \qquad t' = \frac{t}{T},$$

and substitute in, we get

$$\ddot{r}' - \frac{T^2}{R^4}\frac{l^2}{r'^3} + T^2 b^2 r' - \frac{T^2}{R^3}\frac{g}{r'^2} = 0.$$

Now we can define the parameters

$$l' = \frac{Tl}{R^2}, \qquad b' = bT, \qquad g' = \frac{gT^2}{R^3}, \tag{4.11}$$

where l', b', g' are dimensionless. Substituting these in and dropping the primes on the variables leaves

$$\ddot{r} - \frac{l'^2}{r^3} + b'^2 r - \frac{g'}{r^2} = 0. \tag{4.12}$$

This is the dimensionless equation that we will use to plot the particle trajectories. I will also use this again in section 5.5.

4.3 Energy

Before we plot the motion, we can calculate the energy in 3 dimensions. From equation (2.9) we know that

$$E = \frac{1}{2} m \dot{\boldsymbol{R}}^2 + V.$$

Now, substituting in $\dot{\boldsymbol{R}}^2$ and simplifying gives

$$E = \frac{1}{2} m\dot{r}^2 + \frac{1}{2} m\dot{z}^2 + \frac{\left(L - \frac{1}{2}eBr^2\right)}{2mr^2} + V. \tag{4.13}$$

Recall the definition of the potential energy V from (2.8)

$$f(\boldsymbol{r}) = -\frac{\partial V}{\partial r}.$$

This can be combined with the definition of F in equation (4.1) to find the potential as

$$V = -\frac{Ze^2}{4\pi\epsilon_0 \left(r^2 + z^2\right)^{\frac{1}{2}}}. \tag{4.14}$$

Substituting this back into equation (4.13) gives the total energy of the system to be

$$E = \frac{1}{2} m\dot{r}^2 + \frac{1}{2} m\dot{z}^2 + \frac{\left(L - \frac{1}{2}eBr^2\right)}{2mr^2} - \frac{Ze^2}{4\pi\epsilon_0 \left(r^2 + z^2\right)^{\frac{1}{2}}}. \tag{4.15}$$

4.4 Orbits

We can now plot the motion of the particle under certain conditions, using the three equations that we have just derived.

4.4.1 Central Force Only

We will start by observing the motion in a central force without a magnetic field. We can do this by setting $b' = 0$ in equations (4.8) and (4.9).

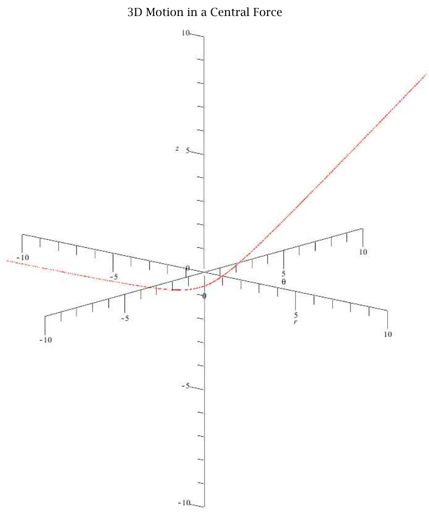

Figure 4.1: 3D motion in only a central force with positive energy.

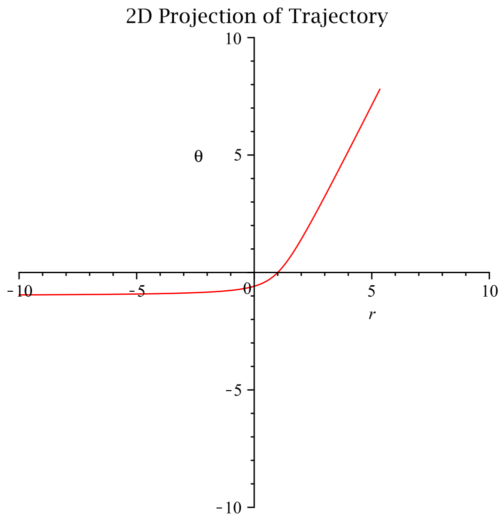

Figure 4.2: 2D projection of motion onto r,θ plane.

Figures 4.1 and 4.3 show the trajectories of a particle under different conditions. Figure 4.1 represents motion under positive energy and 4.3 shows negative energy.

Figure 4.1 shows the particle coming in in a straight line and then changing direction as it gets near the centre. It then leaves again in a straight line. This behaviour is similar to the 2 dimensional case where the trajectory is a hyperbola, as seen in figure 2.2. We have plotted the projection of this onto the r,θ plane, which can be seen in figure 4.2. The similarities between this and the 2D hyperbolic case can be seen easily.

Figure 4.3 shows a simple elliptical orbit in 3 dimensional space. This directly corresponds to the 2 dimensional case where the energy is negative, as can be seen in figures 2.3 and 2.4. A projection of this orbit onto the r,θ plane would show an ellipse like the plots in chapter 2.

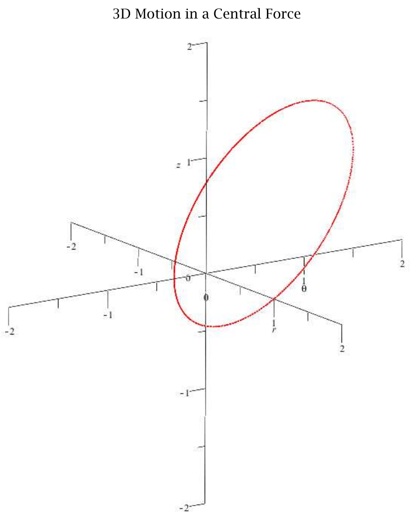

Figure 4.3: 3 dimensional elliptical orbit in a central force only with negative energy.

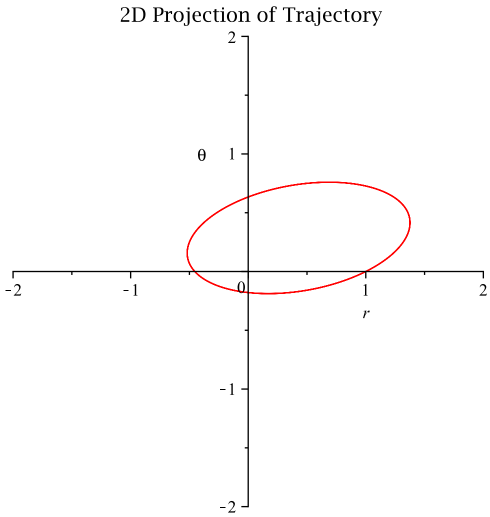

Figure 4.4: 2D projection of elliptical motion onto r,θ plane.

4.4.2 Magnetic Field Only

We are now going to produce a plot of the motion of the particle in only a magnetic field. We will put b' back into equations (4.8), (4.9) and (4.10) and this time set $g' = 0$ to remove the central force.

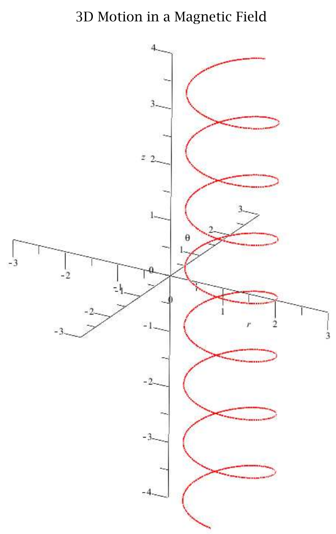

Figure 4.5: 3D orbit in a magnetic field without a central force.

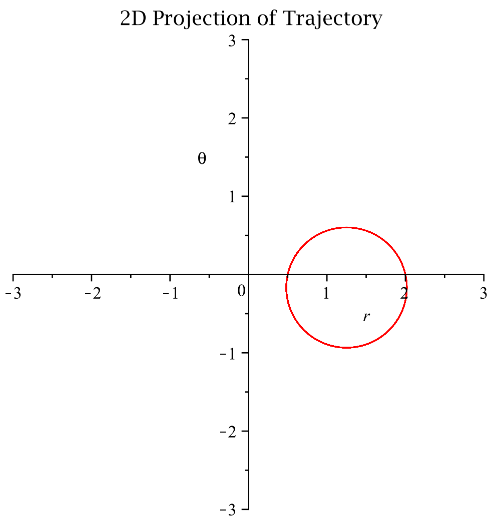

Figure 4.6: 2D projection of figure 4.5 onto r,θ plane.

Figure 4.5 shows the motion in this case. The plot shows that the particle is travelling in a spiral in the vertical direction. This is consistent with what we would expect, since the 2 dimensional plot, figure 3.2, shows circular motion. If we remember that the magnetic field is in the z direction then the obvious thing to expect is circular motion travelling upwards, which is indeed what has been observed.

If we plot the projection of the motion onto the r,θ plane, we would expect to see only a circle as all other external forces have been removed, therefore leaving nothing to force the motion away from the vertical motion of the magnetic field. Figure 4.6 shows this projection and we can see that it is indeed a circle, similar to figure 3.2

4.4.3 Central Force and Magnetic Field

Finally we are going to plot the motion when the particle is in both the central force and the magnetic field. To do this we will use equations (4.8), (4.9) and (4.10) with both b' and g' left in the equations.

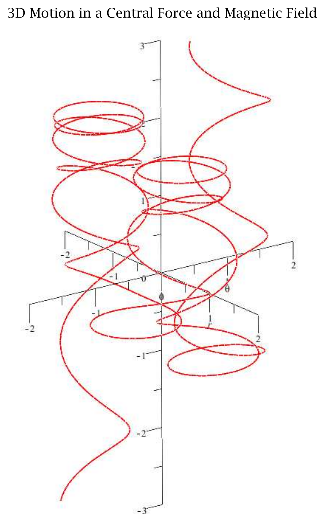

Figure 4.7: 3D motion in central force and magnetic field.

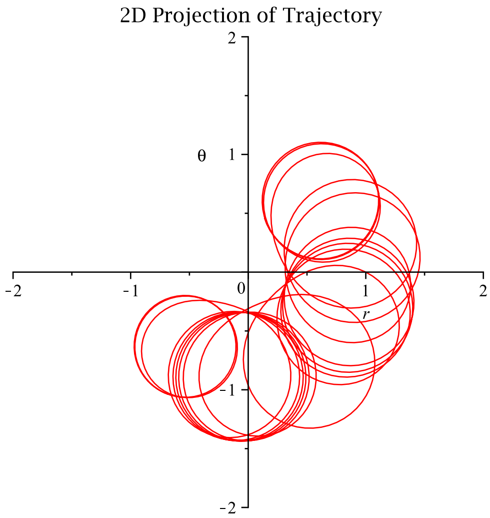

Figure 4.8: Projection of figure 4.7 into 2 dimensions.

Figures 4.7 and 4.9 show trajectories of the particle under different conditions.

Figure 4.7 is one possible trajectory. It shows the particle moving upwards in a spiral, similar to the motion without the central force. As it passes the origin, it changes direction and begins to spiral downwards. The particle continues to change direction, this is because the central force is pulling it towards the centre once it has passed the origin. This force causes the particle to change direction when it is stronger than the magnetic field at some point. Eventually, the particle manages to break free of the central force and then finally spirals upwards in only the magnetic field.

We have plotted the projection of this motion in the r,θ plane, which can be seen in figure 4.8. As you can see, it looks like the particle is spiralling round in a arc. Also, the spirals are not distributed uniformly, they look chaotic.

This seems to imply that motion in both a central force and magnetic field is chaotic. To see if this is the case we can look at the other trajectory.

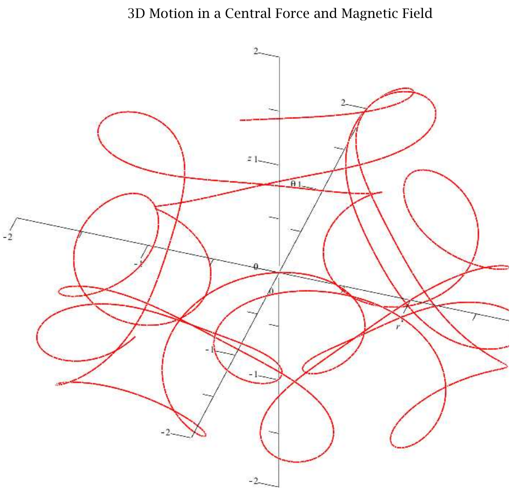

Figure 4.9: Bounded 3D orbit in central force and magnetic field.

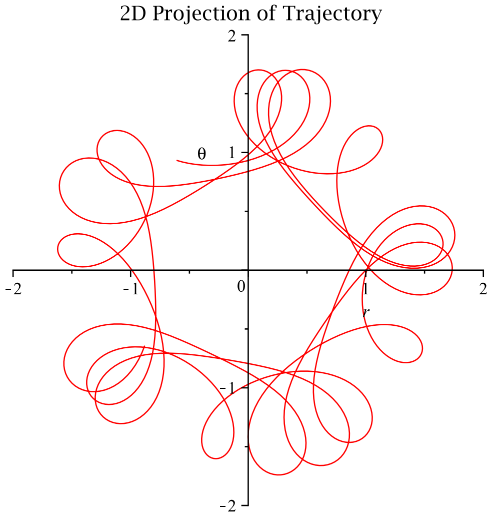

Figure 4.10: 2D projection of the bounded motion onto r,θ plane.

Figure 4.9 shows the motion under different conditions. This time the particle begins inside the central force and moves around inside the force. In this case, the motion is bounded, as the particle never manages to break free of the central force. It simply moves around inside the range of the force. Looking at this plot it is difficult to see any similarities with the 2 dimensional plots in chapter 3.

For this trajectory we have also produced a plot of the projection on the r,θ plane. Figure 4.10 shows this plot. We can see that this plot looks similar to those seen in section 3.4.2, although we couldn't see the link when looking at the 3 dimensional plot. However, this plot looks a lot less predictable those in chapter 3 and also looks quite chaotic. If we continued to observe and plot the motion, we would expect that the particle would continue to loop around, without ever breaking free of the force.

Chapter 5

Chaotic Motion

5.1 Poincaré Map

We have just seen that there are cases where the motion is bound and never escapes a certain area. A useful thing to look at in this case is a Poincaré map.

The Poincaré map is named after the french mathematician and physicist Henri Poincaré. It is a plot of all points where a trajectory in 3 dimensional space intersects with a 2 dimensional plane. The resulting map can have a structured form, showing that the motion is structured and not random. On the other hand, the map can look like a lot of randomly distributed points, showing that the motion is chaotic.

5.1.1 Map in the r,θ Plane

We are going to focus on the motion plotted in figure 4.9. We will produce a Poincaré map of when this trajectory passes through the $z = 0$ plane. In section 4.4.3, when looking at the motion, we pointed out that the particle seemed to be behaving chaotically. Therefore we should expect the Poincaré map to look random.

Figure 5.1 shows the Poincaré map. From the plot we can see that this does indeed look like a lot of points which are distributed randomly. However, we can see that they are actually bounded in the region between two circles centred on the origin. Looking at figure 5.1 we can see that this behaviour could have been predicted as the 2D projection of the motion also lies between 2 circles.

Although the points seem to be randomly distributed, and therefore implying chaos, this is not necessarily proof that the motion is actually chaotic. Consider a particle which moves in a circular orbit but one which is shifted by a small fixed amount each time. After a long time period, we could produce a Poincaré map and this would show a map very similar to

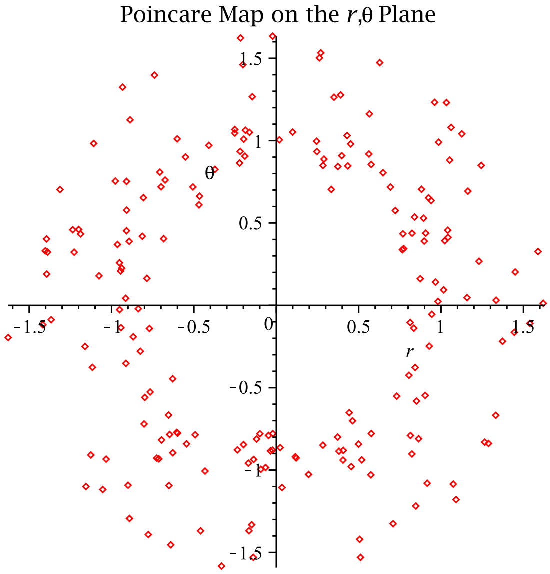

Figure 5.1: Poincaré map on the r,θ plane.

the one above. We know that the motion was not chaotic but uniform all of the time. Even so, it produced a Poincaré map which implied chaotic motion. Therefore we need to find another way to prove that the motion is chaotic.

5.2 Hamiltonian System

A dynamical system is called a Hamiltonian system if it obeys Hamilton's equations

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad i = 1, \ldots, n. \tag{5.1}$$

If a system is a Hamiltonian system then it has a Hamiltonian H which is related to the energy. For our system the Hamiltonian is

$$H = \frac{p_r^2}{2m} + \frac{p_z^2}{2m} + \frac{\left(L - \frac{1}{2}eBq_r^2\right)^2}{2mq_r^2} - \frac{Ze^2}{4\pi\epsilon_0\left(q_r^2 + q_z^2\right)^{\frac{1}{2}}}, \tag{5.2}$$

and we can show that this does indeed satisfy Hamilton's equations.

Suppose we let

$$p_r = A\dot{r}, \quad p_z = B\dot{z}, \quad q_r = r, \quad q_z = z.$$

where A and B are constants. We also have that $q_\theta = \theta$ but we don't need this as the angular momentum L is conserved. These four can be substituted into equation (5.2) to give

$$H = \frac{A^2\dot{r}^2}{2m} + \frac{B^2\dot{z}^2}{2m} + \frac{\left(L - \frac{1}{2}eBr^2\right)^2}{2mr^2} - \frac{Ze^2}{4\pi\epsilon_0\left(r^2+z^2\right)^{\frac{1}{2}}}.$$

Notice that if we set $A = B = m$ then this becomes

$$H = \frac{m\dot{r}^2}{2} + \frac{m\dot{z}^2}{2} + \frac{\left(L - \frac{1}{2}eBr^2\right)^2}{2mr^2} - \frac{Ze^2}{4\pi\epsilon_0\left(r^2+z^2\right)^{\frac{1}{2}}},$$

and this is precisely the energy which was derived in equation (4.15).

Therefore, we have a Hamiltonian defined using the variables

$$p_r = m\dot{r}, \quad p_z = m\dot{z}, \quad q_r = r, \quad q_z = z. \tag{5.3}$$

Therefore we have 4 parameters, p_r, p_z, q_r, q_z. However, since the energy is conserved, only 3 of these parameters are independent. I could freely choose any 3 of the values and substitute them into equation (4.15) to obtain the value of the fourth parameter. Therefore we have a 4 dimensional system consisting of three independent parameters and this can be used to analyse the chaotic motion.

5.3 Showing Chaotic Motion

We are now going to show that the motion is indeed chaotic. To do this we will create more Poincaré maps, but this time we will plot them on different axes, rather than on the r,θ plane. This approach has also been used to analyse the motion of a charged particle around a black hole. [5]

We can ignore motion in the θ direction by remembering that the angular momentum is constant, hence the θ motion can be eliminated. This leaves four variables, so we are working in four dimensional space. We also know that the energy is conserved meaning that we have a constraint on these four dimensions. This means that points can be plotted in 3 dimensional space. There is a possibility that the points could land on a 2 dimensional surface and this surface has the topology of a torus, this is an invariant torus.

As before, there are different behaviours that we can expect to see. There could be just two points on the plot, which corresponds to a circular or elliptical orbit. The points could trace a closed curve, which corresponds to an invariant torus. Finally the points could have no structured form and appear randomly distributed, in this case there is chaos.

Before producing the Poincaré sections, we can first consider Lyapunov exponents. Lyapunov exponents are a way of describing the rate of separation of trajectories after an initial perturbation. The separation rate of two trajectories can be written as

$$|\delta x\,(t)| \approx e^{\lambda t}|\delta x\,(0)|.$$

Here λ is called the Lyapunov exponent. Rearranging this leads to

$$\lambda \approx \frac{1}{t}\ln\left(\frac{\delta x\,(t)}{\delta x\,(0)}\right)$$

This can be written as

$$\lambda = \lim_{\substack{t\to 0\\ \delta x(0)\to\infty}} \frac{1}{t}\ln\left(\frac{\delta x\,(t)}{\delta x\,(0)}\right)$$

In our case there would be two Lyapunov exponents as we have three independent parameters. The signs of these Lyapunov exponents determine the qualities of the motion. There are three main cases that could occur. Firstly, if both exponents are negative, then we would have stable closed orbits, i.e. circular or elliptical orbits. If one was positive and one negative, the orbits would lie on an invariant torus in the phase space. Finally, both exponents could be positive, which corresponds to having unstable orbits which lead to chaos. These three possibilities directly correspond to the three possibilities of the Poincaré sections.

Since we have just a small number of dimensions, there is no need to calculate the Lyapunov exponents. We can see which case we have simply by looking at the Poincaré maps. This means that performing the calculations would be pointless and time consuming. If we had a larger number of dimensions, then calculating the Lyapunov exponents would be necessary as the Poincaré sections alone would not give enough information about what is going on.

We can now produce the Poincaré maps on the r,$\dot{r}$ plane, the r,$\dot{z}$ plane and the $\dot{r}$,$\dot{z}$ plane. These can be seen in figures 5.2, 5.3 and 5.4.

In each of these plots we can see that there are still a lot of randomly distributed points. This shows that the motion is indeed chaotic as there is no well defined shape. Figure 5.2 shows the Poincaré map in the r,$\dot{r}$ plane, this plot was done in the paper by H. Jaio.[1] However the other two plots are new and these plots show how the structure changes when looking at different Poincaré sections.

5.4 Transition to Chaos

We can now reduce the magnetic field to see how the trajectories behave and see if the motion tends to a periodic solution. We are still focusing in the case where the energy is negative and the motion is bounded.

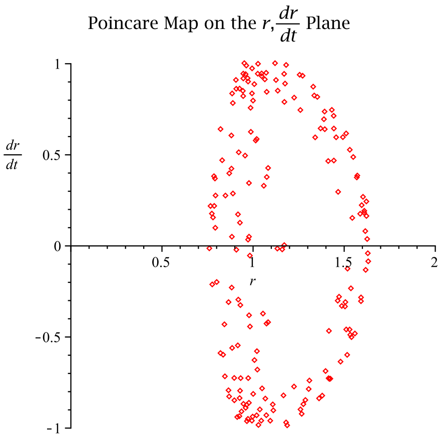

Figure 5.2: Poincaré map on the r,$\dot{r}$ plane.

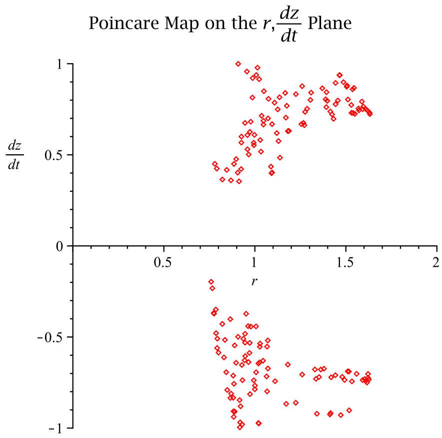

Figure 5.3: Poincaré map on the r,$\dot{z}$ plane.

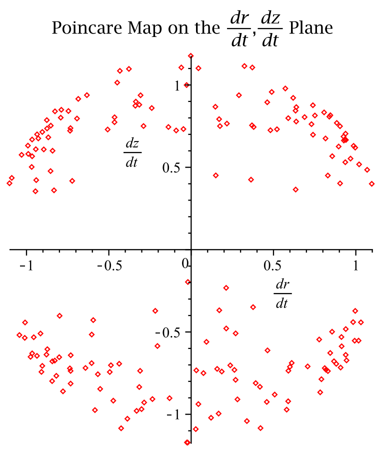

Figure 5.4: Poincaré map on the $\dot{r}$,$\dot{z}$ plane.

We have seen that the motion is ordered when there is no magnetic field and chaotic when there is a magnetic field. Therefore there must be a point in between, where the motion is partly chaotic and partly regular. We have reduced the magnetic field to a value of $b' = 0.01$ and we are going to produce a Poincaré map of the motion on the $\dot{r}$,$\dot{z}$ plane, similar to figure 5.4. As the motion will be mostly structured and almost chaotic, we expect the Poincaré map to look like a closed curve which is sensitive to initial conditions. This is the second case described at the beginning of the previous section, the case where the map traces out an invariant torus. We will produce a lot of these Poincaré sections under different conditions and plot them all on the same axes. This will give an indication of how the motion changes and will identify any fixed points in the motion.

When producing these plots we will be keeping the energy constant for each curve. This is because the energy is conserved and changing it for each plot would give different behaviours, therefore not identifying the correct fixed points. To do this we can use the energy to output the value of $\dot{z}$ when the values of the energy E, r and $\dot{r}$ are inputted. Also, it is easier to set $z = 0$ to reduce the number of dimensions of the problem by 1.

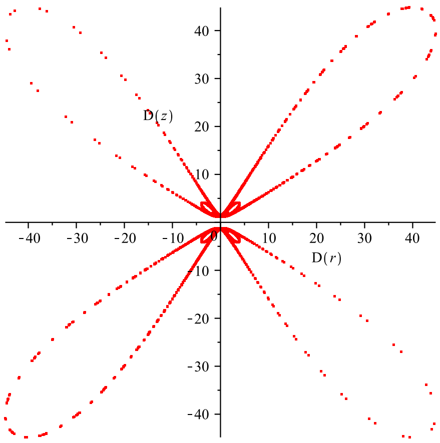

Figure 5.5: Many Poincaré maps of the motion in a small magnetic field.

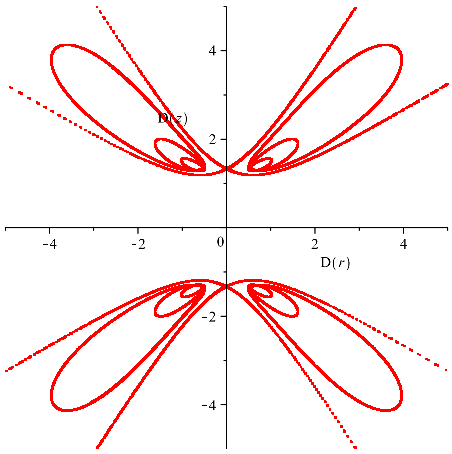

Figure 5.6: Zoomed in image of the Poincaré maps in figure 5.5.

Figure 5.5 shows a plot containing many of these Poincaré maps. There are large loops around the outside but it is difficult to see what is happening towards the middle. Figure 5.6 shows this plot zoomed in to the centre. This plot contains many closed curves. Each of these curves is in fact a non chaotic invariant torus around a fixed point, lying on a surface of constant energy.

We can see that there are four positions where the tori gradually get smaller under different initial conditions. Eventually there will be conditions which cause them to tend to a single point, this is the fixed point. These fixed points correspond to periodic solutions when the magnetic field is small. When producing these plots, it was clear that opposite quadrants of the plot correspond to the same conditions, so at each of the two fixed points, the motion repeatedly passes through the same two points. Clearly this corresponds to elliptical or circular motion, similar to that seen in chapter 2.

Looking at the plot, we can see that there are two more interesting points, in between the two fixed points both above and below the $\dot{r}$ axis. Here we can see the lines seem to be touching or crossing over each other.

These two points are also fixed points, however it is more difficult to see from this plot what is happening. We have zoomed in again to focus on one of these points, which can be seen in figure 5.7. This seems to resemble a saddle point.

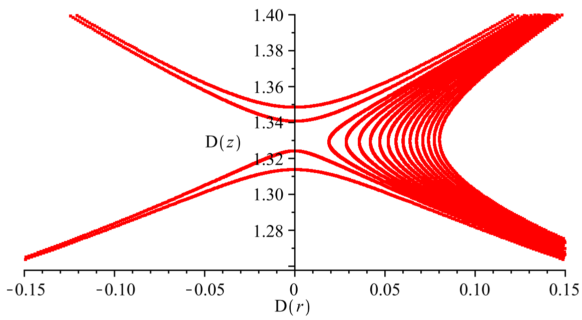

Figure 5.7: Figure 5.6 zoomed in on the top centre fixed point.

We have found 6 fixed points in the motion of the particle. Four of these correspond to periodic solutions and the other two are saddle points. The periodic solutions can be classed as stable fixed points and the saddles can be classed as unstable fixed points.

We can now define two different types of orbit, heteroclinic orbits and homoclinic orbits. We have just seen that the curves which approach the saddle points result in unstable fixed points. If an orbit begins on one of these curves it is called a heteroclinic orbit. On the other hand, a homoclinic orbit is one which starts on one of the closed tori. That is, those corresponding to stable periodic solutions.

Finding these periodic orbits in the small magnetic field is an example of the KAM theorem. The KAM theorem states that when a parameter is slightly perturbed, some of the invariant tori present in the unperturbed case, will still be present but may be slightly distorted. In our case the parameter is the magnetic field. As it was increased slightly, invariant tori have formed around the stable fixed points. So when the magnetic field is small, orbits will lie close to the periodic solutions.

Clearly there is very little chaos in a small magnetic field, so the next thing to do is to increase b' to see what happens during the transition into chaos.

Figure 5.8 shows the Poincaré map of the motion in the case where $b' = 0.5$. There are no longer any clearly defined closed loops, but a few clear lines surrounded by a lot of individual points. The shape is approximately similar to that of the previous case, but much less clear. This shows the midway point between the example above and fully chaotic motion.

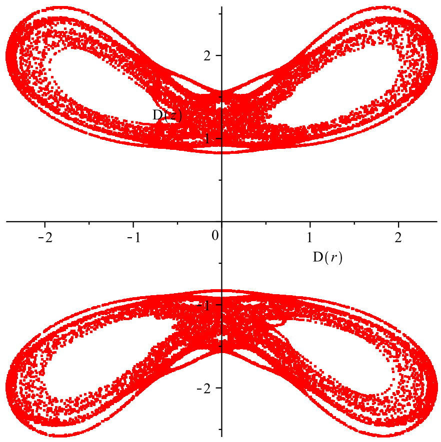

Figure 5.8: Poincaré map showing motion in an increased magnetic field.

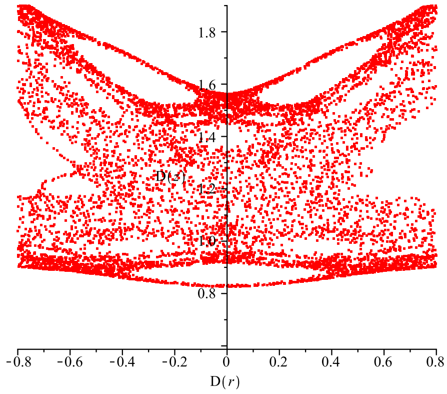

Figure 5.9: Zoomed in figure 5.8 to show the onset of chaos in an increased magnetic field.

We can zoom in on the centre as we did above and this plot can be seen in figure 5.9. The outer regions are well defined but between these regions are a lot of random points indicating that the motion is becoming chaotic.

Figure 5.8 looks similar in structure to figure 5.6. There are four clear areas surrounded by points in the same way that previously there were four fixed points surrounded by invariant tori. This shows that, even in an increased magnetic field, there are still periodic solutions. In the small magnetic field, these solutions were stable fixed points. The fact that they are still present in the increased magnetic field verifies that they are indeed stable. Previously there were also two unstable saddle points. Figure 5.9 shows that these fixed points are no longer present but instead we now have chaos. This also confirms that they were indeed unstable fixed points as they have quickly broken up into chaos with an increase in magnetic field.

If we increase the magnetic field again we will receive a plot which looks identical to figure 5.4 showing that the transition to chaotic motion is complete.

5.5 Magnetic Field Strength

In equation (4.11) we defined some dimensionless parameters l', b' and g'. We can use these values to calculate the size of the magnetic field and see how large the magnetic field needs to be before the motion becomes chaotic.

We can cancel R by considering

$$\frac{l'^3}{g'^2} = \frac{T^3 l^3}{g^2 T^4} = \frac{l^3}{g^2 T},$$

which can also be written as

$$\frac{1}{T} = \frac{g^2 l'^3}{g'^2 l^3}. \tag{5.4}$$

Recall the definition of g from equation (3.9)

$$\frac{gm}{r^2} = \frac{Ze^2}{4\pi\epsilon_0 r^2}.$$

We can set $Z = 1$ since Rydberg atoms behave in a similar way to hydrogen atoms, which leaves

$$g = \frac{e^2}{4\pi\epsilon_0 m}.$$

Using quantum mechanics, the angular momentum can be written as

$$L^2 = l_q \left(l_q + 1\right) \hbar^2,$$

where l_q is the angular quantum number of the Rydberg atom. In a Rydberg atom, $l_q \approx n$ and we know from the definition of the Rydberg atom that n is large. This means that we can rewrite the above equation as

$$L^2 = l_q^2 \hbar^2 \approx n^2 \hbar^2,$$

therefore

$$L = n\hbar,$$

Using this in the parameterization (3.7) leaves

$$l = \frac{n\hbar}{m}.$$

Substituting l and g into equation (5.4) gives

$$\frac{1}{T} = \frac{e^4 m}{16\pi^2 \epsilon_0^2 \hbar^3} \frac{l'^3}{g'^2 n^3}.$$

Now substituting this into b from equation (4.11) gives

$$b = \frac{b'}{T} = \frac{e^4 m}{16\pi^2 \epsilon_0^2 \hbar^3} \frac{l'^3 b'}{g'^2 n^3}.$$

Finally, rearranging equation (3.8) and substituting in b leaves

$$B = \frac{2mb}{e} = \frac{e^3 m^2}{8\pi^2 \epsilon_0^2 \hbar^3} \frac{l'^3 b'}{g'^2 n^3}. \tag{5.5}$$

This can now be used to calculate the strength of the magnetic field. To do this we will need some values for each of the variables in the above equation. We will use the following values written in SI units

$$
\begin{aligned}
e &= 1.6 \times 10^{-19}\mathrm{A\,s}, \\
m &= 9.1 \times 10^{-31}\mathrm{kg}, \\
\epsilon_0 &= 8.85 \times 10^{-12}\mathrm{A^2\,s^4\,m^{-3}\,kg^{-1}}, \\
\hbar &= 1.055 \times 10^{-34}\mathrm{kg\,m^2\,s^{-1}}.
\end{aligned}
$$

These can now be substituted into equation (5.5) to give

$$
\begin{aligned}
B &= 4.67 \times 10^5 \frac{l'^3 b'}{g'^2 n^3}\mathrm{kg\,s^{-2}\,A^{-1}}, \\
&= 4.67 \times 10^5 \frac{l'^3 b'}{g'^2 n^3}\mathrm{Tesla}.
\end{aligned}
$$

The only terms left are the dimensionless terms which we used to produce plots of the orbits. Substituting these in would give the strength of the magnetic field.

For example, using the values for l' and g' that we have kept constant, choosing a large value of n, say $n = 100$, and increasing b' in the same way we have done throughout this chapter, we can calculate the strength of the magnetic field at each stage of the onset of chaos. This gives the strength of the magnetic field when there is very little chaos, as seen in figures 5.5, 5.6 and 5.7, around 0.146Tesla. The strength of the field during the onset of chaos from figures 5.8 and 5.9 is around 7.297Tesla. Finally, when the motion has become fully chaotic, as seen in chapter 4 and the beginning of chapter 5, the magnetic field is around 14.594Tesla.

Chapter 6

Conclusion

We have seen that the motion of a particle can be complicated and chaotic when there is both a central force and magnetic field present.

I began this project by looking at particle motion in two dimensions. In only a central force the particle displayed circular and elliptical orbits. This was as expected since we have observed similar orbits in the physical world, such as in outer space. When the magnetic field was added to the central force, the motion became more complicated. After generalising to three dimensions, I related the trajectories to the 2D orbits and found some similarities. However, I also found that the motion could be either bounded or unbounded. This fact could not be seen in two dimensions. Both cases resulted in chaotic motion. I focused on the bounded case and produced Poincaré maps to show the chaos. I plotted Poincaré sections on many different planes, where previous work has focused on only one. This has been more informative as it has allowed me to see more detailed structure of the motion. I then adjusted the magnetic field to be very small, then plotting more Poincaré maps at fixed energy. This allowed me to find four fixed points which correspond to periodic solutions and two more fixed points which are saddle points. Increasing the magnetic field strength slightly showed the transition of the motion into chaos.

Moving on from this project, the next thing to look at would be the unbound orbits. In particular, the case of scattering would be interesting to consider. A particle would begin by spiralling upwards along the magnetic field lines. As the particle approaches the origin, it would come into the range of the central force causing the motion to become chaotic as we have seen. The case of scattering is where the trajectory as it exits the force is different to the trajectory as it enters. An interesting thing to look at is how much the particle becomes shifted and whether it is possible for the particle to be reflected back the way it came.

This work has also been extended by moving from classical motion to semi classical and even considering quantum mechanics.

Bibliography

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Appendix A

MAPLE Code

```
mysolve := proc(l, b, e, R, d, xm, ym, n) local(eq1, eq2, bc1, bc2, bc3, s);
eq1 := diff(theta(t), t) - l/r(t)^2 + b = 0;
eq2 := diff(diff(r(t), t), t) - l^2/r(t)^3 + b^2·r(t) - e/r(t)^2 = 0;
bc1 := r(0) = R;
bc2 := D(r)(0) = d;
bc3 := theta(0) = 0;
s := dsolve({eq1, eq2, bc1, bc2, bc3}, numeric, output = listprocedure, range = -n..n);
plots[odeplot](s, [r(t)·cos(theta(t)), r(t)·sin(theta(t))], view = [-xm..xm, -ym..ym], labels = ['r',
    'theta'], numpoints = 1000, scaling = constrained);
end proc:
```

Figure A.1:

This is the code that I used to produce the orbits in chapters 2 and 3. I integrated the system of two equations under certain boundary conditions using the command `dsolve`. I then used this to create a plot of the orbit when the conditions were given.

```
Wplot :=proc(l, b, e, rm, wm)local(W);
W := r^2·(l/r^2 − b)^2 - 2e/r;
plot(W, view = [0 .. rm, -wm ..wm], labels = ['r','W']);
end proc:
```

Figure A.2:

Here is the code used to produce the plots of W which can be seen in figures 2.1 and 3.1

```
mysolve :=proc(l, b, e, R, d, Z, xm, ym, zm, n)local(eq1, eq2, eq3, bc1, bc2, bc3, bc4, bc5, s);
eq1 := diff(theta(t), t) - l/r(t)^2 + b = 0;
eq2 := diff(diff(r(t), t), t) - l^2/r(t)^3 + b^2·r(t) - e·r(t)/(r(t)^2 + z(t)^2)^(3/2) = 0;
eq3 := diff(diff(z(t), t), t) − e·z(t)/(r(t)^2 + z(t)^2)^(3/2) = 0;
bc1 := r(0) = R;
bc2 := D(r)(0) = d;
bc3 := theta(0) = 0;
bc4 := z(0) = 0;
bc5 := D(z)(0) = Z;

s := dsolve({eq1, eq2, eq3, bc1, bc2, bc3, bc4, bc5}, numeric, output = listprocedure, range = -n ..n);
plots[odeplot](s, [r(t)·cos(theta(t)), r(t)·sin(theta(t)), z(t)], thickness = 2, axes = normal, view = [
    -xm ..xm, -ym ..ym, -zm ..zm], labels = ['r','theta','z'], scaling = constrained, numpoints = 1000);
plots[odeplot](s, [r(t)·cos(theta(t)), r(t)·sin(theta(t))], axes = normal, view = [-xm ..xm, -ym ..ym],
    labels = ['r','theta'], scaling = constrained, numpoints = 1000);
end proc:
```

Figure A.3:

This code was used to plot the 3 dimensional orbits in chapter 4. I used the `dsolve` command to integrate the 3 equations in the same way as for the 2D orbits. I then used the output to produce two plots, one 3 dimensional trajectory and one 2 dimensional projection onto the r,θ plane.

```
mysolve :=proc(l, b, e, R, d, T, Z, n)local(eq1, eq2, eq3, bc1, bc2, bc3, bc4, bc5, s);
```

$$eq1 := diff(\text{theta}(t), t) - \frac{l}{r(t)^2} + b = 0;$$

$$eq2 := diff(diff(r(t), t), t) - \frac{l^2}{r(t)^3} + b^2 \cdot r(t) - \frac{e \cdot r(t)}{\left(r(t)^2 + z(t)^2\right)^{\frac{3}{2}}} = 0;$$

$$eq3 := diff(diff(z(t), t), t) - \frac{e \cdot z(t)}{\left(r(t)^2 + z(t)^2\right)^{\frac{3}{2}}} = 0;$$

```
bc1 := r(0) = R;
bc2 := D(r)(0) = d;
bc3 := theta(0) = T;
bc4 := z(0) = 0;
bc5 := D(z)(0) = Z;

dsolve( {eq1, eq2, eq3, bc1, bc2, bc3, bc4, bc5}, numeric, range = 0 ..n, events = [ [z(t), halt]]);

end proc:


poincare :=proc(l, b, e, R, d, T, Z, n)local(x, y, p, s, R1, d1, T1, Z1, i);

x := array(1 ..1000);
y := array(1 ..1000);
p := array(1 ..1000);

R1 := R;
d1 := d;
T1 := T;
Z1 := Z;

for i from 1 to n do
s := mysolve(l, b, e, R1, d1, T1, Z1, n);
R1 := eval(r(t), s('last'));
d1 := eval(diff(r(t), t), s('last'));
T1 := eval(theta(t), s('last'));
Z1 := eval(diff(z(t), t), s('last'));
x[i] := R1·cos(T1);
y[i] := R1·sin(T1);
end do:

p := [ [x[m], y[m]]$m = 1 ..n];
plots[pointplot](p, color = red, scaling = constrained, labels = ['r','theta']);

end proc:
```

Figure A.4:

Here I have written some code to produce the Poincaré map seen in figure 5.1. I began by defining a procedure to integrate the 3 dimensional equations using `dsolve`. However, this time I used the `events` parameter to stop the integration whenever the motion passed through the $z = 0$ plane. I then created another procedure which saved the values of r and θ

to an array each time the integration was stopped. Finally I plotted these values, producing the Poincaré map.

This code was easily adjusted to produce figures 5.2, 5.2 and 5.2. I simply had to change the values that were saved to the array.

$$m := 9.1 \cdot 10^{-31}$$

energy :=**proc**(*E*, *R*, *d*, *l*, *b*, *e*)**local**(*Z*);

$$Z := \left(\frac{2\left(\mathrm{E} - \left(-\frac{e \cdot m}{R} \right)\right)}{m} - d^2 - R^2 \cdot \left(\frac{l}{R^2} - b \right)^2 \right)^{\frac{1}{2}};$$

end proc:

```
poincare1 :=proc(l, b, e, R, d, T, E, n)local(x, y, p, s, w, Z, R1, d1, T1, Z1, i);

x := array(1 ..50000);
y := array(1 ..50000);
p := array(1 ..50000);

Z := energy(E, R, d, l, b, e);

R1 := R;
d1 := d;
T1 := T;
Z1 := Z;

for i from 1 to n do
s := mysolve(l, b, e, R1, d1, T1, Z1, n);
R1 := eval(r(t), s('last'));
d1 := eval(diff(r(t), t), s('last'));
T1 := eval(theta(t), s('last'));
Z1 := eval(diff(z(t), t), s('last'));
x[i] := d1;
y[i] := Z1;
end do:

p := [[x[j], y[j]]$j = 1 ..n];
plots[pointplot](p, color = red, labels = ['D(r)','D(z)'], symbol = point);
end proc:
```

Figure A.5:

This is the code that was used to produce the Poincaré maps in section 5.3 which were used to find fixed points and periodic orbits through the transition into chaos. First I created a procedure which outputted the value of z when the energy and all other parameters were inputted, as described in section 5.3 . I then used this in the procedure to create a Poincaré map, looping through and saving the $\dot{r}$ and $\dot{z}$ values after each iteration. I created many of these Poincaré maps and saved them. Finally I used the `display` command to put all plots in one figure resulting in figures 5.5 and 5.8.

This code was then adjusted slightly to zoom in on certain points by modifying the view in the plot command. This allowed me to produce figures 5.6, 5.7 and 5.9.