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Gyroscopic Precessions in Relativity and Gravitoelectromagnetism

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Abstract

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by Paul Bennett

The two main relativistic gyroscopic precessions are discussed. It is shown that the two precessions are related intimately to not just the shape of the object the gyroscope is orbiting, but also its angular momentum. These effects are discussed in detail and then consequences of the angular momentum of a body on light scattering are discussed. The basic equations of gravitoelectromagnetism are derived through the electromagnetism analogy, starting with linearising the Einstein field equations. These equations are solved meticulously to obtain the metric of a slowly rotating spherical object and the Maxwell-like field equations. The results are then used to carefully re-derive one of the earlier precessions in a more general case and afterwards calculate the expected precession values in recent experiments.

Contents

Chapter 1

Introduction

There are several well known tests of general relativity. Two of the less commonly known effects involve spinning bodies in orbits around massive objects. There are two such cases; the geodetic effect which is a spin-orbit coupling, and the frame-dragging effect which is a spin-spin coupling. Both of these phenomenon were predicted in one way or another shortly after Einstein published his theory of general relativity in 1916, and in this paper we discuss, after a few preliminaries in Chapter 2, these two effects with respect to the precession of the axes of an ideal gyroscope. Chapter 3 gives an introduction to the larger of the two precessions, the geodetic precession. We calculate a value for the precession in an idealised situation where the gyroscope is in a stable circular orbit in the equatorial plane. The method closely follow the methods in John Hartle's book [1]. Chapter 4 investigates the second precession of the gyroscope, the Lense-Thirring precession. Again a brief introduction is given to the precession before we get involved in any mathematics. The solution for an idealised situation is once again derived in a very similar manner, this time with the gyroscope free-falling down the rotational axis of a slowly rotating massive object. The light bending effects of a massive spherical body are then derived for two situations: one where the body is stationary and one where the body is slowly rotating.

The most interesting mathematics comes in in Chapter 5. This chapter is based on gravitoelectromagnetism (GEM). We set the scene by discussing the similarities between electromagnetism and GEM and how the concept arose. We take a linear perturbative approach to the Einstein field equations to derive a linearised set of equations. The equations are then fully solved to produce the Maxwell-like GEM equations (aptly named due to the striking resemblance between these equations and the Maxwell equations). Taking into account the aforementioned parallels in electromagnetism and GEM, we re-derive the Lense-Thirring precession by using techniques in electromagnetism and the results we found solving the linearised field equations. The Lense-Thirring precession is then added to a quoted geodetic precession formula to write down the Schiff equation; the equation that governs the mathematics behind Gravity Probe B.

1.1 Gravity Probe B

Gravity Probe B (henceforth known as GP-B) is a NASA based mission to test the theory of general relativity. Specifically, GP-B was commissioned to measure the geodetic and frame-dragging (Lense-Thirring) effects of the Earth and therefore provide another test of the theory of general relativity. Four spinning gyroscopes were housed in a satellite orbiting the poles of the Earth at a height of 642 km. The displacement angles of the spin axes were measured over the course of one year and the results were compared with theoretical predictions.

The results [2] confirmed the predictions of general relativity to great accuracy. The mean geodetic precession rate for the four gyroscopes was -6,601.8 $\pm$ 18.3 mas/year (the minus sign being there due to the method used to measure the precession, we shall ignore it for our purposes) whilst general relativity predicted -6,606.1 mas/year; and the mean frame-dragging precession rate was -37.2 $\pm$ 7.2 mas/year whilst general relativity predicted -39.2 mas/year (where mas is milli-arcsecond). The calculated precessions are therefore accurate to 0.28% and 19% respectively. The frame-dragging precession has been derived in Sec. 5.4. Although we are not too concerned with the actual results of GP-B, they are included as an indicator of the accuracy of general relativity.

Chapter 2

Preliminaries

2.1 Notation

We shall be using standard relativistic notation throughout. Greek indices run from 0 to 3 whilst Latin indices run from 1 to 3 (hence imply purely spatial components). The components are ordered such that $(x^0, x^i) = (t, x^i)$ for whatever coordinate system we are in; this is simply so the zeroth component is time. The metric is taken to be of the form $(-+++)$ and hence has signature (3,1). Contraction shall be performed over the first and third indices of the Riemann tensor to form the Ricci tensor, that is $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$. A comma implies standard derivative and a semi-colon implies covariant derivative as per normal. Numerical indices shall be used interchangeably with symbolic indices, so $x^0 = x^t$. The former implies some general coordinate system, whilst the latter is specific to polar or Cartesian.

In Chapter 5 we shall take $\mathbf{J}$ to be the angular momentum of the massive rotating object, and $\mathbf{S}$ to be the angular momentum of the gyroscope rotating this object. It is for this reason in Chapter 4 we shall use J for the massive rotating objects angular momentum.

2.2 Dimensionless Quantities

In general relativity we normally use geometrised units where we let $c = G = 1$ where c is the speed of light and G is the gravitational constant (although we are not too interested in letting $G = 1$ most of the time, so sometimes G will appear even in these geometrised units). We shall be using this convention throughout, unless otherwise specified. We shall represent the dimension of the physical quantities mass, length and time by M, L and T respectively. The reader should note that the dimension of mass, M, is different from the units metres which is represented by m.

When we later analyse the affects a slowly rotating body has on the surrounding spacetime, one ratio that we shall come across is $4GJ/c^3r^2$. G is the gravitational constant with units N m^2 kg^{-2}, J is the angular momentum with units N m s, c has units m s^{-1} and r has units m. Dimensionally analysing this quantity we obtain

$$\begin{aligned} \frac{4GJ}{c^3r^2} &\sim \frac{\left[\mathrm{N\ m^2\ kg^{-2}}\right]\left[\mathrm{N\ m\ s}\right]}{\left[\mathrm{m\ s^{-1}}\right]^3\left[\mathrm{m}\right]^2} \\ &\sim \frac{\left(\mathrm{M\ L\ T^{-2}L^2M^{-2}}\right)\left(\mathrm{M\ L\ T^{-2}L\ T}\right)}{\left(\mathrm{L^3T^{-3}}\right)\left(\mathrm{L^2}\right)} \\ &\sim 1 \end{aligned} \tag{2.1}$$

and so we can see that this quantity is dimensionless.

2.3 Coordinate System

Throughout this project, we will be using spherical polar coordinates, unless otherwise stated. We shall use the normal convention (r, θ, ϕ) where r represents the radial distance, θ represents the polar angle and ϕ represents the azimuthal angle. The Cartesian coordinates of such a point may be retrieved from the following set of equations:

$$\begin{aligned} x &= r\sin(\theta)\cos(\phi) \\ y &= r\sin(\theta)\sin(\phi) \\ z &= r\cos(\theta) \end{aligned} \tag{2.2}$$

where $0 \leq r < \infty$, $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$.

2.3.1 Differential Equations

From the above sets of equations, (2.2), it is possible to derive the differentials needed to transform line elements from one set of coordinates to another. The total differential of a function is defined in summation notation by

$$dx^i = \frac{\partial x^i}{\partial x^j} dx^j \tag{2.3}$$

So calculating our differentials we obtain

$$\begin{align} dx &= \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \phi} d\phi \nonumber \\ &= \sin(\theta)\cos(\phi)\, dr + r\cos(\theta)\cos(\phi)\, d\theta - r\sin(\theta)\cos(\phi)\, d\phi \tag{2.4a} \\ dy &= \sin(\theta)\sin(\phi)\, dr + r\cos(\theta)\sin(\phi)\, d\theta - r\sin(\theta)\cos(\phi)\, d\phi \tag{2.4b} \\ dz &= \cos(\theta)\, dr - r\sin(\theta)\, d\theta \tag{2.4c} \end{align}$$

Considering (2.4a) and (2.4b) we can see that

$$\begin{align} x\ dy - y\ dx = &\left(r\sin^2(\theta)\cos(\phi)\sin(\phi) - r\sin^2(\theta)\cos(\phi)\sin(\phi)\right) dr \nonumber \\ &+ \left(r^2\sin(\theta)\cos(\theta)\cos(\phi)\sin(\phi) - r^2\sin(\theta)\cos(\theta)\sin(\phi)\cos(\phi)\right) d\theta \nonumber \\ &+ \left(r^2\sin^2(\theta)\sin^2(\phi) + r^2\sin^2(\theta)\cos^2(\phi)\right) d\phi \nonumber \\ &= r^2\sin^2(\theta)\, d\phi \tag{2.5} \end{align}$$

and hence observation shows

$$d\phi = \frac{x\ dy - y\ dx}{r^2\sin^2(\theta)} \tag{2.6}$$

which shall be used later when we consider the Lense-Thirring metric.

2.4 Christoffel Symbols

It is assumed that the reader has a background knowledge of general relativity and will therefore know what Christoffel symbols are; though the definition of the symbols and the calculations used to obtain the symbols are given below. The definition of the Christoffel symbols is

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\sigma} \left(\frac{\partial g_{\sigma\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\sigma\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\sigma}} \right) \tag{2.7}$$

Although it is possible to calculate the Christoffel symbols via the definition, in many cases, for example when we have a non-diagonal metric, it is quicker and easier to calculate them from the Lagrangian of the metric.

2.4.1 Christoffel Symbols from the Lagrangian

Let us start by defining the line element as a Lagrangian. We shall use $L = \frac{1}{2} g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu}$ where the dot denotes s-derivatives ($\dot{x}^{\mu} = dx^{\mu}/ds$). Using this Lagrangian and the Euler-Lagrange equation,

$$\frac{\partial L}{\partial x^{\mu}} - \frac{d}{ds} \frac{\partial L}{\partial \dot{x}^{\mu}} = 0 \tag{2.8}$$

we can easily calculate the non-vanishing Christoffel symbols for a given metric. Let us calculate the non-vanishing $\Gamma^{\theta}_{\mu\nu}$ symbols for the Schwarzschild metric. We know that $g_{\mu\nu} = 0$ when $\mu \neq \nu$, hence we only have $g_{tt} = (1 - 2M/r)$, $g_{rr} = (1 - 2M/r)^{-1}$, $g_{\theta\theta} = r^2$ and $g_{\phi\phi} = r^2 \sin^2(\theta)$. So, for θ we get

$$\begin{aligned} 0 &= \frac{\partial L}{\partial \theta} - \frac{d}{ds} \frac{\partial L}{\partial \dot{\theta}} \\ &= \frac{1}{2} r^2 \cdot 2 \sin(\theta) \cos(\theta) \dot{\phi}^2 - \frac{d}{ds} \left(r^2 \dot{\theta} \right) \\ &= r^2 \sin(\theta) \cos(\theta) \dot{\phi}^2 - 2 r \dot{r} \dot{\theta} - r^2 \ddot{\theta} \end{aligned} \tag{2.9}$$

Factoring out $-r^2$ and changing our notation of $\theta = x^{\theta}$ we find the differential equation

$$\begin{aligned} 0 &= \frac{d^2 x^{\theta}}{ds^2} - \sin(\theta) \cos(\theta) \left(\frac{dx^{\theta}}{ds} \right)^2 + \frac{2}{r} \frac{dr}{ds} \frac{d\theta}{ds} \\ &= \frac{d^2 x^{\theta}}{ds^2} - \Gamma^{\theta}_{\phi\phi} \left(\frac{dx^{\theta}}{ds} \right) \left(\frac{dx^{\theta}}{ds} \right) + \left(\Gamma^{\theta}_{r\theta} + \Gamma^{\theta}_{\theta r} \right) \frac{dr}{ds} \frac{d\theta}{ds} \end{aligned} \tag{2.10}$$

Hence we can read off $\Gamma^{\theta}_{\phi\phi} = -\sin(\theta) \cos(\theta)$ and $\Gamma^{\theta}_{\theta r} = \Gamma^{\theta}_{r\theta} = 1/r$, where in the last step we have simply expanded the geodesic equation (C.1) for the θ-component.

Chapter 3

Geodetic Precession

The following two chapters closely follow the methods use by John Hartle in his book [1].

The geodetic precession/effect (also known as de Sitter precession/effect) is the effect that a spherical mass has on the surrounding spacetime. The effect was predicted initially by Willem de Sitter in 1916 with respect to relativistic corrections to the motions of the Earth-Moon system in the presence of the Sun's gravitational field [3]. Whilst it wasn't exactly the geodetic effect that was calculated, a corresponding effect was proposed independently by Fokker [4] and Eddington [5]. The effect is to do with *parallel transport* along geodesics. For example, if we were to spin a gyroscope about some axis and move it along a geodesic, the spin axis will stay parallel with respect to the affine connection of the manifold. This is fundamentally the definition of parallel transport. If the gyroscope were to move in a circle in flat spacetime (note that this is not a geodesic) the axis would return to its starting position (if we ignore the Thomas precession), but in curved spacetime the axis will not return to its original position.

There are many different ways to obtain the geodetic precession, and complications arise when trying to compare these different methods. It is a well known phenomenon that when a gyroscope has a constant circular orbit with an acceleration perpendicular to this orbit, the axes of the gyroscope obtains a precession called the Thomas precession (as mentioned above). Intuitively one would think that we would therefore obtain a Thomas precession with a gyroscope orbiting the Earth. Different authors have different views on this, but an analysis of this is beyond the scope of this project. For this reason we shall not derive the exact geodetic precession as predicted in the Schiff formula; instead we shall derive an approximate value for the change in angle of the spin axes over one revolution (in an idealised situation), and then quote the exact formula in Sec. 5.4.

Since the geodetic precession is caused by a spherical mass, we can use the Schwarzschild metric to describe the geometry of spacetime around the mass. The Schwarzschild metric is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2(\theta)\, d\phi^2\right) \tag{3.1}$$

Observation shows that the Schwarzschild metric is simple metric with four independent components; there are no cross terms which simplifies the resulting algebra. To simplify the mathematics further we shall consider a gyroscope orbiting the mass in the equatorial plane. The polar angle is therefore $\theta = \pi/2$ hence $g_{\phi\phi} = r^2$. Because of the spherical shape of the object, there is going to be azimuthal symmetry. Let us also assume that the orbit has Schwarzschild radius R. There is no radial movement of the gyroscope as it is in a stable circular orbit, and it remains in the equatorial plane hence $u^r = u^\theta = 0$. We therefore have that $\mathbf{u} = \left(u^t, 0, 0, u^\phi\right)$. By definition,

$$u^\phi \equiv \frac{d\phi}{d\tau} = \frac{d\phi}{dt}\frac{dt}{d\tau} = \Omega u^t \tag{3.2}$$

where we have defined the *orbital angular velocity* as $\Omega \equiv d\phi/dt$.

It is possible for us to calculate u^t by using the condition $\mathbf{u} \cdot \mathbf{u} = -1$, but it turns out that it is not needed hence we shall not do so.

Before we combine the gyroscope equation (C.2) with this Schwarzschild metric, we should also take note that $s^\theta = 0$ initially. Considering the gyroscope equation, one can see that

$$\begin{aligned} \frac{ds^\theta}{d\tau} &= -\Gamma^\theta_{\mu\nu} s^\mu u^\nu \\ &= -\Gamma^\theta_{r\theta}\left(s^r u^\theta + s^\theta u^r\right) - \Gamma^\theta_{\phi\phi} s^\phi u^\phi \end{aligned} \tag{3.3}$$

where the second step has used the non-zero Christoffel symbols for the Schwarzschild metric, as given in Appendix A.1. Since $s^\theta = 0$ initially, and $u^\theta = 0$ due to the orbit being in the equatorial plane, equation (3.3) reduces to

$$\begin{aligned}\frac{ds^{\theta}}{d\tau} &= -\Gamma^{\theta}_{\phi\phi} s^{\phi} u^{\phi} \\ &= \cos(\theta)\sin(\theta) \\ &= 0\end{aligned} \tag{3.4}$$

as the $\cos(\theta)$ term vanishes when $\theta = \pi/2$. Hence s^{θ} starts and remains equal to zero throughout the orbit.

Suppose we consider an inertial frame in which the gyroscope is at rest; that is $\mathbf{u} = (1,0,0,0)$ and $\mathbf{s} = \left(0, s^r, s^{\theta}, s^{\phi}\right)$. Clearly we have $\mathbf{s}\cdot\mathbf{u} = 0$. In fact, it turns out that this condition holds in any frame. Moving back to a general frame, using this condition and $u^r = u^{\theta} = 0$ again we can see that

$$\begin{aligned} 0 &= g_{\mu\nu} s^{\mu} u^{\nu} \\ &= -\left(1 - \frac{2M}{R}\right) s^t u^t + R^2 \sin^2(\theta)\, s^{\phi} u^{\phi} \\ &= -\left(1 - \frac{2M}{R}\right) s^t u^t + R^2 s^{\phi} \Omega u^t \\ \Rightarrow s^t &= R^2 \Omega \left(1 - \frac{2M}{R}\right)^{-1} s^{\phi} \end{aligned} \tag{3.5}$$

We must now solve the gyroscope equation for s^r and s^{ϕ}, starting with the former. Using the appropriate non-zero Christoffel symbols from Appendix A.1 we see that

$$\begin{aligned} 0 &= \frac{ds^r}{d\tau} + \Gamma^r_{tt} s^t u^t + \Gamma^r_{\phi\phi} s^{\phi} u^{\phi} \\ &= \frac{ds^r}{d\tau} + \left(\frac{M}{R^2}\right)\left(1 - \frac{2M}{R}\right) s^t u^t - (R - 2M)\sin^2(\theta)\, s^{\phi} u^{\phi} \\ &= \frac{ds^r}{d\tau} + u^t \left[\left(\frac{M}{R^2}\right)\left(1 - \frac{2M}{R}\right) s^t - (R-2M)\, s^{\phi}\Omega\right] \\ &= \frac{ds^r}{d\tau} + u^t s^{\phi} \left[\left(\frac{M}{R^2}\right)\left(1 - \frac{2M}{R}\right) R^2\Omega \left(1 - \frac{2M}{R}\right)^{-1} - (R-2M)\,\Omega\right] \\ \Rightarrow & \frac{ds^r}{dt} - (R - 3M)\,\Omega s^{\phi} = 0 \end{aligned} \tag{3.6}$$

where we have used $u^t dt = (dt/d\tau)\, d\tau = dt$ in the last step. Using a similar method for s^{ϕ} we can show that

$$\frac{ds^{\phi}}{dt} + \frac{\Omega}{R} s^r = 0 \tag{3.7}$$

Hence we have two coupled equations. Let us differentiate (3.7) and substitute in (3.6) to obtain

$$\frac{d^2 s^\phi}{dt^2} + \left(1 - \frac{3M}{R}\right)\Omega^2 s^\phi = 0 \tag{3.8}$$

This is a second order differential equation describing the time evolution of s^ϕ. In fact, this is just the equation for a simple harmonic oscillator. Let us define the frequency of the solution as

$$\Omega' = \left(1 - \frac{3M}{R}\right)^{\frac{1}{2}} \Omega \tag{3.9}$$

Clearly something strange is happening. If the axes of the gyroscope were to return to their original orientation after one orbit, the solutions frequency would be the same as the orbital frequency, but it is not. The axes of the gyroscope are being affected by the curvature of spacetime and hence are precessing. Using Ω' we are able to write down the general solution of (3.8) as

$$s^\phi = A\sin\left(\Omega' t\right) + B\cos\left(\Omega' t\right) \tag{3.10}$$

where A and B are some constants. At time $t = 0$, the spin is pointing in the r-direction, hence we must have $B = 0$. In the same way that Hartle does, let us normalise the solution such that $(\mathbf{s}\cdot\mathbf{s})^{1/2} = s_*$. Clearly s_* is just the magnitude of the spin (which is also time-independent). Noting that when $t = 0$, we can see that the only spin component that has value is s^r. This normalisation condition must still be satisfied hence substituting our current solution for s^ϕ into equation (3.6) we obtain

$$\begin{aligned} 0 &= \frac{ds^r}{dt} - \Omega\left(R - 3M\right) A\sin\left(\Omega' t\right) \\ \Rightarrow s^r &= -\frac{\Omega}{\Omega'}\left(R - 3M\right) A\cos\left(\Omega' t\right) \\ &= -\frac{R - 3M}{\left(1 - 3M/R\right)^{1/2}} A\cos\left(\Omega' t\right) \end{aligned} \tag{3.11}$$

So when $t = 0$ we have

$$s^r\left(0\right) = -\frac{R - 3M}{\left(1 - 3M/R\right)^{1/2}} A \tag{3.12}$$

Now applying the normalisation condition we find

$$
\begin{aligned}
s_*^2 &= g_{rr} s^r s^r \\
&= -\left(1 - \frac{2M}{R}\right) \frac{(R-3M)^2}{1-3M/R} A^2 \\
\Rightarrow A &= -\left(1 - \frac{2M}{R}\right)^{\frac{1}{2}} \frac{(1-3M/R)^{1/2}}{R-3M} s_*
\end{aligned} \tag{3.13}
$$

Substituting this value for A back into our solutions for s^r and s^ϕ we can now see that

$$
s^r = s_* \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}} \cos(\Omega' t) \tag{3.14a}
$$

$$
s^\phi = -s_* \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}} \left(\frac{\Omega}{\Omega' R}\right) \sin(\Omega' t) \tag{3.14b}
$$

So our full set of equations are given by $s^\theta = 0$ and equations (3.5), (3.14a) and (3.14b).

Using the dot product definition, $\mathbf{A} \cdot \mathbf{B} = AB\cos(\theta)$, we can see that if we take the dot product of a unit vector at time $t = 0$ with a unit vector at time $t = 2\pi/\Omega$ (where $2\pi/\Omega$ is the time it takes for one complete orbit), we will obtain the cosine of the angle the spin has been rotated by. Hence

$$
\begin{aligned}
\hat{\mathbf{e}}_r \cdot \left(\frac{\mathbf{s}(2\pi/\Omega)}{s_*}\right) &= \cos\left(2\pi \frac{\Omega'}{\Omega}\right) \\
&= \cos\left(2\pi \left(1 - \frac{3M}{R}\right)^{\frac{1}{2}}\right)
\end{aligned} \tag{3.15}
$$

The difference between a full rotation of 2π and the value in the cosine term will give us the angle the spin has been rotated by per orbit where the rotation is in the same direction as the orbit. The angle is

$$
\Delta\phi = 2\pi \left[1 - \left(1 - \frac{3M}{R}\right)^{\frac{1}{2}}\right] \tag{3.16}
$$

Chapter 4

Lense Thirring Precession

The Lense-Thirring precession, named after Josef Lense and Hans Thirring [6] (citation is the English translation), is the effect that a rotating spherical mass has on the surrounding spacetime. There are in in fact two effects; the dragging of the orbital plane around the mass, and the precession of the axes of a gyroscope. The effects are often called frame-dragging due to the inertial frames around the mass being dragged by the rotation. We shall only consider the gyroscopic effect. The former effect causes spinning bodies to not follow geodesics as shall be discussed briefly later. The only difference between this effect and the geodetic effect is that the mass is rotating in this case, but the Lense-Thirring precession is orthogonal to the geodetic precession. As we are assuming a very small rotation velocity, we can derive the Lense-Thirring metric which describes the geometry around a slowly rotating spherical object. The metric is

$$ds^2 = ds^2_{\text{Schwarz}} - \frac{4GJ}{c^3 r^2} \sin^2(\theta)\,(rd\phi)\,(cdt) + \mathrm{O}\big(J^2\big) \tag{4.1}$$

By observation one can see that we have acquired cross terms in the metric. It is also straight forward to see that letting the angular momentum $J = 0$ we recover the Schwarzschild metric. This Lense-Thirring metric can be used along with the gyroscope equation (C.2) to analyse the precession of the axis of a gyroscope free falling down the rotation axis of the spinning mass. The metric has been given in polar coordinates, but we find that it is easier to work in regular Cartesian coordinates due to the singularity of the polar coordinates along the z-axis. Using the coordinate transformations that were given in equation (2.2), we can use the differential for $d\phi$ we derived in Sec. 2.3.1 and hence the Lense-Thirring metric in

Cartesian coordinates is

$$ds^2 = ds^2_{\text{Sch-Cart}} - \frac{4GJ}{c^3 r^2}(cdt)\left(\frac{xdy - ydx}{r}\right) + \mathrm{O}\big(J^2\big) \tag{4.2}$$

where $ds^2_{\text{Sch-Cart}}$ is the Schwarzschild metric in Cartesian coordinates. Clearly here we have a more complicated metric than in the case of a spherical mass.

It must be emphasised that we are working to first order in angular momentum, J, since the rotation is slow, so we can ignore the second order angular momentum affects on the shape of the mass. The quantity $4GJ/c^3r^2$, which is dimensionless as was shown in Sec. 2.2, is the ratio that governs the affects of the rotation. It can be shown using dimensional analysis that the amount of curvature is dependant on the objects (rotational) velocity in addition to its mass. We also find that the effects are of order c^{-3}.

If we replace the Schwarzschild part of the metric with the Minkowski metric, we will get a metric described by equation (A.2). There are only four non-vanishing Christoffel symbols for this metric. They are found in Appendix A.2. Using these symbols and noting that $r = z$ since the gyroscope is falling down the z-axis, we find that the gyroscope equation reduces down to two coupled equations

$$\frac{ds^x}{dt} + \frac{2GJ}{c^2 z^3} s^y = 0 \tag{4.3a}$$

$$\frac{ds^y}{dt} - \frac{2GJ}{c^2 z^3} s^y = 0 \tag{4.3b}$$

Combining these two equations gives us

$$\frac{d^2 s^x}{dt^2} + \frac{4G^2 J^2}{c^4 z^6} s^x = 0 \tag{4.4}$$

and the same equation results if we combined the two equations in the other way. Once again this is just an equation of simple harmonic oscillation. By looking at the frequency of the solution, we therefore find that the gyroscope axes precess in the same direction as the object is rotating with an instantaneous rate

$$\Omega_{LT} = \frac{2GJ}{c^2 z^3} \tag{4.5}$$

We revisit this solution in Chapter 5 where we derive a more general result that includes lateral positions.

Although the Lense-Thirring effect is minuscule for objects orbiting the Earth (42 mas/year), for black holes the effect is enormous. The methods used here cannot be used to describe the motions of black holes since the large angular momentum causes geometrical changes to the shape of the object that we cannot simply ignore. The object goes from being a spherical mass to an oblate spheroid (a sphere that has been compressed along its z-axis).

4.1 Light Bending Effects

We shall now consider one of the main implications of the Schwarzschild geometry which is also one of the great tests of relativity. Due to the body's effect on the surrounding spacetime, and also due to the fact that light follows geodesics, we can infer that a massive body must deflect light as it passes by the object. Although it is true that massive bodies follow geodesics, ones that are spinning do not follow geodesics, they follow very slightly modified paths (for example [7, 8]). Say we were considering the precession of Mercury's perihelion; the geodesic equation we use to describe the motion of Mercury is not completely true, but since planets have non-relativistic spin, these approximations describe their orbits to a very great degree of accuracy. In relativistic situations, for example if we had a spinning neutron star orbiting a massive black hole, the path that it follows could be very different from what the geodesic equation would describe.

4.1.1 Spherical Non-Rotating Mass

There are many routes we could take to derive the angle at which light is deflected by an object. The method described here shall derive the motion of the particles of light via the geodesic equation. An alternate method could involve the use of integrals of conserved quantities [1, 9]. Since we are also going to be calculating the extra deflection due to the effect of the rotation of a body, we shall only produce an approximate value for the deflection. A full derivation can be found using Jacobi elliptical functions [10].

We start with the Schwarzschild metric as found in Appendix A.1. Noting that $d\tau^2 = -ds^2$, we define a new variable $\epsilon = (d\tau/d\sigma)^2$ where σ is a parameter along a geodesic. This turns our metric into the Lagrangian

$$\epsilon = \left(1 - \frac{2M}{r}\right)\dot{t}^2 - \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2(\theta)\,\dot{\phi}^2 \tag{4.6}$$

where the dot denotes derivative with respect to σ. Since the Schwarzschild solution describes a spherically symmetric body, we shall orient the body in such a way that we have $\theta = \pi/2$. This reduces the equation down to

$$\epsilon = \left(1 - \frac{2M}{r}\right)\dot{t}^2 - \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 \tag{4.7}$$

From observation we can see that there is no ϕ dependence. We can therefore define the new constant variable $l = r^2\dot{\phi}$. This turns the final term in the equation into l^2/r^2. There is also no t dependence since the solution is static. Let us rescale σ in a way such that $(1 - 2M/r)\,\dot{t} = 1$. Putting both of these back into the equation and solving for $\dot{r}^2$ will give us

$$\dot{r}^2 = 1 - \epsilon\left(1 - \frac{2M}{r}\right) - \frac{l^2}{r^2}\left(1 - \frac{2M}{r}\right) \tag{4.8}$$

To simplify the equation further, we can recast the equation to get r as a function of ϕ. Once we have done this we will define a new variable $u = r^{-1}$.

$$\begin{aligned}\dot{r} &= \frac{dr}{d\sigma} = \frac{dr}{d\phi}\frac{d\phi}{d\sigma} \\ &= \frac{l}{r^2}\frac{dr}{d\phi} \\ &= -\frac{l}{u^2r^2}\frac{du}{d\phi} \\ &= -l\frac{du}{d\phi}\end{aligned} \tag{4.9}$$

Substituting this equation into (4.8) will give us

$$\left(\frac{du}{d\phi}\right)^2 = \frac{1-\epsilon}{l^2} + \frac{2M\epsilon}{l^2}u - u^2 + 2Mu^3 \tag{4.10}$$

It is from this step that we could solve the equation exactly using Jacobi elliptical functions. Instead we shall find an approximate solution. We start by differentiating through with respect to ϕ to get

$$2\left(\frac{du}{d\phi}\right)\left(\frac{d^2u}{d\phi^2}\right) = \frac{2M\epsilon}{l^2}\frac{du}{d\phi} - 2u\frac{du}{d\phi} + 6Mu^2\frac{du}{d\phi} \tag{4.11}$$

We are assuming that the light has a non-circular orbit (it should be noted that in fact light can have a circular orbit, but the orbit is unstable) hence r is non-constant so $(du/d\phi)$ is never zero. This allows us to divide through by $(du/d\phi)$ to get

$$\underbrace{\frac{d^2u}{d\phi^2} + u = \frac{M\epsilon}{l^2}}_{\text{Newtonian gravity}} + \underbrace{3Mu^2}_{\text{GR}} \tag{4.12}$$

This equation describes the approximate orbit of an object around a spherically symmetric body. The non-linear term is due to general relativity; the rest of the equation can be found through Newtonian gravity. Our definition of ϵ means that massive particles have $\epsilon > 0$ and for massless particles we have $\epsilon = 0$. Since light is made up of photons and photons are massless particles, we can remove the term with ϵ from the equation to get

$$\frac{d^2u}{d\phi^2} + u = 3Mu^2 \tag{4.13}$$

Let us also assume that the light passes by the object and at closest approach the radial distance (or the impact parameter) is b. This is illustrated in Figure 4.1. Due to the symmetry it is obvious that the light is going to be deflected by the same amount each side of the object.

We are going to seek a solution in the form $u = u_0 + u_1 + u_2 + \ldots$. We also assume that the non-linear term contributes significantly less than the linear term in the equation. The lowest order solution of the differential equation will therefore be

$$\frac{d^2u_0}{d\phi^2} + u_0 = 0 \tag{4.14}$$

which has the solution $u_0 = A\sin(\theta + \theta_0)$. This is the same as solving the entire equation in special relativity. We orient our axes in a way such that at the point of closest approach is in the plane $\phi = \pi/2$. This gives us our boundary condition that $r = b$ at $\phi = \pi/2$ and so we get $\theta_0 = 0$ and $A = 1/b$. Our next lowest order

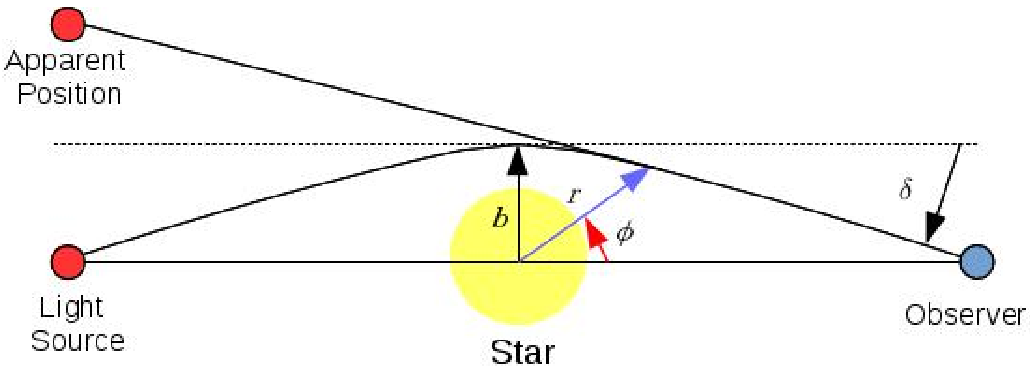

FIGURE 4.1: Light bending around a star

differential equation will be

$$\frac{d^2u_1}{d\phi^2} + u_1 = 3Mu_0^2 = \frac{3M}{b^2}\sin^2(\phi)$$
$$= \frac{3M}{2b^2}(1 - \cos(2\theta)) \tag{4.15}$$

We shall use a trial solution of the form

$$u_1 = \frac{3M}{2b^2} + B\cos(2\theta) \Rightarrow \frac{d^2u_1}{d\phi^2} = -4B\cos(2\theta) \tag{4.16}$$

Substituting this in and solving for we find that $B = M/2b^2$ hence our overall solution is

$$u \simeq \frac{1}{b}sin(\phi) + \frac{M}{2b^2}(3 + \cos(2\theta)) \tag{4.17}$$

Let δ be the angle by which the light is deflected from a straight line on one side of the object. We expect that $\delta \ll 1$. As the radial distance from the object increases, the gravitational effects of the object are going to decrease. Hence as $r \to \infty$ we have $u \to 0$ and $\phi \to -\delta$. Using the small angle formulas in Appendix C we see that

$$0 \simeq -\frac{\delta}{b} + \frac{M}{2b^2}(3 + 1)$$
$$\Rightarrow \delta \simeq \frac{2M}{b} \tag{4.18}$$

So the total bending angle of an object to lowest order and with factors of c and G restored is

$$\theta \simeq \frac{4GM}{rc^2} \tag{4.19}$$

In the case of light grazing the surface of the Sun, we take the values to be

$r = 6.966 \cdot 10^8$ m and $M = 1.99 \cdot 10^30$ kg. Inserting these values and converting the answer from radians to arcseconds, we find a value of approximately 1.75 arcseconds. The exact solution is also 1.75 arcseconds [10].

4.1.2 Spherical Slowly Rotating Mass

Since a rotating body alters the surrounding spacetime, relativity predicts that light passing by this body will be deflected by an amount additional to the amount calculated when considering simple Schwarzschild geometry.

Since the body is slowly rotating the frame dragging effects are extremely small. We are therefore expecting a result of lower order than the one we just calculated. As most of the steps are the same, only the important parts shall be written down.

The metric we are using is described in Appendix A.2. The result we are calculating is to lowest order in c hence we note that the GM/rc^2 terms will not contribute. Therefore, our Lagrangian (using geometrised units) for the system takes the form

$$\epsilon = \dot{t}^2 - \dot{r}^2 - r^2\dot{\theta}^2 - r^2 \sin^2(\theta)\,\dot{\phi} + \frac{4J}{r}\dot{\phi}\dot{t} \tag{4.20}$$

We orient the axes so that we have $\theta = \pi/2$. Our rescaling is different for this metric. We rescale the conserved quantities. If we let $\mathcal{L} = \epsilon/2$ (the factor of 2 being there for convenience, it is possible to do this without redefining in terms of $\mathcal{L}$) then we see that $\partial\mathcal{L}/\partial\dot{t}$ and $\partial\mathcal{L}/\partial\dot{\phi}$ are non-zero. Using these two equations allow us to define again a constant l and rescale σ such that

$$\frac{2J}{r}\dot{t} - r^2\dot{\phi} = l, \qquad \dot{t} + \frac{2J}{r}\dot{\phi} = 1 \tag{4.21}$$

This is our justification for why we scaled the factors in the way we did in Sec. 4.1.1, except in that case we didn't have to solve any simultaneous equations. Solving these two equations to lowest order in J gives us

$$\dot{\phi} = \frac{2J}{r^3} - \frac{l}{r^2}, \qquad \dot{t} = 1 + \frac{2Jl}{r^3} \tag{4.22}$$

Substituting these values into equation (4.20) and solving for $\dot{r}^2$ we find that to lowest order in J

$$\dot{r}^2 = 1 - \epsilon - \frac{l^2}{r^2} - \frac{4Jl}{r^3} \tag{4.23}$$

Once again we reformulate r in terms of ϕ and substitute in $u = r^{-1}$. In this case we get the differential equation

$$\left(\frac{du}{d\phi}\right)^2 = \frac{1-\epsilon}{l^2} - u^2 - \frac{4J}{l}u^3 \tag{4.24}$$

Differentiating and cancelling common factors gives us the second order equation

$$\frac{d^2u}{d\phi^2} + u = -\frac{6J}{l}u^2 \tag{4.25}$$

Solving this in the same way as before with our series solution method we find

$$u = \frac{1}{b}\sin(\phi) - \frac{J}{lb^2}(3 + \cos(2\phi)) \tag{4.26}$$

and hence we find that the total bending angle is

$$\theta = -\frac{8J}{lb} \tag{4.27}$$

The two main differences between this value of bending angle and the value that we found in equation (4.19) are the presence of the l constant and the minus symbol. The rotation of the object in fact causes the light to be bent by an slightly less than if the object were not rotating. We cannot compare the two values properly until we explicitly calculate l.

We find l by considering equation (4.23). For light $\epsilon = 0$ so we can immediately remove this term. We also have $\dot{r} = 0$ when $r = b$ since this is the turning point for light (due to the symmetry of the object). Hence we can rearrange the equation to get

$$l^2 + \frac{4J}{b}l - b^2 = 0 \tag{4.28}$$

Solving this by means of completing the square and then binomially expanding we find

$$\begin{aligned} l &= -\frac{2J}{b} \pm \left(b^2 + \frac{4J}{b^2}\right)^{\frac{1}{2}} \\ &\simeq -\frac{2J}{b} \pm \left(b + \frac{2J}{b^4}\right) \\ &\simeq b \end{aligned} \tag{4.29}$$

We can justify the last line by noticing that $|b| \gg |2J/b| \gg |2J/b^4|$ which becomes even more apparent when the factors of G and c are restored. The square root is taken to be positive for light moving past the object in the same direction as its rotation; we take the negative square root for light moving past the object in the opposite direction to its rotation. This implies that for light passing by in the rotational direction the light is bent by a smaller angle that in the case of no rotation. Substituting this back into equation (4.27) and restoring the constants we find that

$$\theta \simeq -\frac{8GJ}{r^2c^3} \tag{4.30}$$

Comparing the ratio of the magnitudes of equations (4.30) and (4.19), and using a standard Newtonian description of angular momentum

$$J \sim I\Omega \sim MR^2\Omega \sim MRV \tag{4.31}$$

it is straightforward to see

$$\left|\frac{8GJ}{r^2c^3}\right| / \left|\frac{4GM}{rc^2}\right| \sim \frac{J}{Mrc} \sim \frac{v}{c} \ll 1 \tag{4.32}$$

Hence the rotational effect is minute in comparison with the non-rotating spherical body effect. For the Sun $v \simeq 2000$ m s^{-1} which is of many orders lower than c.

In the case of light passing by in the opposite direction to the objects rotation, we simply take the negative value of equation (4.30). In this case the light is therefore bent by a greater angle than in the case of no rotation.

We know from the original calculation that M/r is small (when r is the distance of closest approach); it is half the Schwarzschild radius which for the Sun is around three kilometres; its physical radius is comparison is just shy of seven hundred thousand kilometres. This shows an obvious problem trying to do calculations as such when working with black holes. With stars, the Schwarzschild radius is significantly smaller than the physical radius and so terms like M/r can be ignored, but with black holes the physical radius is the Schwarzschild radius hence the terms are of the same order and cannot be ignored.

Chapter 5

Gravitoelectromagnetism

Historically there has long been speculation of similarities between Newtonian gravity and Coulomb's law of electricity. This led to a theory which was a gravito-electric description of Newtonian gravity. The field of electrodynamics progressed vastly during the nineteenth century, mainly due to the Maxwell equations being published. Once it was realised that electric and magnetic fields are intrinsically related, speculation was made as to an extra (magnetic) force acting in Newtonian gravity. This force would be the corrections to observed quantities that we see; for example the precession of Mercury's perihelion which cannot be correctly predicted by Newtonian gravity. Einstein's theory of relativity provided this correction to Mercury's perihelion precession leaving any Newtonian magnetic forces redundant as the correction was shown to be completely relativistic.

In a weak gravitational field where the velocities are small, we can decompose spacetime in what's called a "3+1 split"; that is we can split the four dimensional metric into a scalar time-time component, a vector time-space component and a tensor space-space component. In our case of the weak field we call the scalar component the gravitoelectic potential, and the vector component the gravito-magnetic potential. We ignore the space-space tensorial components as they are of negligible order. The combination of these is known as gravitoelectromagnetism (which shall henceforth be known as GEM). The individual components describe fields analogous to the electric and magnetic fields in electromagnetism.

Due to the similar nature of GEM and electromagnetism, we are able to determine results in relativity by looking at the paralleled results in electromagnetism, although we may need to rescale some results as they are often out by a factor.

This is due to the fact that approximations in classical electromagnetism involve a spin-1 field, whilst in linearised gravity we have a spin-2 field (this is only true for linearised gravity [11]). From the GEM field equations we are able to derive the Lense-Thirring metric as used in Chapter 4, and also the Schiff equation [12].

The subject of GEM is not free of controversy and whether or not the gravitoelectric and gravitomagnetic fields exist is currently under debate. We shall not be discussing this in this article, we shall simply assume that the fields do exist.

5.1 Gravitoelectromagnetic Field Equations

We start by noting that this solution is only valid in a weak field approximation. Since we are in a weak field, we are able to take a linear perturbation of the Minkowski metric such that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $|h_{\mu\nu}| \ll 1$. We shall work to first order in $h_{\mu\nu}$ throughout. The inverse metric is obviously $g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$ but we can simply ignore the $h^{\mu\nu}$ terms as this will give us non-linear terms. Therefore raising and lowering of indices is performed by the Minkowski metric. Our Christoffel symbols (2.7) are therefore

$$\begin{aligned}\Gamma^{\alpha}_{\mu\nu} &= \frac{1}{2}\eta^{\alpha\sigma}\left(h_{\mu\sigma,\nu} + h_{\sigma\nu,\mu} - h_{\mu\nu,\sigma}\right) \\ &= \frac{1}{2}\left(h_{\mu}{}^{\alpha}{}_{,\nu} + h^{\alpha}{}_{\nu,\mu} - h_{\mu\nu}{}^{,\alpha}\right)\end{aligned} \tag{5.1}$$

Next we must calculate the relevant Ricci tensors. Defining the trace of the Minkowski metric as $h = \eta^{\mu\nu}h_{\mu\nu} = h^{\nu}{}_{\nu}$ we find

$$\begin{aligned}R_{\mu\nu} &= \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} \\ &= \frac{1}{2}\left(h_{\mu}{}^{\alpha}{}_{,\nu\alpha} + h_{\nu}{}^{\alpha}{}_{,\mu\alpha} - h_{\mu\nu,\alpha}{}^{\alpha} - h_{,\mu\nu}\right)\end{aligned} \tag{5.2}$$

And contracting this with the Minkowski inverse metric we find the trace of the Riemann tensor is

$$R = h^{\mu\alpha}{}_{,\mu\alpha} - h_{,\mu}{}^{\mu} \tag{5.3}$$

Substituting all these values into the Einstein field equations (C.5) and relabelling some indices we obtain

$$\begin{aligned} h_{\mu}{}^{\alpha}{}_{,\nu\alpha} + h_{\nu}{}^{\alpha}{}_{,\mu\alpha} - h_{\mu\nu,\alpha}{}^{\alpha} - h_{,\mu\nu} \\ - \eta_{\mu\nu}\left(h^{\alpha\beta}{}_{,\alpha\beta} - h_{,\beta}{}^{\beta}\right) = \frac{16\pi G}{c^4} T_{\mu\nu} \end{aligned} \tag{5.4}$$

Next we define the *trace-reversed perturbation variable*, $\overline{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$. Noting that $\overline{h} = -h$ (hence the apt nomenclature) we find that

$$h_{\mu\nu} = \overline{h}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}h \tag{5.5}$$

$$= \overline{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\overline{h} \tag{5.6}$$

Using this result we can calculate each term in (5.4) and obtain

$$h_{\mu\alpha,\nu}{}^{\alpha} = \left(\overline{h}_{\mu\alpha} - \frac{1}{2}\eta_{\mu\alpha}\overline{h}\right)_{,\nu}{}^{\alpha}$$

$$= \overline{h}_{\mu\alpha,\nu}{}^{\alpha} - \frac{1}{2}\overline{h}_{,\nu\mu} \tag{5.7a}$$

$$h_{\nu\alpha,\mu}{}^{\alpha} = \overline{h}_{\nu\alpha,\mu}{}^{\alpha} - \frac{1}{2}\overline{h}_{,\mu\nu} \tag{5.7b}$$

$$h_{\mu\nu,\alpha}{}^{\alpha} = \overline{h}_{\mu\nu,\alpha}{}^{\alpha} - \frac{1}{2}\eta_{\mu\nu}\overline{h}_{,\alpha}{}^{\alpha} \tag{5.7c}$$

$$h_{,\mu\nu} = -\overline{h}_{,\mu\nu} \tag{5.7d}$$

$$h_{\alpha\beta}{}^{,\alpha\beta} = \overline{h}_{\alpha\beta,}{}^{\alpha\beta} - \frac{1}{2}\overline{h}_{,\beta}{}^{\beta} \tag{5.7e}$$

$$h_{,\beta}{}^{\beta} = -\overline{h}_{,\beta}{}^{\beta} \tag{5.7f}$$

where we have used $h_{\mu}{}^{\alpha}{}_{,\nu\alpha} = h_{\mu\alpha,\nu}{}^{\alpha}$ (simply so we don't have to raise or lower any indices in the $\overline{h}$ terms). This turns the field equations (5.4) into

$$-\overline{h}_{\mu\nu,\alpha}{}^{\alpha} + \overline{h}_{\mu\alpha,}{}^{\alpha}{}_{\nu} + \overline{h}_{\nu\alpha,}{}^{\alpha}{}_{\mu} - \eta_{\mu\nu}\overline{h}_{\alpha\beta}{}^{,\alpha\beta} = \frac{16\pi G}{c^4}T_{\mu\nu} \tag{5.8}$$

We recognise the first term as the d'Alembertian operator in relativistic notation ($\Box = \partial^{\mu}\partial_{\mu}$); the other three terms simply keep the equations gauge-invariant. If we apply the Lorentz gauge condition $\overline{h}^{\mu\alpha}{}_{,\alpha} = 0$, it is immediately obvious that the last three terms vanish (after raising some indices). This allows us to write the linearised Einstein equations to first order in $\overline{h}_{\mu\nu}$ as

$$\overline{h}_{\mu\nu,\alpha}{}^{\alpha} = \Box\overline{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu} \tag{5.9}$$

This is equation only valid if the Lorentz gauge exists. The proof of existence can be found in Appendix B. As a side note we can see the striking resemblance with electromagnetism where we find the Maxwell equations are $\Box A_\mu = -(4\pi/c)\, J_\mu$. The two important equations describing linearised gravity are the field equations (5.9), and the metric

$$g_{\mu\nu} = \eta_{\mu\nu} + \overline{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\overline{h} \tag{5.10}$$

For our calculations we shall now raise the indices of the metric and the linearised field equations. The reason for why is we use the conservation of the stress-energy tensor with upper indices. It can be shown that the exact solution to this equation can be given in terms of retarded potentials of the stress-energy tensor via the use of Green's function (for example see Sec. 7.5 of [13]). It is given by

$$\overline{h}^{\mu\nu} = \frac{4G}{c^4}\int \frac{T^{\mu\nu}\left(t - \left|\mathbf{x}-\mathbf{x}'\right|/c, \mathbf{x}'\right)}{\left|\mathbf{x}-\mathbf{x}'\right|} d^3x' \tag{5.11}$$

To evaluate the integral in equation (5.11), as follows from [14], we shall first use a Taylor-series expansion for $1/\left|\mathbf{x}-\mathbf{x}'\right|$ about $\mathbf{x}' = 0$. This gives

$$\begin{aligned}\frac{1}{\left|\mathbf{x}-\mathbf{x}'\right|} &= \frac{1}{\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{1/2}} \\ &= \frac{1}{r} + \frac{x'^k x^k}{r^3} + \frac{1}{2}\left(3x'^k x'^l - r'^2\delta^l_k\right)\frac{x^k x^l}{r^5} + \cdots \end{aligned} \tag{5.12}$$

where $r = \left(x^2+y^2+z^2\right)^{1/2}$ as usual. We shall keep only the first two terms for our calculation. We also ignore the time completely in the integral since the solution is static, that is we shall consider $T^{\mu\nu}\left(t - \left|\mathbf{x}-\mathbf{x}'\right|/c, \mathbf{x}'\right) = T^{\mu\nu}(\mathbf{x}')$. This means we must integrate the equation

$$\overline{h}^{\mu\nu} = \frac{4G}{c^4 r}\int T^{\mu\nu}(\mathbf{x}')\, d^3x' + \frac{4Gx^k}{c^4 r^3}\int x'^k T^{\mu\nu}(\mathbf{x}')\, d^3x' \tag{5.13}$$

We shall work in geometrised coordinates for the integral calculations and restore the factors of G and c afterwards. Considering the first integral,

$$\int T^{\mu\nu}(\mathbf{x}')\, d^3x' = \int T^{00}(\mathbf{x}')\, d^3x' + \int T^{i\mu}(\mathbf{x}')\, d^3x' \tag{5.14}$$

it is straight forward to see that the first term is simply M. This is because T^{00} is defined as the energy density ($T^{00} = c^2\rho$), hence we are integrating the energy density over the volume of the system, which is by definition the mass of the system M. For the next part of the integral, we start with the stress-energy tensor conservation law. In linearised gravity the covariant derivative is equal to the normal derivative (as we are only retaining the lowest order and we already have small $T^{\mu\nu}$ anyway). The conservation law thus reduces to

$$\begin{aligned} 0 = T^{\mu\nu}{}_{;\nu} &= T^{\mu\nu}{}_{,\nu} \\ &= T^{\mu 0}{}_{,0} + T^{\mu j}{}_{,j} \\ &= T^{\mu j}{}_{,j} \end{aligned} \tag{5.15}$$

where $T^{\mu\nu}{}_{;\nu} = T^{\mu\nu}{}_{;\nu}(\mathbf{x}')$ and we have used the fact that we have no time dependence in the last line. Multiplying this last line by x'^k and integrating over the volume of the system we have

$$0 = \int x'^k T^{\mu j}{}_{,j} d^3x' \tag{5.16}$$

To evaluate this we must integrate by parts. Starting with the total derivative integral we see

$$\begin{aligned} \int \left(x'^k T^{\mu j}\right)_{,j} d^3x' &= \int \delta^k_j T^{\mu j} d^3x' + \int x'^k T^{\mu j}{}_{,j} d^3x' \\ &= \int T^{\mu k} d^3x' + \int x'^k T^{\mu j}{}_{,j} d^3x' \end{aligned} \tag{5.17}$$

Since the left hand side of the equation is a total derivative, this integrates to be zero. Therefore using equations (5.16) and (5.17) we find

$$\int T^{\mu k} d^3x' = 0 \tag{5.18}$$

Hence the first integral term of (5.13) only has value for $\overline{h}^{00}$.

Using the same method as before, we split the second integral in equation (5.13) into

$$\begin{aligned} \int x'^k T^{\mu\nu}(\mathbf{x}') d^3x' = \int x'^k T^{00}(\mathbf{x}') d^3x' &+ \int x'^k T^{0j}(\mathbf{x}') d^3x' \\ &+ \int x'^k T^{ij}(\mathbf{x}') d^3x' \end{aligned} \tag{5.19}$$

The first integral term in this equation is zero since it is simply stating that the coordinate origin is centred at the centre of the mass. An alternate way of looking at this is the integral will produce a rank one tensor, but it is not possible for a rank one tensor to exist in this system, hence it must be equal to zero. For the second term, we start by multiplying equation (5.15) by $x'^k x'^n$ and then integrate over the volume. Using the same integration by parts method we find the identity

$$\begin{aligned} 0 &= \int x'^k x'^n T^{\mu j}{}_{,j} \\ &= -\int x'^n T^{\mu k} d^3x' - \int x'^k T^{\mu n} d^3x' \end{aligned} \tag{5.20}$$

In relativity, angular momentum is defined as the integral of the cross product of the position vector, x^k with the momentum density, T^{i0}. That is,

$$J^n = \int \epsilon^{nki} x'^k T^{i0} \tag{5.21}$$

so for example, the x-component of spin angular momentum is defined as (and similarly for the other components)

$$J^1 = \int \left(x'^2 T^{30} - x'^3 T^{20} \right) d^3x' \tag{5.22}$$

but equation (5.20) reduces this equation to

$$J^1 = 2 \int x'^2 T^{30} d^3x' \tag{5.23}$$

or in general

$$J^n = 2 \int \epsilon^{nki} x'^k T^{i0} d^3x' \tag{5.24}$$

hence to obtain our integral, we must multiply through by $1/2\epsilon^{nki}$. Putting all of this together shows that the second integral of equation (5.19) becomes

$$\int x'^k T^{0j} \left(\mathbf{x}' \right) d^3x' = \frac{1}{2} \epsilon^{kin} J^n \tag{5.25}$$

For the third integral term in equation (5.19), we use the identity

$$T^{ij} = \frac{1}{2} \frac{\partial}{\partial x'^n} \left(x'^j T^{in} + x'^i T^{jn} \right) \tag{5.26}$$

Substituting this into the integral and integrating by parts will give us a term that completely vanishes as a consequence of equation (5.20). Once again it is immediately apparent without the above identity that this term vanishes as it will produce a rank three tensor which is not possible in this system. This is because the stress-energy tensor is symmetric and the natural rank three tensor is the alternating tensor which is anti-symmetric (a similar argument is used in the case of the rank one tensor).

Putting all of these integrals back into equation (5.13) and restoring factors of c and G (for just this set of equations) we find

$$\overline{h}^{00} = \frac{4GM}{c^2 r} \tag{5.27a}$$

$$\overline{h}^{i0} = \overline{h}^{0i} = \frac{2G}{c^3 r^3}\epsilon^{kin} x^k J^n \tag{5.27b}$$

$$\overline{h}^{ij} = 0 \tag{5.27c}$$

There are no space-space $\overline{h}^{ij}$ components as they were omitted in our approximation. If we had used more terms in the Taylor series expansion (5.12) then we would have had these non-zero. Their order makes the effects of them negligible.

Considering first the case of no rotation, we have only $\overline{h}_{00}$ to be non-zero. If we define the *Newtonian gravitational potential* (in this case also the gravitoelectric scalar potential which is essentially the time-time part of the metric), Φ, to be defined by

$$\Phi = -\frac{GM}{r} \tag{5.28}$$

We can immediately see that upon lowering indices $\overline{h}_{00} = -4\Phi$ in the Newtonian limit ($\Phi \to 0$ as $r \to \infty$). Also noting that $\overline{h} = -\overline{h}_{00}$, we can thus determine from the definition of the trace linear perturbation that

$$h_{\mu\nu} = \overline{h}_{\mu\nu} - \frac{1}{2}\overline{h}\eta_{\mu\nu} = \begin{cases} -2\Phi & \text{if } \mu = \nu = 0 \\ -2\Phi & \text{if } \mu = \nu \neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{5.29}$$

We can therefore write the approximate metric for a non-rotating spherical body (the approximate Schwarzschild metric), by making note of the definition of the metric $g_{\mu\nu}$ from equation (5.10), as

$$ds^2 = -(1+2\Phi)\,dt^2 + (1-2\Phi)\left(dx^2 + dy^2 + dz^2\right) \tag{5.30}$$

Now let us consider the rotational effects on the metric. These are the contributions from the non-diagonal terms. The trace linear perturbation for this case is simply $h_{i0} = \overline{h}_{i0}$ as the Minkowski metric is diagonal. If we define the gravitomagnetic vector potential (which is essentially the time-space part of the metric),

$$\mathbf{A} = \frac{G}{c}\frac{\mathbf{J} \times \mathbf{x}}{r^3} \tag{5.31}$$

where $r = |\mathbf{x}|$ then we immediately see, using the anti-symmetric property of the cross product, that equation (5.27b) becomes

$$h_{i0} = h_{0i} = -\frac{2}{c}\mathbf{A} \tag{5.32}$$

once we have lowered the time index which causes a change in sign due to our metric signature. In fact, $h_{i0} = g_{i0}$ due to the Minkowski metric again being diagonal. All we need to do is compute the dot product of this with $(d\mathbf{x}\ dt)$ and we find our non-diagonal metric terms. Although in general $dx^\mu \otimes dx^\nu \neq dx^\nu \otimes dx^\mu$, in this case it is true for obvious reasons. Inserting this rotational component in the non-rotating metric with all factors of c restored we obtain

$$ds^2 = -c^2\left(1 + \frac{2\Phi}{c^2}\right)dt^2 - \frac{4}{c}\left(\mathbf{A} \cdot d\mathbf{x}\right)dt + \left(1 - \frac{2\Phi}{c^2}\right)\delta_{ij}dx^i dx^j \tag{5.33}$$

The split of the metric into a time-time scalar, a time-space vector and a space-space vector (which has negligible order) is the relativity analogue of the decomposition of an electromagnetic four-vector potential into an electric scalar potential and a magnetic vector potential. If we hadn't defined this gravitomagnetic potential, using equation (5.27b) it is possible to derive the Lense-Thirring metric straight away. We derive the Lense-Thirring metric in Sec. 5.3. The reason we omit this for now is because in most articles on GEM they do not explicitly solve the linearised field equations. They write down the retarded time solution and define the potentials (hence they would not derive the spin terms), afterwards simply writing down the linearised metric for a slowly rotating body. Although we could have gone immediately from our cross term definitions to the metric, it therefore makes more sense to start from the metric and work backwards.

Using the Lorentz gauge condition we find that when we substitute in our metric values we shall obtain

$$\begin{aligned} 0 = \overline{h}^{\mu\nu}{}_{,\nu} &= \overline{h}^{00}{}_{,0} + \overline{h}^{0j}{}_{,j} \\ &= \frac{\partial}{\partial t}\left(\frac{4\Phi}{c^2}\right) + \frac{\partial}{\partial x^j}\left(\frac{2A^j}{c}\right) \\ &= \frac{1}{c}\frac{\partial \Phi}{\partial t} + \frac{1}{2}\nabla \cdot \mathbf{A} \end{aligned} \tag{5.34}$$

In the same manner as we would in electromagnetism, we define the time independent GEM fields by the equations

$$\mathbf{g} = -\nabla\Phi, \qquad \mathbf{H} = \nabla \times \mathbf{A} \tag{5.35}$$

The reader should note that in some papers the letter **B** is used instead of **H**. We use **H** in following with [15] and also to emphasise the difference between the gravitomagnetic field and the magnetic field. The letter **g** is used instead of the electromagnetic equivalent **E** once again to emphasise the difference, but mainly because **g** is simply the Newtonian gravitational force.

For the time-time component of the linearised field equations (5.9), using the definition of **g**, $\overline{h}_{00} = -4\Phi$ and $T_{00} = \rho$, we find

$$\begin{aligned} \nabla^2 \overline{h}_{00} = -16\pi\rho &\Leftrightarrow \nabla^2\Phi = 4\pi\rho \\ &\Leftrightarrow \nabla \cdot \mathbf{g} = -4\pi\rho \end{aligned} \tag{5.36}$$

The time-space components imply

$$\begin{aligned} \nabla^2 \overline{h}_{0i} = -16\pi T_{0i} &\Leftrightarrow \nabla^2 A_i = -16\pi j_i \\ &\Leftrightarrow \nabla \times \left(\frac{1}{2}\mathbf{H}\right) = -4\pi\mathbf{j} \end{aligned} \tag{5.37}$$

where we have used identity (C.6) in the last line. **j** is the current-mass density defined as $\mathbf{j} = \rho\mathbf{v}$. The final two Maxwell-like equations arise directly from taking the dot and cross products of **H** and **g** respectively. We thus arrive at the time-independent, first order GEM field equations (with constant factors restored):

$$\nabla \times \mathbf{g} = 0 \tag{5.38a}$$

$$\nabla \cdot \left(\frac{1}{2}\mathbf{H}\right) = 0 \tag{5.38b}$$

$$\nabla \cdot \mathbf{g} = -4\pi G\rho \tag{5.38c}$$

$$\nabla \times \left(\frac{1}{2}\mathbf{H}\right) = -\frac{4\pi G}{c}\mathbf{j} \tag{5.38d}$$

These equations are consistent with [16] removing the time dependence. There are a few important differences between our GEM field equations and the Maxwell equations. The presence of the factor $1/2$ is once again due to the spin-2 field. The obvious presence of the minus symbols simply indicate that gravity is attractive, rather than repulsive. If the equations weren't limited to first order (and also not time independent), equation (5.38a) would have $-(1/c)(\partial\mathbf{H}/\partial t)$ on the right hand side.

Using very similar methods, one can find the time-dependent Maxwell-like field equations [16, 17]. We are not interested in these as all of our work assumes no time dependence.

To complete our GEM theory we must have some kind of equation that is the Lorentz force equivalent in electromagnetism. We compute this by starting with the geodesic equation. Non-relativistic motion implies that $dx^0/ds \simeq 1$ and so the velocity of the particle is $v^i/c \simeq dx^i/ds$. We shall ignore terms of order c^{-2} and apply the time independence condition. Once we have calculated the relevant Christoffel symbols we can produce an equation that turns out to be the same as the electromagnetic Lorentz force, except for replacing q with m [17] (the proof of which has been omitted).

$$\mathbf{F} = m\mathbf{g} + m\mathbf{v} \times \mathbf{H} \tag{5.39}$$

A similar method is used in [16], except a Lagrangian is used to produce a slightly modified Lorentz force equation. A gravitational version of Larmor's theorem [18] is then used to calculate the Lense-Thirring precession.

Before we proceed, we can note some properties about these GEM fields. In electromagnetism the electric field is a monopole radial field, and the magnetic field is a dipolar field. In the analogy we have been using, we therefore find that the gravitoelectric field is a monopole (inwards) radial field and the gravitomagnetic field is a dipolar field. We can also infer that rotating gravitoelectric fields induce gravitomagnetic fields that have a strength proportional to the angular momentum.

5.2 Lense-Thirring Precession

We now consider once again the Lense-Thirring precession, but now from a more general viewpoint considering the mathematics in this chapter. To understand the mathematics in this section, we must briefly re-visit electromagnetism. The Lense-Thirring precessions is the most commonly known gravitomagnetic effect, and lots of research has been done on it. For a gyroscope it is what's known as spin-spin coupling since both the gyroscope and the object it is orbiting are rotating. The precession in GEM is the analogue of the precession of the angular momentum of a charged test particle orbiting around some magnetic dipole in electromagnetism.

In electromagnetism, the magnetic field applies a torque on the magnetic dipole moment in the situation as described above. This torque is given by $\mathbf{\Gamma} = \boldsymbol{\mu} \times \mathbf{B}$ and causes the angular momentum of the particle to precess. The precession amount is given by the Larmor frequency $\omega = -\gamma B$ where $\boldsymbol{\mu} = \gamma \mathbf{J}$. The GEM equivalent of this is given by $\boldsymbol{\mu} \to \mathbf{S}/2c$ (because $\boldsymbol{\mu} = \frac{1}{2} \int \mathbf{x} \times \mathbf{j} d^3x$ whereas $\mathbf{S} = c \int \mathbf{x} \times \mathbf{j} d^3x$ where $T^{i0} = cj^i$) and $\mathbf{B} \to \mathbf{H}$. Hence the torque on the angular momentum of the gyroscope, $\mathbf{S}$, is

$$\tau = \frac{1}{2c} \mathbf{S} \times \mathbf{H} \tag{5.40}$$

Torque is defined as the first derivative of angular momentum with respect to time. Hence

$$\begin{aligned} \frac{d\mathbf{S}}{dt} &= \frac{\mathbf{S}}{2c} \times \mathbf{H} \\ &= \mathbf{\Omega}_{LT} \times \mathbf{S} \end{aligned} \tag{5.41}$$

where we have defined the precession $\mathbf{\Omega}_{LT} = -\mathbf{H}/2c$. All that we need to do is calculate the gravitomagnetic field. Noting our definition of $\mathbf{H}$ in equation (5.35), we find that

$$\begin{aligned}\mathbf{H} = \nabla \times \mathbf{A} &= -\frac{G}{c}\nabla \times \left(\frac{\mathbf{J} \times \mathbf{x}}{r^3}\right) \\ &= -\frac{G}{c}\left[\mathbf{J}\left(\nabla \cdot \frac{\mathbf{x}}{r^3}\right) - (\mathbf{J} \cdot \nabla)\frac{\mathbf{x}}{r^3}\right]\end{aligned} \tag{5.42}$$

and by using some simple vector analysis we obtain

$$\begin{aligned}\nabla \cdot \frac{\mathbf{x}}{r^3} &= \frac{\partial}{\partial x^i}\left(\frac{x^i}{r^3}\right) \\ &= \frac{3}{r^3} - \frac{3}{2}\frac{2x^i x^i}{r^5} = 0\end{aligned} \tag{5.43}$$

$$\begin{aligned}\left[(\mathbf{J} \cdot \nabla)\frac{\mathbf{x}}{r^3}\right]_i &= J^j \frac{\partial}{\partial x^i}\left(\frac{x^j}{r^3}\right) \\ &= J^j\left(\frac{\delta_{ij}}{r^3} - \frac{3}{2}\frac{2x^i x^j}{r^5}\right) \\ &= \frac{J^i}{r^3} - \frac{3\left(\mathbf{J} \cdot \mathbf{x}\right) x^i}{r^5} \\ \Rightarrow (\mathbf{J} \cdot \nabla)\frac{\mathbf{x}}{r^3} &= \frac{1}{r^3}\left(-\frac{3}{r^2}\left(\mathbf{J} \cdot \mathbf{x}\right)\mathbf{x} + \mathbf{J}\right)\end{aligned} \tag{5.44}$$

Substituting these values into equation (5.42) gives us the equation describing the magnetic field in electromagnetism. Due to the presence of the factor 1/2 in the Maxwell-like equations (5.38), it makes sense to multiply our result by 2 to find the GEM analogue of the magnetic field equation, the gravitomagnetic field equation. Our gravitomagnetic field is therefore

$$\mathbf{H} = -\frac{2G}{r^3 c}\left(\frac{3}{r^3}\left(\mathbf{J} \cdot \mathbf{x}\right)\mathbf{x} + \mathbf{J}\right) \tag{5.45}$$

Using our definition of $\mathbf{\Omega}_{LT}$, we arrive at the conclusion that the Lense-Thirring precession of the gyroscope obeys the equation

$$\mathbf{\Omega}_{LT} = \frac{G}{r^3 c^2}\left[\frac{3\left(\mathbf{J} \cdot \mathbf{x}\right)\mathbf{x}}{r^2} - \mathbf{J}\right] \tag{5.46}$$

The positive sign confirms our earlier statement that the precession is in the rotational direction of the object.

5.3 Lense-Thirring Metric

In Sec. 5.1 we derived the general metric of linearised gravity for a slowly rotating body. This was

$$ds^2 = -c^2\left(1+\frac{2\Phi}{c^2}\right)dt^2 - \frac{4}{c}\left(\mathbf{A}\cdot d\mathbf{x}\right)dt + \left(1-\frac{2\Phi}{c^2}\right)\delta_{ij}dx^i dx^j \tag{5.47}$$

Clearly, when we set the angular momentum, $\mathbf{J}$, to be equal to zero, we recover the linearised gravity solution for the Schwarzschild metric.

First let us consider the cross term in the metric. In our discussion of the Lense-Thirring precession in Chapter 4, we assumed the object to be rotating slowly, and we oriented the axes in a manner such that the spin was about the z-axis, or $\mathbf{J} = (0,0,J^z) = J^z\hat{\mathbf{z}}$. This means that we only need the $\hat{\mathbf{z}}$ component of $\mathbf{r}\times d\mathbf{r} = x\ dy - y\ dx$ which we identified in Sec. 2.3.1 to be $r^2\sin^2(\theta)$. So substituting our definition of the gravitomagnetic potential, $\mathbf{A}$, into the above metric, the cross term becomes

$$\begin{aligned}
\left(\mathbf{A}\cdot d\mathbf{x}\right)dt &= \frac{G}{c}\left(\frac{\mathbf{J}\times\mathbf{x}}{r^3}\cdot d\mathbf{x}\right)dt \\
&= \frac{G}{cr^3}\left(\mathbf{J}\cdot\mathbf{x}\times d\mathbf{x}\right)dt \\
&= \frac{G}{cr^3}\left(J^z\hat{\mathbf{z}}\cdot\left(x\ dy - y\ dx\right)\hat{\mathbf{z}}\right)dt \\
&= \frac{G}{cr}\left(J^z\sin^2(\theta)\,d\phi\right)dt
\end{aligned} \tag{5.48}$$

Substituting (5.48) back into (5.47) we get

$$ds^2 = -c^2\left(1+\frac{2\Phi}{c^2}\right)dt^2 - \frac{4GJ}{c^3r^2}\sin^2(\theta)\,(rd\phi)\,(cdt) + \left(1-\frac{2\Phi}{c^2}\right)\delta_{ij}dx^i dx^j$$

where we have dropped the z index from the angular momentum.

It has already been shown that the diagonal terms in the metric are simply the linear approximation of the Schwarzschild metric. We can now see that the metric here is the same as we had for the metric in Chapter 4. One could also derive this metric from the Kerr metric by applying the weak field and non-relativistic motion limits.

5.4 Schiff Equation

The predictions of Gravity Probe B are based on the precession described by the so called Schiff equation [19]. The Schiff equation was originally derived from the equations of motion, but the analysis gets very complicated and lengthy. The method we used to derive the Lense-Thirring precession is a lot more accessible.

Adding together the geodetic precession [12, 19] with the Lense-Thirring (5.46) precession we obtain the Schiff equation

$$\boldsymbol{\Omega} = \frac{3GM}{2c^2r^3}(\mathbf{x}\times\mathbf{v}) + \frac{G}{c^2r^3}\left(\frac{3\mathbf{x}}{r^2}(\mathbf{J}\cdot\mathbf{x}) - \mathbf{J}\right) \tag{5.49}$$

Let us discuss the last term in the equation. We shall consider a gyroscope orbiting the Earth in a polar orbit. It is obvious that the angular momentum of the Earth only has a z-component, and we shall also orient the x-axis such that polar orbit is in the xz-plane. The position of the satellite is thus

$$\mathbf{x} = (r\sin(\theta), 0, r\cos(\theta)) \tag{5.50}$$

Substituting this into the second term in the Schiff equation leaves us with only the z term (due to the single angular momentum component)

$$\boldsymbol{\Omega} = \left(0, 0, \frac{GJ}{c^2r^3}\left(3\cos^2(\theta) - 1\right)\right) \tag{5.51}$$

where we have used $\mathbf{J} = (0, 0, J)$. We take the average over one orbit giving us

$$\begin{aligned}\Omega_{av} &= \frac{\int_0^{2\pi}\Omega d\theta}{\int_0^{2\pi}d\theta} = \frac{GJ}{c^2r^3}\frac{\int_0^{2\pi}3\cos^2(\theta) - 1}{2\pi} \\ &= \frac{GJ}{2c^2r^3}\end{aligned} \tag{5.52}$$

This is a very similar answer to the one we found earlier in equation (4.5). The factor of 1/4 comes from the fact that in Chapter 4 we had the gyroscope falling down the z-axis, whilst we now have a polar orbit hence there are latitude effects to consider.

Using the values for the Earth of $r = 7020.1$ km (6378.1 km + 642 km) and $J = 2.661\cdot 10^{30}$ J s in equation (5.52) and multiplying by 365 days, we find the average precession over one year to be 37.5 mas/year.

Chapter 6

Conclusion

In this project we have shown that instead of using the complicated equations of motion to derive the precessions, we can simply use the analogies between electromagnetism and GEM. Approximate methods can also be used, as in Chapters 3 and 4, to obtain fairly accurate results. The project showed how linearised gravity can be used to describe many situation, and though it was not discussed, we can use the linearised metrics to discuss phenomenon such as light bending and gravitational waves [13]. The theory of GEM is a controversial theory and is only true in the weak field limit. For this reason, in much of the top level research, GEM is not applicable. For this reason, GEM is not a very widely studied subject. The simple act of working to lowest order vastly reduces the number of avenues that we are able to proceed down.

The obvious next route to take on the precession of a gyroscope would be to consider more general (elliptical) orbits and orbits around more general objects. More complicated mathematics arise when we consider relativistically rotating and charged black holes, but the effects are much more pronounced [20, 21]. Another option would be to look at the orbital effects of frame-dragging. One can show that there is an induced orbital precession of identical amount as the gyroscopic precession. The frame-dragging causes the orbit to very slowly rotate with the Earth. Some interesting phenomenon occur such as the perigee in an elliptical equatorial orbit precesses a similar amount, except in the opposite direction to the object's rotation. Another effect in such an orbit is that the orbital speed decreases if its in the same direction as the objects rotation, and increases in the opposite direction, which is at first counter-intuitive.

Appendix A

Christoffel Symbols

The general equation for a metric is $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$.

A.1 Schwarszchild Metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + sin^2\left(\theta\right) d\phi^2\right)$$

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2M}{r}\right) & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2M}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2(\theta) \end{pmatrix} \tag{A.1}$$

Christoffel Symbols:

$$\Gamma^t_{tr} = \left(\frac{M}{r^2}\right)\left(1 - \frac{2M}{r}\right)^{-1} \qquad \Gamma^r_{tt} = \left(\frac{M}{r^2}\right)\left(1 - \frac{2M}{r}\right)$$

$$\Gamma^r_{rr} = -\left(\frac{M}{r^2}\right)\left(1 - \frac{2M}{r}\right)^{-1} \qquad \Gamma^r_{\theta\theta} = -\left(r - 2M\right)$$

$$\Gamma^r_{\phi\phi} = -\left(r - 2M\right) sin^2\left(\theta\right) \qquad \Gamma^\theta_{r\theta} = \frac{1}{r}$$

$$\Gamma^\theta_{\phi\phi} = -cos\left(\theta\right) sin\left(\theta\right) \qquad \Gamma^\phi_{r\phi} = \frac{1}{r}$$

$$\Gamma^\phi_{\theta\phi} = cot\left(\theta\right)$$

A.2 Lense-Thirring Metric (Flat, Cartesian)

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 - \frac{4GJ}{c^3r^2}(cdt)\left(\frac{xdy - ydx}{r}\right) + \mathrm{O}\big(J^2\big)$$

$$g_{\mu\nu} = \begin{pmatrix} -c^2 & \frac{4GJ}{c^2r^3}y & -\frac{4GJ}{c^2r^3}x & 0 \\ \frac{4GJ}{c^2r^3}y & 1 & 0 & 0 \\ -\frac{4GJ}{c^2r^3}x & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{A.2}$$

Christoffel Symbols (to lowest order):

$$\Gamma^x_{ty} = \Gamma^x_{yt} = \frac{2GJ}{c^2r^3} \tag{A.3}$$

$$\Gamma^y_{tx} = \Gamma^y_{xt} = -\frac{2GJ}{c^2r^3} \tag{A.4}$$

Appendix B

Existence of the Lorentz Gauage

Considering a change in coordinates such that $x'^{\mu} = x^{\mu} + \xi^{\mu}$ where ξ^{μ} is small (in the sense that $\left|\xi^{\mu}{}_{,\nu}\right| \ll 1$), we can relate the metrics of the two coordinate systems via

$$\begin{aligned}
g'_{\mu\nu} &= \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta} \\
&= \frac{\partial}{\partial x'^{\mu}} \left(x'^{\alpha} - \xi^{\alpha}\right) \frac{\partial}{\partial x'^{\nu}} \left(x'^{\beta} - \xi^{\beta}\right) g_{\alpha\beta} \\
&= \left(\delta^{\alpha}_{\mu} - \xi^{\alpha}{}_{,\mu}\right) \left(\delta^{\beta}_{\nu} - \xi^{\beta}{}_{,\nu}\right) g_{\alpha\beta} \\
&= g_{\mu\nu} - \xi_{\nu,\mu} - \xi_{\mu,\nu}
\end{aligned} \tag{B.1}$$

where we have removed the non-linear term in the final line. This is called our gauge transformation. In linearised gravity when we transform $h_{\mu\nu}$ like this it remains small provided that ξ^{μ} is small as was required.

We defined our Lorentz gauge as $\overline{h}^{\mu\nu}{}_{,\nu} = 0$. Suppose that in some coordinate system $\overline{h}'^{\mu\nu}{}_{,\nu} \neq 0$. Using the above gauge transformation, we find that

$$\overline{h}'_{\mu\nu} = \overline{h}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \xi\eta_{\mu\nu} \tag{B.2}$$

where $\xi = \xi^{\mu}{}_{\mu}$. Taking the derivative and raising the μ and ν indices by multiplying through by $\eta^{\mu\beta}\eta^{\nu\gamma}$ gives us

$$\overline{h}'^{\beta\gamma}{}_{,\alpha} = \overline{h}^{\beta\gamma}{}_{,\alpha} - \xi^{\beta,\gamma}{}_{\alpha} - \xi^{\gamma,\beta}{}_{\alpha} + \xi_{,\alpha}\eta^{\beta\gamma} \tag{B.3}$$

Contracting over γ and α will give us

$$\begin{aligned} \overline{h}'^{\beta\alpha}{}_{,\alpha} &= \overline{h}^{\beta\alpha}{}_{,\alpha} - \xi^{\beta,\alpha}{}_{\alpha} - \xi^{\alpha,\beta}{}_{\alpha} + \xi_{,\alpha}\eta^{\beta\gamma} \\ &= \overline{h}^{\beta\alpha}{}_{,\alpha} - \Box\xi^{\beta} \end{aligned} \tag{B.4}$$

If we can choose $\Box\xi^{\beta} = \overline{h}^{\beta\alpha}{}_{,\alpha}$ then the gauge exists. For any well behaved function F we can always find a solution to $F = \Box f$ [22]. Hence we can always find some ξ^{μ} to transform our metric to the Lorentz gauge.

We can in fact take some vector η^{μ} that satisfies $\Box\eta^{\mu} = 0$ and add this to ξ^{μ} and the result will still hold. Hence, the Lorentz gauge is technically a class of gauges.

Appendix C

Other Equations

Geodesic Equation:

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \tag{C.1}$$

Gyroscope Equation:

$$\frac{ds^\alpha}{d\tau} + \Gamma^\alpha_{\mu\nu}s^\mu u^\nu = 0 \tag{C.2}$$

s^μ is the spin 4-vector, u^μ is the velocity 4-vector.

Small Angle Formulae:

$$\sin(\theta) \simeq \theta \tag{C.3}$$

$$\cos(\theta) \simeq 1 \tag{C.4}$$

Einstein Field Equations:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{C.5}$$

We have taken the cosmological constant, Λ, to be zero. In the case of using geometrised units we simply omit the terms G and c from the equation.

Vector Triple Product:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \tag{C.6}$$

$$\nabla \times (\nabla \times \mathbf{A}) = -\nabla^2\mathbf{A} \tag{C.7}$$

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