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JOHN ERIK FORNÆSS

Contents

1. Introduction

In this lecture series I will give an introduction to complex analysis in higher dimension. I will introduce 3 topics. I will also mention open problems.

1. The Levi problem
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2. Holomorphic Functions

We introduce some notation.
Let $\mathbb{C}$ denote the complex plane, with complex variable $z = x + iy$.

We recall that a holomorphic function in several variables, $z = (z_1, \dots, z_n)$ is a function $f(z) : \mathbb{C}^n \to \mathbb{C}$, such that f is analytic in each variable separately. It is a classical result that such a function is real analytic and locally is given by a power series:

$$f(z) = \sum_\alpha a_\alpha z^\alpha.$$

We use the following notation: $\alpha = (\alpha_1, \dots, \alpha_n)$ where each $\alpha_j = 0, 1, \dots$. Each a_α is a complex number and $z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}$.

We have the n-dimensional analogue of the Cauchy integral formula.

$$f(z_1, \dots, z_n) = \frac{1}{(2\pi i)^n} \int_{|\eta_1|=1,\dots,|\eta_n|=1} \frac{f(\eta_1, \dots, \eta_n)}{(\eta_1 - z_1) \cdots (\eta_n - z_n)} d\eta_1 \cdots d\eta_n$$

Notation: Let $f(x, y) = u(x, y) + iv(x, y)$. Then f is analytic if and only if

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

It is convenient to define $\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}(f_x + if_y)$. Then

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left((u_x - v_y) + i(u_y + v_x)\right)$$

Hence f is holomorphic if and only if $\frac{\partial f}{\partial \overline{z}} = 0$. Similarly, in $\mathbb{C}^n$, $f(z_1, \dots, z_n)$ is holomorphic if and only if

$$\frac{\partial f}{\partial \overline{z}_1} = \cdots = \frac{\partial f}{\partial \overline{z}_n} = 0.$$

For ease of notation, it is easier to write this as $\overline{\partial} f = 0$ where we write $\overline{\partial} f = \sum_i \frac{\partial f}{\partial \overline{z}_i} d\overline{z}_i$. We say that the expression $\sum_i a_i d\overline{z}_i = 0$ if all $a_i = 0$. We can write $d\overline{z} = dx + idy$.

3. Domains of holomorphy

Let Ω denote a domain in the complex plane, a connected open set. If z_0 is a boundary point, then the function $\frac{1}{z-z_0}$ is a function on Ω which is singular at the boundary point z_0. Using this one can actually find a holomorphic function $f(z)$ on Ω which is singular at every boundary point. For this reason we say that every domain in the complex plane is a domain of holomorphy.

The problem is to generalize this to higher dimension. This is the Levi problem.

Let $\mathbb{B}^n$ denote the unit ball in $\mathbb{C}^n$, $\mathbb{B}^n(0,1) = \{(z_1, \dots z_n); \|z\|^2 = |z_1|^2 + \cdots + |z_n|^2 < 1\}$

Lemma 3.1. *(Hartogs extension) Let f be a holomorphic function on the punctured unit ball: $\mathbb{B}^n \setminus \{0\}, n > 1$. Then f extends to a holomorphic function on $\mathbb{B}^n$.*

Proof. Use the formula

$$f(z_1, \dots, z_n) \quad = \quad \frac{1}{2\pi i} \int_{|\eta|=1/2} \frac{f(\eta, z_2, \dots, z_n)}{\eta - z_1} d\eta$$

If $(z_2, \cdots, z_n) \neq 0$, then the Cauchy integral formula says that this equality holds. Then it holds by continuity also for $(z_2, \dots, z_n) = 0$. Moreover this formula provides an analytic extension across $0 \in \mathbb{C}^n$. $\square$

We say that a bounded domain Ω in $\mathbb{C}^n$ with smooth boundary is strongly convex if for every boundary point, the tangent plane is tangent to second order. Basic example is the ball.

Let Ω be a strongly convex domain in $\mathbb{C}^n$ and let p be a boundary point. Then we know that there is a real linear functional $L(z) = \sum_j (a_j x_j + b_j y_j)$ so that L takes it maximum value on $\overline{\Omega}$ exactly at p. We can then write $L(z) = \mathrm{Re}(\sum (a_j - ib_j)(x_j + iy_j)) = \mathrm{Re}(\hat{L}(z))$. Here $\hat{L}$ is a holomorphic function. Then the function $\frac{1}{\hat{L}(z) - \hat{L}(p)}$ is singular at p. The function $f = e^{\hat{L}(z) - \hat{L}(p)}$ is also a useful holomorphic function. It is a peak function, $f(p) = 1, |f(q)| < 1, q \in \overline{\Omega} \setminus \{p\}$. We say then that every such domain is a domain of holomorphy.

In complex analysis, one prefers to study properties which are independent of holomorphic coordinate system.

Definition 3.2. A bounded domain Ω in $\mathbb{C}^n$ with smooth boundary is called strongly pseudoconvex if there is a biholomorphic map defined in a neighborhood of any given boundary point such that the local image of Ω becomes strongly convex.

The following problem was a main motivator of several complex variables. It was solved in this form about 1950 or so. More general versions are still open, for example in complex spaces with singular points.

Problem 3.3. (LEVI PROBLEM) If Ω is strongly pseudoconvex and p is a boundary point, does there exist a holomorphic function on Ω which is singular at p? (A peak function would suffice).

A more general definition of pseudoconvex domain in $\mathbb{C}^n$.

Definition 3.4. A domain Ω in $\mathbb{C}^n$ is pseudoconvex if there exists for every compact $K \subset \Omega$, a strongly pseudoconvex domain U with $K \subset U \subset \Omega$.

The classical formulation of the Levi problem was to find functions with boundary singularities for any pseudoconvex domain. So this was solved affirmatively.

Here is an open question:

Problem 3.5. Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^3$ which has real analytic boundary. Then is every boundary point a peak point?

4. Hartogs figures

Definition 4.1. (Hartogs skeleton) Let $S = S_1 \cup S_2$ be the union of two compact sets in $\mathbb{C}^2$. $S_1 = \{(z,w); |z| \leq 1, w = 0\}$ is a disc in the z axis. $S_2 = \{(z,w); |z| = 1, |w| \leq 1\}$, is a cylindrical wall. The set $\hat{S} := \overline{\Delta}^2 := \{(z,w); |z|, |w| \leq 1\}$ is called the hull of S. It is also the convex hull.

Lemma 4.2. *Suppose that $f(z,w)$ is a holomorphic function defined in an open set containing S. Then f extends to $\hat{S}$.*

Proof. One uses the Cauchy integral formula $f(z,w) = \frac{1}{2\pi i}\int_{|\eta|=1}\frac{f(\eta,w)d\eta}{\eta - z}$. This expression defines a holomorphic function. Because S_1 is in the skeleton, the function agrees with the given f when $|w|$ is small. Then the rest follows from the uniqueness theorem. $\square$

5. Envelopes of Holomorphy

The Levi problem is about characterizing the domains of holomorphy. Not all domains are domains of holomorphy, as we have seen, for example the punctured ball.

Lemma 5.1. *Assume that a domain U in $\mathbb{C}^2$ contains the Hartogs skeleton S. If U is a domain of holomorphy, then the domain must also contain $\hat{S}$.*

Proof. Not every boundary point in the interior can be singular by the previous lemma. $\square$

If one has a domain Ω and take a Hartogs skeleton S contained in Ω, then $\Omega \cup \hat{S}$ is contained in the envelope of holomorphy. It can be shown that if one does this countably many times one gets the envelope of holomorphy. Not much is known about the shape of the envelope of holomorphy. Explain non schlichtness.

Problem 5.2. Let Ω be a domain in $\mathbb{C}^2$ with real analytic boundary. Is the envelope of holomorphy finitely sheeted?

6. Jorickes Theorem

Joricke has a recent result on envelopes of holomorphy: (Inventiones 2009). This concerns the question about how necessary it is to use addition of Hartogs figures infinitely many times.

Let Ω be a domain in $\mathbb{C}^2$, and let V denote the envelope of holomorphy. Then if $p \in V \setminus \Omega$, there is a disc in V through p such that the boundary is in Ω.

7. Subharmonic Functions

A harmonic function $h(x, y)$ in the complex plane is one for which $\Delta h = h_{xx} + h_{yy} = 0$. A smooth function is said to be subharmonic if $\Delta h \geq 0$. The function is called strongly subharmonic if $\Delta h > 0$. Subharmonicity can be generalized to nonsmooth functions. In that case one can define subharmonicity by the condition $\Delta h \geq 0$ in the sense of distributions.

Let f be a function in L^1. Then the derivative $\frac{\partial f}{\partial x}$ in the sense of distributions are given by $\frac{\partial f}{\partial x}(\psi) = -int f \frac{\partial \psi}{\partial x}$ for all smooth functions ψ with compact support.

In that case the condition of subharmonicity can also be stated as Δh being a nonnegative measure.

One can approximate subharmonic functions by smooth ones. h_ϵ converging down to h pointwise. This is done by convolution. Let $\chi(z) = \chi(|z|) \geq 0$ be a smooth function with compact support in the unit disc. Assume that $\int_{\mathbb{C}} \chi = 1$.

Then define

$$h_\epsilon(z) = \frac{1}{\epsilon^2} \int h(z + w) \chi(\frac{w}{\epsilon}) dw.$$

8. Plurisubharmonic Functions

A smooth function $\rho(z_1, \ldots, z_n)$ in $\mathbb{C}^n$ is called plurisubharmonic if the restriction to any complex line is subharmonic, $h(z + \tau w)$ is subharmonic as a function of the complex variable $\tau \in \mathbb{C}$ for any given $z, w \in \mathbb{C}^n$. This definition also applies to nonsmooth functions. But one adds the condition that the function is upper semicontinuous. Similarly to one variable one can smooth plurisubharmonic functions.

Lemma 8.1. .
1. Convex functions are plurisubharmonic.
2. If ρ is plurisubharmonic and f is biholomorphic, then $\rho \circ f$ is plurisubharmonic.

9. $-\log d$

The crucial connection to several complex variables comes from the following result. Let Ω be a domain in $\mathbb{C}^n$. Let $d : \Omega \to \mathbb{R}^+$ be the distance to the boundary:

$$d(z) = \min_{w \in \partial\Omega} \|z - w\|.$$

Theorem 9.1. *The domain Ω is pseudoconvex if and only if the function $-\log d$ is plurisubharmonic.*

Proof. (One dimension). Let $w \in \partial\Omega$. Then $-\log|z-w|$ is harmonic in Ω. Hence $d(z) = \sup_{w \in \partial\Omega}(-\log|z-w|)$ is subharmonic. □

Proof. (Several variables) We assume first that Ω is strongly pseudoconvex.

Lemma 9.2. *Let Ω be a bounded smooth strongly pseudoconvex domain. Then there exists a continuous function ρ on $\overline{\Omega}$ such that $\rho = 0$ on $\partial\Omega$ and $\rho < 0$ and plurisubharmonic on Ω.*

Proof. Let r be a smooth function which is 0 on the boundary and negative inside. Also assume that $\nabla r \neq 0$ on the boundary. Locally we can compose with a biholomorphic map f so that the domain is strongly convex. Then $(r + Ar^2) \circ f$ will be a convex function if $A > 0$ is a large constant. Hence the function $r + Ar^2$ will be plurisubharmonic. Next define $\rho = \max\{r + Ar^2, -\epsilon\}$ which extends to all of Ω and satisfies the requirements of the Lemma. □

Next pick a unit vector ξ. We define a distance in the ξ direction: For $z \in \Omega$, we set $d_\xi(z) := \sup\{t; z + \tau\xi \in \Omega, \forall\ \tau \in \mathbb{C}, |\tau| < t\}$. We show that $-\log(d_\xi)$ is plurisubharmonic. Assume not. Then there is a complex line L and a disc $\overline{D} \subset L$ so that d_ξ has value at the center strictly larger than the average value on the boundary. We can assume that L is the z_1 axis and D is the unit disc. We can choose a holomorphic polynomial $P(z)$ with $h = \Re(P(z))$ so that $h > -\log d_\xi$ on the boundary of the disc and $h(0) < -\log d_\xi(0)$. Consider the complex discs $D_t(\zeta)$ for $t \in \mathbb{C}, |t| \leq 1$ and for $\zeta \in \mathbb{C}, |\zeta| \leq 1$. $D_t(\zeta) = (\zeta, 0, \dots, 0) + t\xi e^{-P(\zeta)}$. If ζ is on the boundary of the unit disc, we have that $-\log d_\xi(\zeta) < h(\zeta)$ and hence $|te^{-P(\zeta)}| = |t|e^{-h(\zeta)} < e^{\log d_\xi(\zeta)} = d_\xi(\zeta)$. It follows that the boundaries of the discs are all in Ω. For $t = 0$ the interior of the disc is in Ω. Consider the function

ρ restricted to the discs. For those t for which the whole disc is in Ω, we have by the maximum principle a uniform strictly negative upper bound. Hence no such disc gets too close to the boundary. So it follows that also for $|t| = 1$ the disc is in Ω. Let $\zeta = 0, |t| \leq 1$. Then $D_t(0) = t\xi e^{-P(0)} \in \Omega$. Hence $|e^{-P(0)}| < d_\xi(0)$, so $-\log d_\xi(0) < h(0)$, a contradiction.

So we have shown that if Ω is a strongly pseudoconvex domain, then $-\log(d_\xi)$ is plurisubharmonic. Hence when we take the sup over all unit vectors, we see that $-\log d$ is plurisubharmonic. Finally, if Ω is an increasing union of strongly pseudoconvex domains, this is still true.

It remains to show that if $-\log d$ is plurisubharmonic then Ω can be exhausted by strongly pseudoconvex domains.

To see this, we can smooth the function with convolutions. This gives smooth plurisubharmonic functions. Then add $\epsilon\|z\|^2$ to make the smoothings strongly plurisubharmonic. Then for generic sublevel sets these are strongly pseudoconvex domains. Here one uses Sard's Lemma which ensures that the gradient is nonzero on almost level sets of the function.

□

Corollary 9.3. *Let Ω be a pseudoconvex domain. If $s : \Omega \to \mathbb{R}$ is a continuous function, then there is a smooth plurisubharmonic function ρ on Ω so that $\rho > s$. Also there is a sequence of smooth plurisubharmonic function ρ_n so that $\rho_n \searrow 0$ uniformly on compact sets so that $\rho_n > s$ close enough to the boundary.*

Proof. The key fact is that if ξ is convex and increasing and ρ is plurisubharmonic, then $\xi \circ \rho$ is also plurisubharmonic. □

10. Hilbert spaces with weights

Let Ω be a domain in $\mathbb{C}^n$ and let ϕ be a plurisubharmonic function on Ω.

We let $L^2(\Omega, \phi) := \{f; \int_\Omega |f|^2 e^{-\phi} dV =: \|f\|_\phi^2 < \infty\}$. This is a Hilbert space of measurable functions. We can introduce an inner product

$$< f, g >_\phi := \int_\Omega f\overline{g} e^{-\phi} dV.$$

We note that $\mathcal{C}_0^\infty$, the smooth functions with compact support, are dense in $L^2(\Omega, \phi)$.

Another useful fact is that if f is a measurable function on a pseudoconvex domain Ω, and f is in L^2 on each compact subset of Ω, then there exists a plurisbharmonic function ϕ on Ω such that $f \in L^2(\Omega, \phi)$. We say that such functions are in $L^2(\Omega)_{loc}$.

11. Hormanders theorem

Lemma 11.1. *Suppose that u is a smooth function on Ω in $\mathbb{C}^n$. Set $f_i = \frac{\partial u}{\partial \bar{z}_i}$. Then $\frac{\partial f_i}{\partial \bar{z}_j} = \frac{\partial f_j}{\partial \bar{z}_i}$ for all i, j.*

Proof. This is true because it holds for derivatives with respect to real variables. □

The following is a theorem by Hormander (1964).

Theorem 11.2. *Let $\{f_i\}_{i=1}^n$ be smooth L^2 functions on a bounded pseudoconvex domain Ω. Suppose also that $\frac{\partial f_i}{\partial \bar{z}_j} = \frac{\partial f_j}{\partial \bar{z}_i}$ for all i, j. Then there exists a smooth L^2 function u on Ω such that $\frac{\partial u}{\partial \bar{z}_i} = f_i$ for all i.*

12. unbounded operators on Hilbert space

Given two complex Hilbert spaces, H_1, H_2 and a dense linear subspace $D \subset H_1$. We assume that $T : D \to H_2$ is a linear operator. Here T is not assumed to be continuous. We can write D_T instead of D if we have several operators. Let $<,>_1, <,>_2$ denote the inner products and $\|\|_j$ denote the norms.

Definition 12.1. The operator T is said to be **closed** if the graph $G_T = \{(x, Tx) \in H_1 \times H_2\}$ is a closed subspace.

Adjoint operator $T^* : H_2 \to H_1$:

Definition 12.2. A $\psi \in H_2$ is in D_{T^*} if there exists a constant C such that $| < T\phi, \psi >_2 | \leq C\|\phi\|_1$ for all $\phi \in D_T$.

Proposition 12.3. *If $\psi \in D_{T^*}$, then there exists a unique element $T^*\psi \in H_1$ so that*

$$< \phi, T^*\psi >_1 = < T\phi, \psi >_2$$

for all $\phi \in D_T$.

We get a new linear operator $T^* : H_2 \to H_1$.

Lemma 12.4. *The operator T^* is closed. If D_{T^*} is dense, then $T^{**} = T$.*

Lemma 12.5. *Let $T : H_1 \to H_2$ be a closed, densely defined operator. Suppose also that T^* is densely defined. Let $F \subset H_2$ denote a closed subspace containing $T(H_1)$. Then $T(H_1) = F$ if and only if there is a constant C such that*

$$(1) \quad \|f\|_{H_2} \leq C\|T^*(f)\|_{H_1} \;\; \forall\, f \in D_{T^*} \cap F.$$

The crucial point in Hormanders theorem is that (1) implies that T is onto. We explain the main idea:

Let $z \in F$. We need to write $z = Tx$. Define a linear functional ϕ by $\phi(T^*y) =< y, z >_2$. By the inequality (1) this is bounded. Hence by Hahn-Banach it extends to a continuous linear functional ϕ on H_1. A linear functional on a Hilbert space can be identified with one element of the Hilbert space, x. So $< T^*y, x >_1 =< y, z >_2$ for all $y \in D_{T^*} \cap F$. This implies that $x \in D_{T^{**}}$ and therefore $x \in D_T$. Hence $< y, z >_2 =< T^*y, x >_1 =< y, Tx >_2$ which implies that $z = Tx$.

13. Hormander in L^2 spaces

Theorem 13.1. *Let $\{f_i\}_{i=1}^n$ be L^2 functions on a bounded pseudoconvex domain Ω. Suppose also that $\frac{\partial f_i}{\partial \bar{z}_j} = \frac{\partial f_j}{\partial \bar{z}_i}$ for all i, j Then there exists an L^2 function u on Ω such that $\frac{\partial u}{\partial \bar{z}_i} = f_i$ for all i. Here all derivatives are in the sense of distributions.*

Let ϕ denote a plurisubharmonic function on a pseudoconvex domain Ω. We can then define weighted Hilbert spaces:

$$L^2(\Omega, \phi) = \{u; \int_\Omega |u|^2 e^{-\phi} = \|u\|_\phi^2 < \infty\}.$$

Let $f = (f_1, \dots, f_n)$ denote an n-tuple of measurable functions.

$$L^2(\Omega, \phi, n) = \{f; \sum_{i=1}^n \int_\Omega |f_i|^2 e^{-\phi} = \|f\|_\phi^2 < \infty\}.$$

The Theorem of Hormander works in weighted L^2 spaces: Let ϕ be any plurisubharmonic function (not necessarily smooth) on a pseudoconvex domain Ω in $\mathbb{C}^n$ (not necessarily bounded.)

Let ϕ be a plurisubharmonic function in a pseudoconvex domain Ω. Assume that κ is a continuous function which is a lower bound for the plurisubharmonicity of ϕ. ($\sum_{jk} \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} t_j \bar{t}_k - e^\kappa |t|^2$ is a nonnegative measure for all $t \in \mathbb{C}^n$.)

Theorem 13.2. *Let $f = (f_1, \dots, f_n) \in L^2(\Omega, \phi + \kappa, n)$. Suppose also that $\frac{\partial f_i}{\partial \bar{z}_j} = \frac{\partial f_j}{\partial \bar{z}_i}$ for all i, j. Then there is a $u \in L^2(\Omega, \phi)$ such that $\frac{\partial u}{\partial \bar{z}_i} = f_i$ for all i and*

$$q \int_\Omega |u|^2 e^{-\phi} dV \leq \int_\Omega |f|^2 e^{-(\phi+\kappa)} dV$$

Recall that if $f = (f_1, \dots, f_n)$ are in L^2_{loc} then there is a plurisubharmonic weight ϕ going to infinity fast enough that $f \in L^2(\Omega, \phi, n) \subset L^2(\Omega, \phi + \kappa, n)$. Hence one can also find u when the data f are in L^2_{loc}. The solution u is also in L^2_{loc}.

14. The proof of the Theorem of Hormander

For the proof one needs to use weights that go suffiently fast to infinity at the boundary. One get constants independent of the weight used. Then pass to weak limits to get solutions for any weight: Let ϕ be any weight. Write ϕ as limit of ϕ_n where ϕ_n are smooth plurisubharmonic and changed near boundary to go suffiently fast to infinity there.

For the proof we use two suitable plsurisubharmonic weight functions.

$$\begin{aligned} H_1 &= L^2(\Omega, \phi_1) \\ H_2 &= L^2(\Omega, \phi_2, n) \\ F \subset H_2 &= \{f = (f_1, \dots, f_n); \frac{\partial f_i}{\partial \overline{z}_j} = \frac{\partial f_j}{\partial \overline{z}_i} \; \forall \, i, j.\} \end{aligned}$$

We set $Tu = (\frac{\partial u}{\partial \overline{z}_1}, \dots, \frac{\partial u}{\partial \overline{z}_n})$. This is densely defined and linear. We notice that F is a closed subspace which contains the image of T.

Then one proves by integration by parts that

$$\|f\|_2 \leq \|\overline{\partial}^* f\|_1 \; \forall \; f \in D_{T^*} \cap F.$$

In this argument is it crucial to be able to calculate with smooth functions and then pass to limits in L^2 spaces. Here using plurisubharmonic functions going to infinity sufficiently fast is essential.

Then one can use Lemma 2.5 to show existence of solutions.

15. Solution of the Levi problem

15.1. An extension theorem.

Theorem 15.1. *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$. Let $g(z_1, z_2, \dots, z_{n-1})$ be a holomorphic function on $H = \{(z_1, \dots, z_{n-1}, 0) \in \Omega\}$. Then g extends to a holomorphic function on Ω.*

Proof. Let G be a smooth extension of g to Ω so that $G(z_1, \dots, z_n) = g(z_1, \dots, z_{n-1}, 0)$ in a neighborhood U of H. Define the $f_i' = \frac{\overline{\partial} G}{z_i}, i = 1, \dots, n$. The f_i' vanish in a neighborhood of H. Also, $\frac{\partial f_i'}{\partial \overline{z}_j} = \frac{\partial f_j'}{\partial \overline{z}_i}$ for all i, j.

We next define $f_i = \frac{f_i'}{z_n}$ on Ω by defining $f_i = 0$ on H. Then the f_i are smooth functions on Ω and $\frac{\partial f_i}{\partial \overline{z}_j} = \frac{\partial f_j}{\partial \overline{z}_i}$ for all i, j. They are also in L^2_{loc}. Hence there exists a function u in L^2_{loc} such that $\frac{\partial u}{\partial \overline{z}_i} = f_i$ for all i.

Define next $G - z_n u$ on Ω. This extends g from H. Moreover, we see that $\frac{\partial (G - z_n u)}{\partial \overline{z}_i} = f_i' - z_n f_i = 0$ for all i. Hence $G - z_n u$ is holomorphic.

□

15.2. **The Levi problem.** Recall that a domain Ω is a domain of holomorphy if there is for every boundary point a holomorphic function which is singular there. (Pay attention to precise statement when boundary is not smooth) We can restrict to the smooth case for simplicity.

Theorem 15.2. *Let Ω be a pseudoconvex domain. Then Ω is a domain of holomorphy.*

Proof. In dimension 1, this is true because all domains are domains of holomorphy. Suppose true in dimension $n-1$. Let Ω be a pseudoconvex domain in $\mathbb{C}^n$. Pick a point p in the boundary. Assume for simplicity that the boundary is smooth near p and that $p = 0$ and that $H = \Omega \cap \{z_n = 0\}$ clusters at p. Then H is pseudoconvex in $\mathbb{C}^{n-1}$ and there is a holomorphic function f on H which is singular at p. Then any extension to Ω is singular at p. (If the boundary is not smooth, then we can still do this on a dense set in the boundary, which will suffice) $\square$

Department for Mathematical Sciences, Norwegian University of Science and Technology, Trondheim, Norway

E-mail address: john.fornass@math.ntnu.no