Problem Set 8, Physics 622 (Fall 2001)
Solutions

1(4.2) Let $\psi(\vec{x})$ be an arbitrary wavefunction.

(a) $T_{\vec{d}} T_{\vec{d'}} \psi(\vec{x}) = \psi(\vec{x} + \vec{d'} + \vec{d}) = \psi(\vec{x} + \vec{d} + \vec{d'}) = T_{\vec{d'}} T_{\vec{d}} \psi(\vec{x})$.
Therefore $[T_{\vec{d}}, T_{\vec{d'}}] = 0$. ■

(b) $[D(\hat{n}, \phi), D(\hat{n'}, \phi')] \neq 0$, i.e. finite rotations do not commute, an example is when $\hat{n} = \hat{x}$ and $\hat{n'} = \hat{y}$. ■

(c) $T_{\vec{d}} \Pi \psi(\vec{x}) = \psi(-\vec{x} + \vec{d}) \neq \psi(-\vec{x} - \vec{d}) = \Pi T_{\vec{d}} \psi(\vec{x})$.
Hence, $[T_{\vec{d}}, \Pi] \neq 0$. ■

(d) $\Pi D(\hat{n}, \phi) \psi(\vec{x}) = \Pi \psi(\vec{x'}) = \psi(-\vec{x'})$ where $\vec{x'} = D(\hat{n}, \phi)\vec{x}$
$D(\hat{n}, \phi) \Pi \psi(\vec{x}) = D(\hat{n}, \phi) \psi(-\vec{x}) = \psi(-\vec{x'})$
Hence, $[D(\hat{n}, \phi), \Pi] = 0$. ■

2(4.5) Notice that $\vec{S} \cdot \vec{x}$ is invariant under rotations but changes sign under parity, so it is a pseudoscalar. Since $\delta^{(3)}(\vec{x})$ is a scalar, hence the perturbation V is a pseudoscalar. We can then conclude that

(i) the perturbed state must have the same j and m as the unperturbed one, so $j' = j$ and $m' = m$.

(ii) from perturbation theory, $C_{n'l'j'm'}$ is proprotional to $\langle n'l'j'm'|V|nljm\rangle$, since $|nljm\rangle$ has parity $(-1)^l$, for the matrix element to be non-vanishing, $|n'l'j'm'\rangle$ and $|nljm\rangle$ must have different parities, so $l' = l \pm 1$, ($|\Delta l| \geq 2$ is not allowed since j must remain unchanged).

There is further restriction on $C_{n'l'j'm'}$. Due to the presence of $\delta^{(3)}(\vec{x})$ in V, the matrix element is non-zero only for those states whose radial wavefunctions are finite at the origin, therefore we must have $S_{\frac{1}{2}}$ or $P_{\frac{1}{2}}$ for $|nljm\rangle$. ■

3(4.6) Let $k_\alpha^2 = \dfrac{2mE}{\hbar^2}$ and $\kappa_\alpha^2 = \dfrac{2m(V_0 - E)}{\hbar^2}$ where $\alpha = 1, 2$ refering to the symmetric and antisymmetric cases respectively. Due to the symmetry in the potential, the wavefunctions will have definite parity and are given by:

$$\psi_s = \begin{cases} A \sin[k_1(x + a + b)] \\ B \cosh \kappa_1 x \\ -A \sin[k_1(x - a - b)] \end{cases} \qquad \psi_a = \begin{cases} C \sin[k_2(x + a + b)] & -a - b < x < -a \\ D \sinh \kappa_2 x & -a \leq x \leq a \\ C \sin[k_2(x - a - b)] & a < x < a + b \end{cases}$$

Matching boundary conditions at the $x = a$ gives the following eigenvalues conditions:

$$\begin{aligned} \kappa_1 \tan k_1 b &= -k_1 \coth \kappa_1 a \\ \kappa_2 \tan k_2 b &= -k_2 \tanh \kappa_2 a \end{aligned}$$

Since $V_0 \gg E$, for the lowest lying states we can write $bk_{1,2} = \pi + \epsilon_{1,2}$ where $\epsilon_{1,2}$ are small perturbations to the ground state of the unperturbed infinite square well problem. Also, let $\kappa_{1,2} = \lambda \approx \sqrt{\frac{2mV_0}{\hbar^2}}$. Substitute all these into the above equations and using $\tan x \approx x$, we can solve for $\epsilon_{1,2}$ and the results are

$$\begin{aligned}\epsilon_1 &= \frac{-\pi \coth \lambda a}{\lambda b + \coth \lambda a} \\ \epsilon_2 &= \frac{-\pi \tanh \lambda a}{\lambda b + \tanh \lambda a}\end{aligned}$$

The difference in energy is given by

$$\begin{aligned}\Delta E &= \frac{\hbar^2}{2m}(k_2^2 - k_1^2) \\ &= \frac{\hbar^2 \pi^2}{2mb^2}\left[\left(1 - \frac{\tanh \lambda a}{\lambda b + \tanh \lambda a}\right)^2 - \left(1 - \frac{\coth \lambda a}{\lambda b + \coth \lambda a}\right)^2\right] \quad \blacksquare\end{aligned}$$

4(4.7) (a) The plane wave is given by

$$\psi(\vec{x}, t) = \exp\left[-\frac{i}{\hbar}(\vec{p} \cdot \vec{x} - Et)\right]$$

$$\begin{aligned}\text{then } \psi^*(\vec{x}, -t) &= \exp\left[\frac{i}{\hbar}(\vec{p} \cdot \vec{x} + Et)\right] \\ &= \exp\left[-\frac{i}{\hbar}(-\vec{p} \cdot \vec{x} - Et)\right]\end{aligned}$$

which is the wavefunction for the place wave with the momentum direction reversed. $\blacksquare$

(b) In explicit form,

$$\begin{aligned}\chi(\hat{n}) &= \cos\frac{\beta}{2}e^{-\frac{i\gamma}{2}}\begin{pmatrix}1\\0\end{pmatrix} + \sin\frac{\beta}{2}e^{\frac{i\gamma}{2}}\begin{pmatrix}0\\1\end{pmatrix} \\ -i\sigma_2\chi^*(\hat{n}) &= -i\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\left[\cos\frac{\beta}{2}e^{\frac{i\gamma}{2}}\begin{pmatrix}1\\0\end{pmatrix} + \sin\frac{\beta}{2}e^{-\frac{i\gamma}{2}}\begin{pmatrix}0\\1\end{pmatrix}\right] \\ &= \cos\frac{\beta}{2}e^{\frac{i\gamma}{2}}\begin{pmatrix}0\\1\end{pmatrix} + \sin\frac{\beta}{2}e^{-\frac{i\gamma}{2}}\begin{pmatrix}-1\\0\end{pmatrix}\end{aligned}$$

It can then be easily verified that

$$\hat{\sigma} \cdot \hat{n}(-i\sigma_2\chi^*(\hat{n})) = (-1)[-i\sigma_2\chi^*(\hat{n})]$$

so the spin direction is indeed reversed. $\blacksquare$

5(4.12) For spin-1 particle,

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and the Hamiltonian is then given by

$$H = \hbar^2 \begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ B & 0 & A \end{pmatrix}$$

Diagonalizing the above matrix gives the eigenvalues and eigenvectors (in the S_z-basis):

$$\begin{aligned} E &= 0 & &|1,0\rangle \\ E &= \hbar^2(A+B) & &\tfrac{1}{\sqrt{2}}(|1,1\rangle + |1,-1\rangle) \\ E &= \hbar^2(A-B) & &\tfrac{1}{\sqrt{2}}(|1,1\rangle - |1,-1\rangle) \end{aligned}$$ ■

Assume A and B are real. Since, for $\alpha = x, y, z$,

$$\Theta S_\alpha \Theta^{-1} = -S_\alpha \qquad \Theta S_\alpha^2 \Theta^{-1} = \Theta S_\alpha \Theta^{-1} \Theta S_\alpha \Theta^{-1} = S_\alpha$$

Therefore $\Theta H \Theta^{-1} = H$, so H is time reversal invariant. The eigenvectors transform as follow:

$$\begin{aligned} |1,0\rangle &\rightarrow |1,0\rangle \\ \tfrac{1}{\sqrt{2}}(|1,1\rangle + |1,-1\rangle) &\rightarrow -\tfrac{1}{\sqrt{2}}(|1,1\rangle + |1,-1\rangle) \\ \tfrac{1}{\sqrt{2}}(|1,1\rangle - |1,-1\rangle) &\rightarrow \tfrac{1}{\sqrt{2}}(|1,1\rangle - |1,-1\rangle) \end{aligned}$$ ■

6 (a) Consider the reaction: $\Omega^- \rightarrow \Lambda + K^-$

The orbital angular momentum of the hyperons Ω and Λ is zero.

Initial state:

$$\Omega^-(Y_{3/2}^{3/2}) \; : \; l = 0, \; J = \frac{3}{2}, \; m_J = \frac{3}{2} \; ; \; P_\Omega = +1$$

Final state:

$$\begin{aligned} \Lambda \; &: \; s = \frac{1}{2}, \; l = 0 \; ; \; P_\Lambda = +1 \\ K^- \; &: \; s = 0 \; ; \; P_K = -1 \end{aligned}$$

To have the final $J = \frac{3}{2}$, l can be 1 or 2 and the parity of the final state is given by $P_\Lambda P_K(-1)^l = (-1)^{l+1}$. Let χ_{s,m_s} be the spin wavefunction of Λ, since $m_J = \frac{3}{2}$, the final state of the system is given by:

$$|\psi\rangle = AY_{11}\,\chi_{\frac{1}{2}\frac{1}{2}} + B\left(-\frac{1}{\sqrt{5}}Y_{21}\,\chi_{\frac{1}{2}\frac{1}{2}} + \frac{2}{\sqrt{5}}Y_{22}\,\chi_{\frac{1}{2}-\frac{1}{2}}\right)$$

Note that the final state is a mixture of states of different parities, thus $|\psi\rangle$ does not have definite parity. The angular momentum distribution is given by

$$\begin{aligned} & |A|^2|Y_{11}|^2 + |B|^2(|Y_{21}|^2 + |Y_{22}|^2) && (|A|^2 + |B|^2 = 1) \\ = \; & \frac{3\alpha}{8\pi}\sin^2\theta + (1-\alpha)\left(\frac{1}{5}\frac{15}{8\pi}\sin^2\theta\cos^2\theta + \frac{4}{5}\frac{15}{32\pi}\sin^4\theta\right) && (\alpha = |A|^2) \\ = \; & \frac{3}{8\pi}\sin^2\theta \quad \blacksquare \end{aligned}$$

(b) If parity is conserved, the only contribution comes from $l = 1$ so that the parity of the final state equals $+1$. The angular momentum distribution is then given by

$$|Y_{11}|^2 = \frac{3}{8\pi}\sin^2\theta \quad \blacksquare$$

7 Given the interaction: $\pi^- + d^+ \longrightarrow n + n$

Initial state:

$$\begin{aligned} \pi^- \; &: \; s = 0,\ l = 0\ ;\ P_\pi = -1 \\ d^+(^3S_1) \; &: \; s = 1,\ l = 0\ ;\ P_d = +1 \end{aligned}$$

total angular momentum, $J = 1$
total orbital angular momentum, $L = 0$
parity $= P_\pi P_d(-1)^L = -1$

Final state:
total spin of the neutron pair can either be $S = 0$ or $S = 1$, since the total wavefunction of two fermions must be antisymmetric, we have

$$S = 0 \text{ (odd spin wavefunction)} \qquad L \text{ is even}$$

$$S = 1 \text{ (even spin wavefunction)} \qquad L \text{ is odd}$$

Therefore, the possible states for the neutron pair are $^1S_0$, $^3P_{012}$, $^1D_2$, $^3F_{234}, \ldots$etc. The only one with $J = 1$ is $^3P_1$, therefore

(a) the total orbital angular momentum of the neutron pair is $L = 1$ $\blacksquare$

(b) the total spin angular momentum of the neutron pair is $S = 1$ $\blacksquare$

(c) Align the z-axis along the direction of the spin of the deuteron. Then the initial $m_J = m_S = 1$, therefore we must have in the final state:

$$m_L + m_S = 1$$

For both neutrons spins directed opposite to that of the deuteron, $m_S = -1$ which implies $m_L = 2 > L$ which is impossible. Therefore, the required probability is zero. ■

(d) For a neutron with spin directed opposite to that of the deuteron, $m_S = 0$ which implies $m_L = 1$, thus the angular distribution is given by

$$|Y_{1,1}(\theta, \phi)|^2 = \frac{3}{8\pi} \sin^2 \theta \quad \blacksquare$$

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