Isomorphism and non-isomorphism for interval groups of type D_n

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Abstract

We consider presentations that were derived in [3] for the interval groups associated with proper quasi-Coxeter elements of the Coxeter group $W(D_n)$. We use combinatorial methods to derive alternative presentations for the groups, and use these new presentations to show that the interval group associated with a proper quasi-Coxeter element of $W(D_n)$ cannot be isomorphic to the Artin group of type D_n. While the specific problems we solve arise from the study of interval groups, their solution provides an illustration of how techniques indicated by computational observation can be used to derive properties of all groups within an infinite family.

1. Introduction

This article uses combinatorial methods to answer a question that arises from the results of [3]. In particular, we prove that the presentations in an infinite family associated with the Coxeter group $W(D_n)$ define groups that are not isomorphic to the Artin groups of the same type. Our proof of that non-isomorphism was driven by observations made during computation with those presentations, and provides an illustration of how a collection of computer results that strongly indicate the correctness of a general conjecture for infinite classes of groups can be used in writing a theoretical proof of those general results.

The presentations that we consider are those of interval groups associated with

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quasi-Coxeter elements of the Coxeter group $W(D_n)$. The article [3] constructs these presentations, defining each interval group as a one-relator quotient of an Artin group $A(\Delta)$, for a Carter diagram Δ associated with the associated conjugacy class of quasi-Coxeter elements. The question of whether a group so defined is isomorphic to the Artin group $A(D_n)$ is a natural one arising out of the theory of interval groups associated with Coxeter (rather then quasi-Coxeter) elements of $W(D_n)$ [5]. However the question could only be partially answered using that theory (an answer is provided for even n in [4]). The proof in this article, valid for all integers n, uses none of that theory, and works purely from the presentations derived in [3], using the method of Reidemeister-Schreier [10, Section 2.5].

The main results of this article are Theorem 1.1, which uses combinatorial methods to find alternative presentations for the interval groups considered in this article, as one-relator quotients of a different Artin group (the affine Artin group $A(\tilde{A}_{n-1})$), and Theorem 1.3, which uses those new presentations to prove that none of those interval groups is isomorphic to the Artin group of type D_n (for appropriate n). Our proof of Theorem 1.3 finds subgroups of groups in the two infinite families that would have to correspond under an isomorphism between a pair of groups, uses Reidemeister-Schreier techniques to compute presentations of those subgroups, and then demonstrates that these presentations have different abelianisations, thereby establishing the claimed non-isomorphism.

While the questions that are answered in this article arise from the study of interval groups in [3], that article provides motivation only, and no knowledge of it is necessary for understanding of this current article. We have chosen not to define Coxeter elements or quasi-Coxeter elements of the Coxeter group $W(D_n)$ in this article, or to explain the construction of the interval groups associated to those elements, because we do not need this information; we merely study presentations that are found in [3].

Each of the presentations from [3] that we study in this article is associated with a *Carter diagram* associated with the Coxeter group $W(D_n)$. Those Carter diagrams are classified in [7], where they are called *admissible diagrams*. It is the diagrams $D_n(a_i)$ $(1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1)$ of [7] that are associated with the quasi-Coxeter elements of $W(D_n)$.

In this article we use the notation $\Delta_{t,n}$ from [3] for the diagram with n vertices that is formed by attaching paths containing $t-1$ and $n-3-t$ edges to vertices at opposite ends of a diagonal of a square, where $1 \leq t \leq n-3$. For $1 \leq t \leq (n-2)/2$, this is the Carter diagram $D_n(a_t)$.

We use the notation Δ_n to denote the n-gon on n vertices; when n is even this is a Carter diagram, denoted by $D_n(b_{\frac{n}{2}-1})$ in [7]. We note that it is also the Coxeter diagram of affine type $\tilde{A}_{n-1}$

Each of the diagrams $\Delta_{t,n}$ can be found as the result of applying a sequence of mutations to the diagram D_n, and hence it is covered by [9, Theorem 1.1] and [8, Theorem A]. Those results prove that, for any diagram Γ' derived from a finite type Coxeter diagram Γ by a sequence of mutations, the Artin group $A(\Gamma)$ is isomorphic to the quotient of the Artin group $A(\Gamma')$ by the normal closure of a set of *cycle commutators* (defined in Section 2 below) associated with the set of chordless cycles of Γ', each of which relates generators that correspond to the vertices of a chordless cycle.

Each $\Delta_{t,n}$ contains a single 4-cycle, and Δ_n a single n-cycle. So each diagram $\Delta = \Delta_{t,n}$ or Δ_n provides us with a presentation of $A(D_n)$ as a one-relator quotient of $A(\Delta)$. Similarly, by [6, Theorem 3.10] (or alternatively by later work of [2] that is cited by [9]), the Coxeter group $W(D_n)$ can be found as a one-relator quotient of $W(\Delta)$.

In Section 3 of this paper, we re-prove some of the isomorphisms proved as corollaries of [9, Theorem 1.1], by presenting explicit isomorphisms, and use the same methods to prove the following new result. Note that we define *twisted cycle commutators* in Section 2 below. See also Figure 1 below for our numbering of the vertices of the Coxeter diagrams, and hence our labelling of the generators of the corresponding Artin group referred to in the theorem.

Theorem 1.1. *For each t with $1 \leq t \leq n-3$, the following two groups are isomorphic:*

(1) the quotient $Q = Q_{n,t}$ of $A(\Delta_{t,n})$ by the normal closure of the twisted cycle commutator $\mathrm{TC}(b_1, b_2, b_3, b_4)$, and

(2) the quotient $G = G_{n,t}$ of $A(\Delta_n)$ by the t-twisted cycle commutator $\mathrm{TC}(a_1, a_2, \ldots, a_n)_t$.

Remark 1.2. *It is clear, by rotating the diagram $\Delta_{t,n}$ through 180°, that $A(\Delta_{t,n}) \cong A(\Delta_{n-2-t,n})$, and it follows from Lemma 2.5 below that $Q_{n,t} \cong Q_{n,n-2-t}$ for all $1 \leq t \leq n-3$. So we could assume that $t \leq \lfloor (n-2)/2 \rfloor$, but this is not necessary for the proof.*

It follows from the theorem that the groups $G_{n,t}$ and $G_{n,n-2-t}$ are isomorphic; this can also be proved directly by applying the automorphism of $A(\Delta_n)$ induced by $a_i \mapsto a_i^{-1}$ for $1 \leq i \leq n$.

It is natural to ask if the groups Q and G have soluble word problems, but we have been unable to answer that question.

In Section 4 we prove Theorem 1.3 (stated below) that the groups $Q_{n,t}$ and $G_{n,t}$ are not isomorphic to the Artin group of type D_n. This theorem contrasts with the results of [9] that we have already mentioned, and which we re-prove in Section 3 as Propositions 3.1, 3.2 and 3.3, which find isomorphisms between $A(D_n)$ and each of the quotients $A(\Delta_{t,n})$ by the normal closure of the cycle commutator $\mathrm{CC}(b_1, b_2, b_3, b_4)$, and of $A(\Delta_n)$ by the normal closure of the cycle commutator $\textsc{cc}(a_1, a_2, \ldots, a_n)$. We note that each of these presentations for quotients by cycle commutators can be derived in a natural way as a presentation for the interval group of a Coxeter (rather than quasi-Coxeter) element of type D_n, on a subset of the set of all reflections (using methods described in [3]).

Theorem 1.3. *The groups $Q_{n,t}$ and $G_{n,t}$ referred to in Theorem 1.1 are not isomorphic to $A(D_n)$ for any n, t with $n \geq 4$ and $1 \leq t \leq n-3$.*

This result for n up to about 70 was originally proved by computer calculations, in which we found subgroups of the groups G of Theorem 1.1 and of $A(D_n)$ that would have to correspond under a putative isomorphism, but which had different abelianisations. We were able to observe the details of the steps in the computer calculations, and to use these to construct a proof for general n.

We note that the article [4] contains a proof that the groups $Q_{n,t}$ are non-isomorphic to $A(D_n)$ when n is even, but that proof does not extend to the case where n is odd. We have not resolved the question of whether, for a given value of n, the groups $G_{n,t}$ are isomorphic for different values of t, except for the isomorphism $G_{n,t} \cong G_{n,n-2-t}$ mentioned in the remark above.

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2. Notation and preliminary lemmas

Throughout this article we denote the identity element of a group by 1. Given words u, v we write $u = v$ to indicate that words are identical as strings, and $u =_G v$ to indicate that u, v represent the same element of the group G.

The labelling of the vertices in the diagrams that we shall now describe can be seen in Figure 1.

We label the vertices of the Coxeter diagram of D_n as $1, 2, \ldots, n$, with 1 and 2 labelling the two vertices at ends of the fork, 3 joined to both 1 and 2, and then $4, 5, 6, \ldots, n$ labelling the successive vertices from 3 to the end of the diagram. We label the generators of the Artin group $A(D_n)$ as $x_1, \ldots, x_n$. So x_1 and x_2 commute, x_1 and x_2 braid with x_3, the generator x_i braids with x_{i+1} for $3 \leq i \leq n-1$, and all other pairs of generators commute.

We label the vertices of the Δ_n diagram (this is $\tilde{A}_{n-1}$) as $1, 2, \ldots, n$ around the cycle. We label the generators of the Artin group $A(\Delta_n)$ as $a_1, \ldots, a_n$. So, for

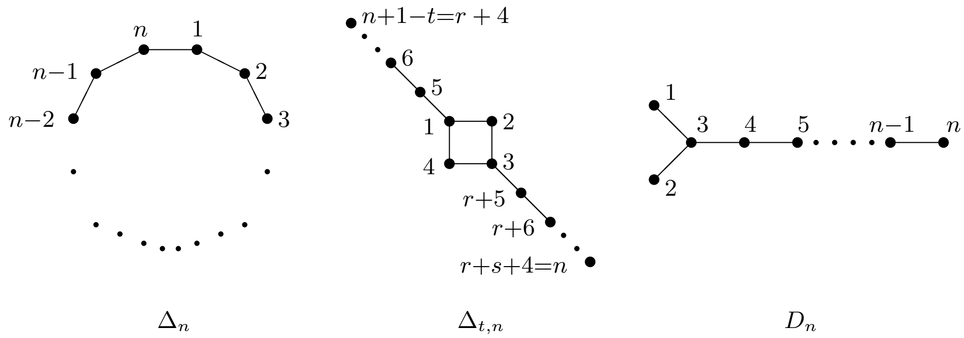

Figure 1: The labelled diagrams

$1 \leq i \leq n-1$, a_i braids with a_{i+1}, a_n braids with a_1 and all other pairs of generators commute.

We label the vertices of the $\Delta_{1,n}$ diagram $1, 2, \ldots, n$ with $1, 2, 3, 4$ around a square, and $1, 5, 6, \ldots, n$ labelling consecutive vertices on a path. We label the generators of the Artin group $A(\Delta_{1,n})$ as $b_1, \ldots, b_n$. So the pairs $\{b_1, b_2\}$, $\{b_2, b_3\}$, $\{b_3, b_4\}$, $\{b_4, b_1\}$, $\{b_1, b_5\}$ and then $\{b_i, b_{i+1}\}$ for $5 \leq i \leq n-1$ braid, and all other pairs of generators commute.

Define $r := n-3-t$, and $s := t-1$, so that the two arms of $\Delta_{t,n}$ contain r and s edges respectively. Some of our constructions may seem clearer when expressed in terms of the parameters r, s rather than t. For $t > 1$, we label the vertices of the $\Delta_{t,n}$ diagram $1, 2, \ldots, n$, with $1, 2, 3, 4$ around a square, and $1, 5, 6, \ldots, 4+r$ and $3, 5+r, \ldots, n = 4+r+s$ each labelling sequential vertices on a path (where we recall that $r = n-3-t$ and $s = t-1$). We label the generators of the Artin group $A(\Delta_{t,n})$ as $b_1, \ldots, b_{4+r}$, $c_{5+r}, \ldots, c_n$. So the pairs $\{b_1, b_2\}$, $\{b_2, b_3\}$, $\{b_3, b_4\}$, $\{b_4, b_1\}$; $\{b_1, b_5\}$ and $\{b_i, b_{i+1}\}$ for $5 \leq i \leq 3+r$; and $\{b_3, c_{5+r}\}$ and $\{c_i, c_{i+1}\}$ for $5+r \leq i \leq n-1$ braid, while all other pairs of vertices commute.

Definition 2.1. For $n \geq 3$, we define the cycle commutator $\textsc{cc}(y_1, y_2, \ldots, y_n)$ to be the commutator $[y_1, y_2 \ldots y_{n-1} y_n y_{n-1}^{-1} \cdots y_2^{-1}]$. So we have, for example, $\textsc{cc}(y_1, y_2, y_3, y_4) = [y_1, y_2 y_3 y_4 y_3^{-1} y_2^{-1}]$.

We define the twisted cycle commutator $\textsc{tc}(y_1, y_2, y_3, y_4)$ to be the commutator $[y_1, y_2^{-1} y_3 y_4 y_3^{-1} y_2]$ and, for $n \geq 4$ and $1 \leq t \leq n-2$, we define the t-twisted cycle commutator $\textsc{tc}(y_1, y_2, \ldots, y_n)_t$ to be the commutator

$$[y_1, y_2^{-1} \cdots y_{t+1}^{-1} y_{t+2} \cdots y_{n-1} y_n y_{n-1}^{-1} \cdots y_{t+2}^{-1} y_{t+1} \cdots y_2].$$

So in particular $\textsc{tc}(y_1, y_2, y_3, y_4)_1 = \textsc{tc}(y_1, y_2, y_3, y_4)$, and $\textsc{tc}(y_1, y_2, \ldots, y_n)_{n-2}$ is the commutator

$$[y_1, y_2^{-1} \cdots y_{n-1}^{-1} y_n y_{n-1} \cdots y_2].$$

In the above definitions, the reader should feel free to use whichever of the two possible definitions of the commutator $[g,h]$ they prefer - it will make no difference! We shall show in Lemmas 2.4 and 2.5 below that, in the context of our applications, these definitions have various equivalent formulations.

For group elements g, h, we will write $\mathrm{CM}(g,h)$ to mean that g and h commute (i.e. $gh =_G hg$) and $\mathrm{BR}(g,h)$ to mean that g and h braid (i.e. $ghg =_G hgh$.) We will sometimes write statements such as $\mathrm{BR}(g,h) = \mathrm{BR}(g',h')$. By this we mean that the two statements $\mathrm{BR}(g,h)$ and $\mathrm{BR}(g',h')$ are logically equivalent. We will sometimes refer to the *conjugate* of such a relation, by an element k; by this we mean a relation of the form $k^{-1}ghk =_G k^{-1}hgk$ or $k^{-1}ghgk =_G k^{-1}hghk$.

The following technical lemmas will be used repeatedly in Section 3.

Lemma 2.2. *Let G be a group, and suppose that $f, g, h \in G$ satisfy* $\mathrm{BR}(f,g)$, $\mathrm{BR}(g,h)$ *and* $\mathrm{CM}(f,h)$. *Then*

(i) $\mathrm{BR}(f, ghg^{-1})$, $\mathrm{BR}(f, g^{-1}hg)$, *and* $\mathrm{BR}(f, fghg^{-1}f^{-1})$;
(ii) $\mathrm{CM}(g, fghg^{-1}f^{-1})$, $\mathrm{CM}(f^{-1}gf, h^{-1}gh)$, *and* $\mathrm{CM}(fgf^{-1}, hgh^{-1})$.

Proof. Since $hghg^{-1}h^{-1} =_G g$ and h commutes with f, we have

$$\mathrm{BR}(f,g) \Rightarrow \mathrm{BR}(f, hghg^{-1}h^{-1}) \Rightarrow \mathrm{BR}(f, ghg^{-1})$$

and similarly $h^{-1}g^{-1}hgh =_G g$ implies $\mathrm{BR}(f, g^{-1}hg)$. Then $\mathrm{BR}(f, fghg^{-1}f^{-1})$ follows by conjugating $\mathrm{BR}(f, ghg^{-1})$ by f.

Furthermore, putting $w = fghg^{-1}f^{-1}$, we have

$$gw = gfghg^{-1}f^{-1} =_G fgfhg^{-1}f^{-1} =_G fghfg^{-1}f^{-1} =_G fghg^{-1}f^{-1}g = wg,$$

proving $\mathrm{CM}(g, fghg^{-1}f^{-1})$. Also, conjugating $\mathrm{CM}(f,h)$, we derive the two equalities:

$$\begin{aligned} \mathrm{CM}(g^{-1}fg, g^{-1}hg) &= \mathrm{CM}(fgf^{-1}, hgh^{-1}), \\ \mathrm{CM}(gfg^{-1}, ghg^{-1}) &= \mathrm{CM}(f^{-1}gf, h^{-1}gh). \end{aligned}$$

∎

Lemma 2.3. *Let G be a group, and suppose that $y_1, \ldots, y_k \in G$ (with $k \geq 2$) satisfy* $\mathrm{BR}(y_i, y_{i+1})$ *for* $1 \leq i < k$ *and* $\mathrm{CM}(y_i, y_j)$ *for* $1 \leq i < j-1 \leq k-1$ *(as in the braid group). Let* $w := y_1 y_2 \cdots y_{k-1} y_k y_{k-1}^{-1} \cdots y_2^{-1} y_1^{-1}$. *Then*

(i) $w =_G y_k^{-1} y_{k-1}^{-1} \cdots y_2^{-1} y_1 y_2 \cdots y_{k-1} y_k$;
(ii) $\mathrm{BR}(y_1, w)$, $\mathrm{BR}(y_k, w)$, *and (if* $k \geq 3$*)* $\mathrm{BR}(y_k, v)$, *where* $v := y_{k-1}^{-1} \cdots y_2^{-1} y_1 y_2 \cdots y_{k-1}$;
(iii) $\mathrm{CM}(y_i, w)$ *for* $2 \leq i \leq k-1$.

Proof. (i) We prove this by induction on k. It follows directly from $\mathrm{BR}(y_1, y_2)$ when $k = 2$. For $k > 2$, we use $\mathrm{BR}(y_{k-1}, y_k)$ to replace the subword $y_{k-1} y_k y_{k-1}^{-1}$

of w by $y_k^{-1}y_{k-1}y_k$, and then use $\mathrm{CM}(y_i, y_k)$ for $1 \leq i \leq k-2$ and induction to complete the proof.

(ii) To get $\mathrm{BR}(y_k, w)$, note that $\mathrm{BR}(y_k, y_{k-1})$ implies $\mathrm{BR}(y_k, y_k^{-1}y_{k-1}y_k) = \mathrm{BR}(y_k, y_{k-1}y_ky_{k-1}^{-1})$, and conjugating this by $y_1 \cdots y_{k-2}$, which commutes with y_k, gives $\mathrm{BR}(y_k, w)$. This implies $\mathrm{BR}(y_k, y_kwy_k^{-1}) = \mathrm{BR}(y_k, v)$. Similarly, it follows from $\mathrm{BR}(y_1, y_2)$ that $\mathrm{BR}(y_1, y_1y_2y_1^{-1}) = \mathrm{BR}(y_1, y_2^{-1}y_1y_2)$ and (when $k \geq 3$) conjugating by $y_k^{-1} \cdots y_3^{-1}$ gives $\mathrm{BR}(y_1, y_k^{-1} \cdots y_2^{-1}y_1y_2 \cdots y_k)$, and then $\mathrm{BR}(y_1, w)$ follows from Part (i). When $k = 2$, $\mathrm{BR}(y_1, w)$ follows from $\mathrm{BR}(y_1, y_2)$.

(iii) For a given i with $2 \leq i \leq k-1$, let $f = y_{i-1}$, $g = y_i$, and $h = y_{i+1} \cdots y_{k-1}y_ky_{k-1}^{-1} \cdots y_{i+1}^{-1}$. Then we have $\mathrm{CM}(f, h)$ and $\mathrm{BR}(f, g)$, and $\mathrm{BR}(g, h)$ follows from (ii), so $\mathrm{CM}(y_i, y_{i-1}y_i \cdots y_{k-1}y_ky_{k-1}^{-1} \cdots y_i^{-1}y_{i-1}^{-1})$ follows from Lemma 2.2 (ii). Conjugating by $y_1 \cdots y_{i-2}$ gives $\mathrm{CM}(y_i, w)$. □

The next lemma is also proved as [8, Lemma 2.4]. We repeat the proof here because we will need to use essentially the same argument in the following lemma on twisted cycle commutators.

Lemma 2.4. *Let $n \geq 3$, and suppose that $a_1, a_2, \ldots, a_n$ are elements of a group G that satisfy the defining relations of $A(\Delta_n)$. Then the n properties*

$$\mathrm{CC}(a_1, a_2, \ldots, a_n) =_G 1,\ \mathrm{CC}(a_2, a_3, \ldots, a_1) =_G 1, \ldots, \mathrm{CC}(a_n, a_1, \ldots, a_{n-1}) =_G 1$$

are all equivalent.

Proof. We have

$$\begin{aligned}
\mathrm{CC}(a_1, a_2, \ldots, a_n) =_G 1 &\quad \Rightarrow \\
\mathrm{CM}(a_1, a_2 \ldots a_{n-1}a_na_{n-1}^{-1} \ldots a_2^{-1}) &\quad \Rightarrow \\
\mathrm{CM}(a_na_1a_n^{-1}, a_na_2 \ldots a_{n-1}a_na_{n-1}^{-1} \ldots a_2^{-1}a_n^{-1}) &\quad \Rightarrow \\
\mathrm{CM}(a_na_1a_n^{-1}, a_2 \ldots a_{n-2}a_na_{n-1}a_na_{n-1}^{-1}a_n^{-1}a_{n-2}^{-1} \ldots a_2^{-1}) &\quad =_G \\
\mathrm{CM}(a_na_1a_n^{-1}, a_2 \ldots a_{n-2}a_{n-1}a_{n-2}^{-1} \ldots a_2^{-1}) &\quad \Rightarrow \\
\mathrm{CM}(a_1a_na_1a_n^{-1}a_1^{-1}, a_1a_2 \ldots a_{n-2}a_{n-1}a_{n-2}^{-1} \ldots a_2^{-1}a_1^{-1}) &\quad =_G \\
\mathrm{CM}(a_n, a_1a_2 \ldots a_{n-2}a_{n-1}a_{n-2}^{-1} \ldots a_2^{-1}a_1^{-1}) &
\end{aligned}$$

and so $\mathrm{CC}(a_n, a_1, \ldots, a_{n-1}) =_G 1$, and we derive the equivalence of the properties by iterating this argument. □

Lemma 2.5. *Let $n \geq 3$ and $1 \leq t \leq n-2$, and suppose that $a_1, a_2, \ldots, a_n$ are elements of a group G that satisfy the defining relations of $A(\Delta_n)$. Then the following statements hold.*

(i) The n properties

$$\begin{aligned}
\mathrm{TC}(a_1, a_2, \ldots, a_n)_t &=_G 1,\ \mathrm{TC}(a_2, a_3, \ldots, a_1)_t =_G 1, \ldots, \\
\mathrm{TC}(a_n, a_1, \ldots, a_{n-1})_t &=_G 1
\end{aligned}$$

are all equivalent.

(ii) Let $I := \{2,3,\ldots,n-1\}$ and let S be a subset of I with $|S| = t$. Define ϵ_i for $i \in I$ by $\epsilon_i = -1$ if $i \in S$ and $\epsilon_i = 1$ if $i \notin S$.

$$\textsc{tc}(a_1,a_2,\ldots,a_n)_t =_G 1 \iff [a_1, a_2^{\epsilon_2}\cdots a_{n-1}^{\epsilon_{n-1}} a_n a_{n-1}^{-\epsilon_{n-1}} \cdots a_2^{-\epsilon_2}] =_G 1.$$

Proof. We prove (ii) first. There is nothing to prove if $t = n-2$, so suppose that $t < n-2$ (and hence $n \geq 4$). Let $k \in I$ with $k < n-2$, and for $i \in I \setminus \{k, k+1\}$, choose $\epsilon_i = \pm 1$. Define

$$\begin{aligned} c_1 &:= a_2^{\epsilon_2}\cdots a_{k-1}^{\epsilon_{k-1}} a_k^{-1} a_{k+1} a_{k+2}^{\epsilon_{k+2}} \cdots a_{n-1}^{\epsilon_{n-1}}, \\ c_2 &:= a_2^{\epsilon_2}\cdots a_{k-1}^{\epsilon_{k-1}} a_k a_{k+1}^{-1} a_{k+2}^{\epsilon_{k+2}} \cdots a_{n-1}^{\epsilon_{n-1}}. \end{aligned}$$

We shall show below that, for any choice of k and ϵ_i, we have

$$[a_1, c_1 a_n c_1^{-1}] =_G 1 \iff [a_1, c_2 a_n c_2^{-1}] =_G 1. \quad (*)$$

Once $(*)$ is proved, (ii) will follow. For suppose S of size t is given. There must exist $i, i+1 \in I$ such that exactly one of i and $i+1$ is in S, and then the set S' formed from S by deleting from S one of those elements of $\{i, i+1\}$ and adding in the other one, also has size t; a sequence of such 'moves' can be found to define a sequence of sets $S = S_0, S_1, \ldots S_n = \{2,3,\ldots,t+1\}$, for which any two sets that are adjacent in the sequence differ by a single pair of elements $i, i+1$ of I. We associate the commutator $[a_1, a_2^{\epsilon_2}\cdots a_{n-1}^{\epsilon_{n-1}} a_n a_{n-1}^{-\epsilon_{n-1}} \cdots a_2^{-\epsilon_2}]$ with the set S and $\textsc{tc}(a_1,a_2,\ldots,a_n)_t$ with the set $\{2,\ldots,t+1\}$. We use $(*)$ to prove equivalence of the relations defined by commutators associated with successive terms in that sequence, and then (ii) follows.

To avoid horrific notation, we shall write out the proof of $(*)$ in the specific case $n = 6$, $k = 3$, $t = 2$, $\epsilon_2 = 1$, $\epsilon_5 = -1$, which we hope will make the general case clear. In this case $c_1 = a_2 a_3^{-1} a_4 a_5^{-1}$ and $c_2 = a_2 a_3 a_4^{-1} a_5^{-1}$. Using the braid and commutativity relations in G, we have

$$\begin{aligned} [a_1, c_1 a_n c_1^{-1}] = [a_1, a_2 a_3^{-1} a_4 a_5^{-1} a_6 a_5 a_4^{-1} a_3 a_2^{-1}] =_G 1 &\iff \\ [a_1, a_4^{-1} a_2 a_3^{-1} a_4 a_5^{-1} a_6 a_5 a_4^{-1} a_3 a_2^{-1} a_4] =_G 1 &\iff \\ [a_1, a_2 a_4^{-1} a_3^{-1} a_4 a_5^{-1} a_6 a_5 a_4^{-1} a_3 a_4 a_2^{-1}] =_G 1 &\iff \\ [a_1, a_2 a_3 a_4^{-1} a_3^{-1} a_5^{-1} a_6 a_5 a_3 a_4 a_3^{-1} a_2^{-1}] =_G 1 &\iff \\ [a_1, a_2 a_3 a_4^{-1} a_5^{-1} a_6 a_5 a_4 a_3^{-1} a_2^{-1}] = [a_1, c_2 a_n c_2^{-1}] =_G 1. & \end{aligned}$$

For (i) with $t < n-2$, we can prove as in the proof of Lemma 2.4 that $\textsc{tc}(a_1,a_2,\ldots,a_n)_t =_G 1$ if and only if

$$\mathrm{CM}(a_n, a_1 a_2^{-1} \cdots a_{t+1}^{-1} a_{t+2} \cdots a_{n-2} a_{n-1} a_{n-2}^{-1} \cdots a_{t+2}^{-1} a_{t+1} \cdots a_2 a_1^{-1}),$$

and the result now follows from (ii). The proof when $t = n-2$ is similar, but in that case we conjugate the commuting relation by a_n^{-1} and then by a_1^{-1} rather than by a_n and a_1. $\square$

3. Specific transformations to show isomorphisms between $A(D_n)$ and the quotients of $A(\Delta)$

Although Propositions 3.1, 3.2, and 3.3 can be found as corollaries of [9, Theorem 1.1], we have provided our own proofs, in order to see explicit isomorphisms. Our construction of the proofs of those propositions also helped us to construct the proof of Theorem 1.1, which is a new result.

Our proofs of these three propositions and of Theorem 1.1 below all have the following structure. We are trying to establish that two groups A and B defined by presentations on generators $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ are isomorphic. To do this, we first define homomorphisms $\phi : A \to B$ and $\psi : B \to A$, and then prove that ϕ and ψ are mutually inverse, which implies that they are isomorphisms. We construct ϕ by first defining $\phi(\alpha_1), \ldots, \phi(\alpha_n)$ as words over the β_i (and we abuse notation by denoting these words by $\alpha_1, \ldots, \alpha_n$) and then verify that these images satisfy the defining relations of the group A, which proves that the map ϕ defined on the generators extends to a homomorphism $A \to B$. The homomorphism $\psi : B \to A$ is defined similarly. To prove that ϕ and ψ are mutually inverse, it is sufficient to verify this on the generators; that is, we show that $\psi(\phi(\alpha_i)) = \alpha_i$ and $\phi(\psi(\beta_i)) = \beta_i$ for $1 \le i \le n$.

Proposition 3.1. *If $n \ge 4$, then the Artin group $G := A(D_n)$ is isomorphic to the quotient Q of the Artin group $A(\Delta_n)$ on generators $a_1, \ldots, a_n$ by the normal closure of the cycle commutator $\mathrm{CC}(a_1, a_2, \ldots, a_n)$.*

Proof. Where $x_1, \ldots, x_n$ are generators of the group G in its standard presentation, we define elements $a_1, \ldots, a_n$ of G as words over $x_1, \ldots, x_n$ as follows.

$$a_1 := x_1,\ a_2 := x_3,\ a_3 := x_4, \ldots, a_{n-1} := x_n,$$
$$a_n := x_n^{-1} x_{n-1}^{-1} \cdots x_3^{-1} x_2 x_3 \cdots x_{n-1} x_n.$$

Given the Artin relations between the x_i, we need to verify that all of the relations of $A(\Delta_n)$ hold between the a_i, and also verify the cycle commutator relation $\mathrm{CC}(a_1, a_2, \ldots, a_n) =_G 1$. That will prove that the map from Q to G defined by mapping each generator a_i of Q to the element a_i of G that we have just defined extends to a homomorphism from Q to G.

We have

$$a_n = a_{n-1}^{-1} a_{n-2}^{-1} \cdots a_2^{-1} x_2 a_2 \cdots a_{n-2} a_{n-1} \text{ so } x_2 = a_2 a_3 \cdots a_{n-1} a_n a_{n-1}^{-1} \cdots a_2^{-1},$$

and $\mathrm{CC}(a_1, a_2, \ldots, a_n) =_G 1$ follows immediately from $\mathrm{CM}(x_1, x_2)$.

We see that, since $x_4, \ldots, x_n$ all commute with x_1, the relator $\mathrm{BR}(a_1, a_n)$ is equivalent to $\mathrm{BR}(x_1, x_3^{-1} x_2 x_3)$, and that this (conjugating) is equivalent to $\mathrm{BR}(x_2, x_3 x_1 x_3^{-1})$, which follows from Lemma 2.2 (i) applied with $f = x_2$, $g = x_3$, $h = x_1$.

The other relations are immediate apart from those that involve a_n: i.e. $\mathrm{BR}(a_{n-1}, a_n)$ and $\mathrm{CM}(a_i, a_n)$ for $2 \le i \le n-2$. They all follow from Lemma 2.3 (ii) and (iii).

Conversely, given generators $a_1, \dots, a_n$ of Q, we define elements $x_1, \dots, x_n$ of Q as products of those.

$$x_1 := a_1,\ x_2 := a_2 a_3 \cdots a_{n-1} a_n a_{n-1}^{-1} \cdots a_3^{-1} a_2^{-1}, x_3 := a_2, \dots, x_n := a_{n-1}$$

Then we need to verify that all the relations of G hold between the x_i. That will prove that the map from G to Q defined by mapping each generator x_i of G to the element x_i of Q that we have just defined extends to a homomorphism from G to Q.

The relation $\mathrm{CM}(x_1, x_2)$ is precisely $\mathrm{CC}(a_1, a_2, \cdots, a_n) =_Q 1$. All other relations not involving x_2 follow immediately from those of the n-gon. So that leaves $\mathrm{BR}(x_2, x_3)$ and $\mathrm{CM}(x_2, x_k)$ for $4 \le k \le n$. These all follow from Lemma 2.3 (ii) and (iii).

We also need to check that the two maps between the two groups are mutually inverse, but this is clear from their actions on the generators. $\square$

Proposition 3.2. *If $n \ge 4$, then the Artin group $G := A(D_n)$ is isomorphic to the quotient Q of the Artin group $A(\Delta_{1,n})$ on generators $b_1, \dots, b_n$ by the normal closure of the cycle commutator $\mathrm{CC}(b_1, b_2, b_3, b_4)$.*

Proof. Where $x_1, \dots, x_n$ are the generators of G in its standard presentation we define elements $b_1, \dots, b_n$ of G as products of those as follows.

$$b_1 := x_4,\ b_2 := x_3,\ b_3 := x_1,\ b_4 := x_1^{-1} x_3^{-1} x_2 x_3 x_1,$$
$$b_k := x_k \text{ for } 5 \le k \le n.$$

We need now to check that the relations of Q hold between these elements of G. All relations of Q involving b_k for $k \ge 5$ are immediate, as are $\mathrm{BR}(b_1, b_2)$, $\mathrm{BR}(b_2, b_3)$ and $\mathrm{CM}(b_1, b_3)$ whereas, since b_4 freely reduces to $x_2 x_3 x_1 x_3^{-1} x_2^{-1}$ by Lemma 2.3 (i), $\mathrm{BR}(b_3, b_4)$ and $\mathrm{CM}(b_2, b_4)$ come from Lemma 2.3 (ii) and (iii). Furthermore, since $b_4 =_G x_2 x_1^{-1} x_3 x_1 x_2^{-1}$ and x_1 and x_2 commute with x_4, we see that $\mathrm{BR}(b_4, b_1)$ follows from $\mathrm{BR}(x_3, x_4)$, whereas, since $b_2 b_3 b_4 b_3^{-1} b_2^{-1} =_G x_2$, $\mathrm{CC}(b_1, b_2, b_3, b_4) =_G 1$ follows immediately from $\mathrm{CM}(x_2, x_4)$.

In the other direction, we define elements x_i of Q in terms of the generators b_i of Q as follows.

$$x_1 := b_3,\ x_2 := b_2 b_3 b_4 b_3^{-1} b_2^{-1},\ x_3 := b_2,\ x_4 := b_1,$$
$$x_k := b_k \text{ for } 5 \le k \le n.$$

All relations of G involving x_k for $k \ge 5$ are immediate, as are $\mathrm{BR}(x_1, x_3)$, $\mathrm{BR}(x_4, x_3)$ and $\mathrm{CM}(x_1, x_4)$, whereas $\mathrm{BR}(x_2, x_3)$ and $\mathrm{CM}(x_1, x_2)$ come from

Lemma 2.3 (ii) and (iii). Finally, $\mathrm{CM}(x_2, x_4)$ is precisely the relation $\textsc{cc}(b_1, b_2, b_3, b_4) =_G 1$.

The fact that the two homomorphisms that we have defined are mutually inverse follows immediately from their definitions. ☐

Proposition 3.3. *For $1 \le t \le n-3$, the quotient Q_1 of the Artin group $A(\Delta_{1,n})$ on generators $b_1, \ldots, b_n$ by the normal closure of the cycle commutator $\textsc{cc}(b_1, b_2, b_3, b_4)$ is isomorphic to the quotient Q_2 of the Artin group $A(\Delta_{t,n})$ on generators $b_1, \ldots, b_{4+r}, c_{5+r}, \ldots, c_n$ by the normal closure of the cycle commutator $\textsc{cc}(b_1, b_2, b_3, b_4)$.*

Proof. In the homomorphisms that we shall define between these groups, the generators b_k in the two groups will correspond for $1 \le k \le r+4$, that is each map will map each such generator b_k of the first group to the generator b_k of the second group, so there should be no danger of confusion. In order to find a homomorphism from Q_2 to Q_1, we define elements c_k of Q_1 as products of the generators $b_1, \ldots, b_n$ of Q_1 as follows.

$$c_{5+r} := b_{5+r} b_{4+r} \cdots b_5 b_1 b_2 b_3 b_2^{-1} b_1^{-1} b_5^{-1} \cdots b_{5+r}^{-1}$$

and $c_k := b_k$ for $6 + r \le k \le n$. Note that, by Lemma 2.3 (i), we have

$$c_{5+r} =_{Q_1} b_3^{-1} b_2^{-1} b_1^{-1} b_5^{-1} \cdots b_{4+r}^{-1} b_{5+r} b_{4+r} \cdots b_5 b_1 b_2 b_3.$$

We only need to verify the relations of Q_2 that involve c_{5+r}. Of these $\mathrm{CM}(c_k, c_{5+r})$ is immediate for $k > 6 + r$. Since $b_{4+r} \cdots b_5 b_1 b_2 b_3$ commutes with $c_{6+r} = b_{6+r}$ (when $s > 1$), $\mathrm{BR}(c_{6+r}, c_{5+r})$ follows from $\mathrm{BR}(b_{6+r}, b_{5+r})$, whereas $\mathrm{BR}(b_3, c_{5+r})$ and $\mathrm{CM}(b_k, c_{5+r})$ for $k = 1, 2$ and $5 \le k \le 4+r$ follow from Lemma 2.3 (ii) and (iii).

Finally, applying Lemma 2.4 to the cycle commutator relation $\textsc{cc}(b_1, b_2, b_3, b_4) =_{Q_1} 1$ gives $\mathrm{CM}(b_4, b_1 b_2 b_3 b_2^{-1} b_1^{-1})$ and, since b_4 commutes with $b_{5+r} \cdots b_5$, we get $\mathrm{CM}(b_4, c_{5+r})$.

In the other direction, in order to define a homomorphism from Q_1 to Q_2, we define elements b_k of Q_2 for $k \ge 5 + r$ as follows.

$$b_{5+r} := b_{4+r} b_{3+r} \cdots b_5 b_1 b_2 b_3 c_{5+r} b_3^{-1} b_2^{-1} b_1^{-1} b_5^{-1} \cdots b_{4+r}^{-1}$$

and $b_k := c_k$ for $6 + r \le k \le n$. Note that, by Lemma 2.3 (i), we have

$$b_{5+r} =_{Q_2} c_{5+r}^{-1} b_3^{-1} b_2^{-1} b_1^{-1} b_5^{-1} \cdots b_{3+r}^{-1} b_{4+r} b_{3+r} \cdots b_5 b_1 b_2 b_3 c_{5+r}.$$

We only need to verify the relations of Q_1 that involve b_{5+r}. Of these $\mathrm{CM}(b_k, b_{5+r})$ is immediate for $k > 6 + r$. Since $b_{4+r} \cdots b_5 b_1 b_2 b_3$ commutes with $c_{6+r} = b_{6+r}$, $\mathrm{BR}(b_{6+r}, b_{5+r})$ follows from $\mathrm{BR}(c_{6+r}, c_{5+r})$, whereas $\mathrm{BR}(b_{4+r}, b_{5+r})$, $\mathrm{BR}(b_{6+r}, b_{5+r})$, and $\mathrm{CM}(b_k, b_{5+r})$ for $k = 1, 2, 3$ and $5 \le k \le 3 + r$ follow from Lemma 2.3 (ii) and (iii).

We can also use Lemma 2.3 (i) to give

$$b_{5+r} =_{Q_2} c_{5+r}^{-1} b_{4+r} b_{3+r} \cdots b_5 b_1 b_2 b_3 b_2^{-1} b_1^{-1} b_5^{-1} \cdots b_{3+r}^{-1} c_{5+r},$$

and then, since $c_{5+r}^{-1} b_{4+r} \cdots b_5$ commutes with b_4, $\mathrm{CM}(b_4, b_{5+r})$ follows from $\mathrm{CM}(b_4, b_1 b_2 b_3 b_2^{-1} b_1^{-1})$.

It follows immediately from their definitions and Lemma 2.3 (i), that the two homomorphisms that we have defined are mutually inverse. □

Proof of Theorem 1.1. Let $a_1, \ldots, a_n$ denote the generators of G. The presentations for G and Q clearly match when $n = 4$, so we may assume that $n \geq 5$. Then we define elements $b_1, \ldots, b_{r+4}, c_{5+r}, \ldots, c_n$ of G as products of the generators $a_1, \ldots, a_n$, as follows.

$$\begin{aligned}
b_1 &:= a_2 a_3 \cdots a_{r+1} a_{r+2} a_{r+1}^{-1} \cdots a_3^{-1} a_2^{-1}, \\
b_2 &:= a_2 \cdots a_{r+2} a_{r+3}^{-1} \cdots a_{n-1}^{-1} a_n a_{n-1} \cdots a_{r+3} a_{r+2}^{-1} \cdots a_2^{-1}, \\
b_3 &:= a_{r+4}^{-1} \cdots a_{n-1}^{-1} a_n a_{n-1} \cdots a_{r+4} \ (= a_n \text{ when } s = 0), \\
b_4 &:= a_1, \\
b_k &:= a_{r+7-k} \text{ for } 5 \leq k \leq r+4, \\
c_k &:= a_{k-1} \text{ for } 5 + r \leq k \leq n\ .
\end{aligned}$$

We need to verify that the relations of Q hold between the elements $b_1, \ldots, b_{4+r}$, $c_{r+5}, \ldots, c_n$. The relations of Q among the b_j and c_k that do not involve b_1, b_2 or b_3 are all immediate, as are $\mathrm{CM}(b_1, b_3)$, $\mathrm{CM}(b_1, c_k)$ for $5 + r \leq k \leq n$ and $\mathrm{CM}(b_3, b_k)$ for $5 \leq k \leq r + 4$.

Also, since a_1 commutes with $a_{n-1} \cdots a_{r+4}$, $\mathrm{BR}(b_3, b_4)$ follows immediately from $\mathrm{BR}(a_1, a_n)$. Similarly, since $b_1 =_G a_{r+2}^{-1} \cdots a_3^{-1} a_2 a_3 \cdots a_{r+2}$ by Lemma 2.3 (i), we see that $\mathrm{BR}(b_1, b_4)$ follows from $\mathrm{BR}(a_1, a_2)$ and $\mathrm{CM}(a_1, a_3 \cdots a_{r+2})$.

Now $\mathrm{BR}(b_1, b_5)$, $\mathrm{CM}(b_1, b_k)$ for $6 \leq k \leq r + 4$, $\mathrm{BR}(b_3, c_{5+r})$, and $\mathrm{CM}(b_3, c_k)$ for $6 + r \leq k \leq n$ all follow from Lemma 2.3 (ii) and (iii).

That leaves the relations involving b_2. Let $g := a_{3+r}^{-1} \cdots a_{n-1}^{-1} a_n a_{n-1} \cdots a_{r+3}$. Then, since

$$b_1 = a_2 \cdots a_{r+1} a_{r+2} a_{r+1}^{-1} \cdots a_2^{-1} \text{ and } b_2 = a_2 \cdots a_{r+1} a_{r+2} g a_{r+2}^{-1} a_{r+1}^{-1} \cdots a_2^{-1},$$

$\mathrm{BR}(b_1, b_2)$ is equivalent to $\mathrm{BR}(a_{r+2}, a_{r+2} g a_{r+2}^{-1})$ and hence to $\mathrm{BR}(a_{r+2}, g)$, which follows from Lemma 2.3 (ii).

To verify $\mathrm{BR}(b_2, b_3)$, again note that $b_2 = a_2 \cdots a_{r+2} g a_{r+2}^{-1} \cdots a_2^{-1}$ Then, since $b_3 =_G a_{r+3} g a_{r+3}^{-1}$ and $a_2, \ldots, a_{r+2}$ commute with b_3, we see that $\mathrm{BR}(b_2, b_3)$ is equivalent to $\mathrm{BR}(g, a_{r+3} g a_{r+3}^{-1})$. Now Lemma 2.3 (ii) gives $\mathrm{BR}(a_{r+3}, g)$, so $a_{r+3}ga_{r+3}^{-1} =_G g^{-1}a_{r+3}g$, from which $\mathrm{BR}(g, a_{r+3}ga_{r+3}^{-1})$ (and hence also $\mathrm{BR}(b_2, b_3)$) follows immediately.

By Lemma 2.5 (ii), the relation $\mathrm{CM}(b_2, b_4)$ follows from the relator $\textsc{tc}(a_1, a_2, \ldots, a_n)_{s+1}$ of G.

As we saw above, Lemma 2.3 (ii) gives $\mathrm{BR}(a_{r+2}, g)$ and, since we also have $\mathrm{CM}(a_i, g)$ for $2 \leq i \leq r+1$, we can apply Lemma 2.3 to the sequence $a_2, \ldots, a_{r+2}, g$, and then Lemma 2.3 (iii) implies $\mathrm{CM}(b_2, b_k)$ for $5 \leq k \leq r+4$. Furthermore (since $a_2 \cdots a_{r+2}$ commutes with c_k), $\mathrm{CM}(b_2, c_k)$ is equivalent to $\mathrm{CM}(g, c_k)$ for $5 + r \leq k \leq n$, which also follows from Lemma 2.3 (iii).

It remains to verify that $\textsc{tc}(b_1, b_2, b_3, b_4)_1 =_G 1$. Since b_3 commutes with a_k for $2 \leq k \leq a_{r+2}$, and $\mathrm{BR}(b_3, a_{r+3})$ by Lemma 2.3 (ii), while also $b_2 = a_2 \cdots a_{r+2}(a_{r+3}^{-1}b_3a_{r+3})a_{r+2}^{-1} \cdots a_2^{-1}$, we have

$$\begin{aligned} b_2b_3b_2^{-1} &=_G a_2 \cdots a_{r+2}(a_{r+3}^{-1}b_3a_{r+3})b_3(a_{r+3}^{-1}b_3^{-1}a_{r+3})a_{r+2}^{-1} \cdots a_2^{-1} \\ &=_G a_2 \cdots a_{r+2}a_{r+3}a_{r+2}^{-1} \cdots a_2^{-1} \end{aligned}$$

Now, using this expression for $b_2b_3b_2^{-1}$, we find that $b_1^{-1}b_2b_3b_2^{-1}b_1$ reduces to a_{r+3} using free reduction and the commuting relations between a_{r+3} and $a_2, \ldots, a_{r+1}$. Hence $\mathrm{CM}(b_4, b_1^{-1}b_2b_3b_2^{-1}b_1)$ follows from $\mathrm{CM}(a_1, a_{r+3})$, and now $\textsc{tc}(b_1, b_2, b_3, b_4)_1 =_G 1$ follows from this, by Lemma 2.5 (i).

In the other direction, let $b_1, \ldots, c_n$ denote the generators of Q, and define elements $a_1, \ldots, a_n$ in Q as follows.

$$\begin{aligned} &a_1 := b_4, \\ &a_2 := b_{r+4}b_{r+3} \cdots b_5b_1b_5^{-1} \cdots b_{r+3}^{-1}b_{r+4}^{-1}, \\ &a_k := b_{r+7-k} \text{ for } 3 \leq k \leq r+2, \\ &a_{r+3} := b_1^{-1}b_2b_3b_2^{-1}b_1, \\ &a_k := c_{k+1} \text{ for } r+4 \leq k \leq n-1, \\ &a_n := c_n \cdots c_{r+5}b_3c_{r+5}^{-1} \cdots c_n^{-1} \ (= b_3 \text{ when } s = 0). \end{aligned}$$

We need to verify that the relations of G hold between the elements $a_1, \ldots, a_n$ of Q.

The relations of G among the a_i that do not involve a_2, a_{r+3} or a_n are all immediate, as are $\mathrm{CM}(a_2, a_k)$ for $r+4 \leq k \leq n-1$, $\mathrm{CM}(a_n, a_k)$ for $3 \leq k \leq r+2$, and $\mathrm{CM}(a_{r+3}, a_k)$ for $3 \leq k \leq r+1$ and $r+5 \leq k \leq n-1$.

Now $\mathrm{BR}(a_1, a_n)$ follows from $\mathrm{BR}(b_4, b_3)$ since b_4 commutes with $c_n \cdots c_{r+5}$, and $\mathrm{CM}(a_2, a_n)$ follows similarly from $\mathrm{CM}(b_1, b_3)$, whereas $\mathrm{CM}(a_1, a_{r+3})$ comes from $\textsc{tc}(b_1, b_2, b_3, b_4) =_G 1$ and Lemma 2.5 (i).

Since a_{r+3} commutes with $b_{r+4} \cdots b_6$, $\mathrm{CM}(a_2, a_{r+3})$ reduces to $\mathrm{CM}(b_5 b_1 b_5^{-1}, a_{r+3})$, which is equivalent to $\mathrm{CM}(b_1^{-1} b_5 b_1, a_{r+3})$, and this follows from $\mathrm{CM}(b_5, b_2 b_3 b_2^{-1})$. Similarly, since a_1 commutes with $b_{r+4} \cdots b_5$, $\mathrm{BR}(a_1, a_2)$ reduces to $\mathrm{BR}(b_4, b_1)$.

Next we see that $\mathrm{BR}(a_2, a_3)$ comes from Lemma 2.3 (ii), and $\mathrm{CM}(a_2, a_k)$ for $4 \le k \le r+2$ come from Lemma 2.3 (iii). Similarly, $\mathrm{BR}(a_{n-1}, a_n)$ comes from Lemma 2.3 (ii) and $\mathrm{CM}(a_k, a_n)$ for $r+4 \le k \le n-2$ come from Lemma 2.3 (iii).

Furthermore, we have $\mathrm{BR}(b_1, b_2 b_3 b_2^{-1})$ by Lemma 2.2 (i) and, since we also have $\mathrm{CM}(b_5, b_2 b_3 b_2^{-1})$, we get $\mathrm{BR}(a_{r+2}, a_{r+3}) = \mathrm{BR}(b_5, b_1^{-1} b_2 b_3 b_2^{-1} b_1)$ by applying Lemma 2.2 (i) with $f = b_5$, $g = b_1$ and $h = b_2 b_3 b_2^{-1}$.

From $\mathrm{BR}(c_{r+5}, b_3)$ and $\mathrm{CM}(c_{r+5}, b_1^{-1} b_2)$, we get $\mathrm{BR}(c_{r+5}, b_1^{-1} b_2 b_3 b_2^{-1} b_1) = \mathrm{BR}(a_{r+4}, a_{r+3})$.

Since b_1 commutes with a_n and $b_2 b_3 b_2^{-1}$ commutes with $c_n \cdots c_{r+6}$, $\mathrm{CM}(a_n, a_{r+3})$ reduces to $\mathrm{CM}(c_{r+5} b_3 c_{r+5}^{-1}, b_2 b_3 b_2^{-1})$, which we get by applying Lemma 2.2 (ii) with $f = c_{r+5}$, $g = b_3$ and $h = b_2$.

It remains to check that $\textsc{tc}(a_1, a_2, \ldots, a_n)_{s+1} =_G 1$. We find that

$$a_2 \cdots a_{r+2} a_{r+3}^{-1} \cdots a_{n-1}^{-1} a_n a_{n-1} \cdots a_{r+3} a_{r+2}^{-1} \cdots a_2^{-1}$$

freely reduces to $b_{r+4} \cdots b_5 b_2 b_3^{-1} b_2^{-1} b_1 b_3 b_1^{-1} b_2 b_3 b_2^{-1} b_5^{-1} \cdots b_{r+4}^{-1}$ which, since b_1 and b_3 commute, is equal in G to

$$\begin{aligned} b_{r+4} \cdots b_5 b_2 b_3^{-1} b_2^{-1} b_3 b_2 b_3 b_2^{-1} b_5^{-1} \cdots b_{r+4}^{-1} &=_G \\ b_{r+4} \cdots b_5 b_2 b_5^{-1} \cdots b_{r+4}^{-1} &=_G \quad b_2, \end{aligned}$$

and this commutes with $a_1 = b_4$, so the relation $\textsc{tc}(a_1, a_2, \ldots, a_n)_{s+1} =_G 1$ follows from Lemma 2.5 (ii).

Finally, we need to check that both composites of the homomorphisms between the two groups are equal to the identity.

First consider the images of the a_i under the composite mapping $G \to Q \to G$. They map back to a_i immediately except when $i = 2, r+3$ or n, and for a_n this follows using a free reduction. Note that by Lemma 2.3 (i), the definition of b_1 in G is equivalent to $b_1 := a_{r+2}^{-1} \cdots a_3^{-1} a_2 a_3 \cdots a_{r+2}$ and using this, we find that a_2 also maps back to itself by using a free reduction. As for a_{r+3}, we saw earlier that the image of $b_1^{-1} b_2 b_3 b_2^{-1} b_1$ under the map $Q \to G$ reduces to a_{r+3} in G by using free reduction and the commuting relations, and so it too maps back to itself.

The images of b_i under the composite mapping $Q \to G \to Q$ are immediately equal to b_i except for b_1, b_2 and b_3. The same applies to b_3 using a free reduction, and also for b_1 after rewriting it as $a_{r+2}^{-1} \cdots a_3^{-1} a_2 a_3 \cdots a_{r+2}$ as above.

Finally, for b_2, note first that the image of $a_2 \cdots a_{r+2}$ under the map $G \to Q$ freely reduces to $b_{r+4} \cdots b_5 b_1$, whereas the image of $a_{r+4}^{-1} \cdots a_{n-1}^{-1} a_n a_{n-1} \cdots a_{r+4}$

freely reduces to b_3, and hence that of $a_{r+3}^{-1} \cdots a_{n-1}^{-1} a_n a_{n-1} \cdots a_{r+3}$ reduces to

$$b_1^{-1} b_2 b_3^{-1} b_2^{-1} b_1 b_3 b_1^{-1} b_2 b_3 b_2^{-1} b_1 =_Q b_1^{-1} b_2 b_3^{-1} b_2^{-1} b_3 b_2 b_3 b_2^{-1} b_1 =_Q b_1^{-1} b_2 b_1$$

and, since b_2 commutes with $b_{r+4} \cdots b_5$, we find that the image of b_2 under the composite is indeed b_2. □

4. Proving non-isomorphism

The aim of this section is to prove Theorem 1.3. By Proposition 3.1 and Theorem 1.1, this is equivalent to proving that the quotients of $A(\Delta_n)$ by the normal closures of the cycle and twisted cycle commutators $\mathrm{CC}(a_1, a_2, \ldots, a_n)$ and $\mathrm{TC}(a_1, a_2, \ldots, a_n)_t$ are not isomorphic for any t with $1 \leq t \leq n-3$, which is what we shall prove here. (Note that these quotients are isomorphic when $t = n-2$, which can be seen by considering the automorphism of $A(\Delta_n)$ induced by $a_i \mapsto a_i^{-1}$ for $1 \leq i \leq n$.)

When $n = 4$, the result is easily proved by computer. For example, the two quotients have 9 and 8 conjugacy classes of subgroups of index 4. So we shall assume from now on that $n > 4$.

We start with a result that is probably already known. In general, two surjective homomorphisms $\sigma_1, \sigma_2 : G \to H$ are said to be *equivalent* if they have the same kernels or, equivalently, if there is an automorphism α of H with $\alpha(\sigma_1(g)) = \sigma_2(g)$ for all $g \in G$.

Proposition 4.1. *All surjective homomorphisms $\sigma : A(\Delta_n) \to \mathrm{Sym}(n)$ are equivalent when $n > 4$.*

Proof. Let $\sigma : A(\Delta_n) \to \mathrm{Sym}(n)$ be a surjective homomorphism. We consider the restriction of σ to the subgroup $B := \langle a_1, a_2, \ldots, a_{n-1} \rangle$ of $A(\Delta_n)$, which is isomorphic to the braid group B_n. If $\sigma(B)$ is abelian then, since a_2 and a_n commute, we have $\sigma(a_2) \in Z(\mathrm{Sym}(n)) = 1$. But the generators a_i are all conjugate in $A(\Delta_n)$ so this is impossible. Hence $\sigma(B)$ is nonabelian.

In [11, Theorem A] Lin proved that, for $n > 4$, all homomorphisms of B_n to $\mathrm{Sym}(k)$ with $k < n$ have cyclic image. So, if $\sigma(B)$ is intransitive then it is abelian, contrary to what we just proved. Artin proved in [1] that, for $n > 4$, all homomorphisms from B_n to a transitive subgroup of $\mathrm{Sym}(n)$ are surjective and equivalent to the homomorphism in which the generators map to the transpositions $(i, i+1)$ for $1 \leq i < n$.

So we may assume that $\sigma(a_i) = (i, i+1)$ for $1 \leq i < n$. Now, since $\sigma(a_n)$ commutes with the images of $\sigma(a_i)$ for $2 \leq i \leq n-2$ and these images generate the subgroup $\mathrm{Sym}(n-2)$ of $\mathrm{Sym}(n)$ acting on $\{2, 3, \ldots, n-1\}$, which has trivial centre, we see that the only possible image of σ_n is $(1, n)$. □

4.1. Subgroups of $A(\Delta_n)$ and its quotients

Let $G := A(\Delta_n) = \langle X \mid R\rangle$ be our standard presentation of Δ_n with $X = \{a_1, \ldots, a_n\}$. Let G_0 and G_t for $1 \le t \le n-2$ be the quotients of G by the normal closures of the cycle commutator $\textsc{cc}(a_1, a_2, \ldots, a_n)$ and the twisted cycle commutator $\textsc{tc}(a_1, a_2, \ldots, a_n)_t$, respectively. Define $\sigma : G \to \mathrm{Sym}(n)$ by $\sigma(a_i) = (i, i+1)$ for $1 \le i < n$ and $\sigma(a_n) = (1, n)$. Then, since we are assuming that $n > 4$, Proposition 4.1 tells us that σ is a representative of the unique equivalence class of surjective homomorphisms $G \to \mathrm{Sym}(n)$.

Since $\textsc{cc}(a_1, a_2, \ldots, a_n)$ and $\textsc{tc}(a_1, a_2, \ldots, a_n)_t$ are both contained in $\ker(\sigma)$, it follows that there is also a unique equivalence class of surjective homomorphisms $G_i \to \mathrm{Sym}(n)$ for $0 \le i \le n$, with the map σ_i induced by σ as representative.

Let $H < G$ be the inverse image in G of the stabiliser in $\mathrm{Sym}(n)$ of the unordered pair $\{1, 2\}$ under σ. So $|G : H| = n(n-1)/2$. Define H_i for $1 \le i \le n-2$ to be the corresponding subgroups of G_i, and note that they also have index $n(n-1)/2$ in G_i. In fact the uniqueness of the equivalence classes of surjective homomorphisms $G_i \to \mathrm{Sym}(n)$ implies that an isomorphism $G_0 \to G_t$ for $1 \le t \le n-2$ would map H_0 to an image of H_t under an automorphism of $\mathrm{Sym}(n)$, and so there would be such an isomorphism mapping H_0 to H_t. The remainder of this section will be devoted to the proof of Proposition 4.3, stated below, which shows that H_0 and H_t have different abelianisations when $1 \le t \le n-3$, and so they cannot be isomorphic. This will also complete the proof of Theorem 1.3.

Remark 4.2. *We initially carried out corresponding calculations using the inverse image in G of the stabiliser of 1 in $\mathrm{Sym}(n)$, which has index n in G, but we found that the corresponding subgroups H_0 and H_t both had the abelianisation $\frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \mathbb{Z}^2$, so that did not work.*

Proposition 4.3. *For all $n > 4$ isomorphisms exist between $H/[H,H]$ and $\mathbb{Z}^4$, between $H_0/[H_0,H_0]$ and $\frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \mathbb{Z}^3$, and between $H_t/[H_t,H_t]$ and $\frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \mathbb{Z}^2$ for $1 \le t \le n-3$.*

The proof of this proposition will be completed only at the very end of Section 4.5.

4.2. Computing presentations of subgroups

We shall now present a quick summary of the methods based on the Reidemeister-Schreier algorithm that are used in computer calculations of presentations of subgroups of finite index of groups defined by a finite presentation. In the following subsections we shall use this theory to carry out a calculation of this type by hand for an infinite family of examples, although this calculation was based on observations of the results of computer calculations in small cases. This theory can be found in many textbooks; our description here is based on that in [10, Section 2.5.2].

Let $\mathbb{G} := \langle X \mid R\rangle$ be a group defined by a presentation, let $\mathbb{H} \le \mathbb{G}$, and let T be a right transversal of $\mathbb{H}$ in $\mathbb{G}$ that contains the empty word ϵ. For $t \in T$ and $g \in \mathbb{G}$, we denote the unique element of $\mathbb{H}tg \cap T$ by $\overline{tg}$. (So $\overline{tg}$ is the image of t under g in the action of $\mathbb{G}$ on T induced by its natural action on the right cosets of $\mathbb{H}$ in $\mathbb{G}$.) In the remainder of this section, we shall apply the results of this section to various pairs $\mathbb{G}, \mathbb{H}$ that have just been introduced, namely to the pair $(\mathbb{G}, \mathbb{H}) = (G, H) = (A(\Delta_n), H)$ as well as the pairs (G_i, H_i) of subgroups of those. Hence we have been careful to distinguish between $(\mathbb{G}, \mathbb{H})$ and (G, H) typographically.

Let Y be a subset of $\mathbb{H}$ and suppose that, for each $t \in T$ and $x \in X$, there exists a word $\rho(t, x) \in (Y^{\pm})^*$ with $tx =_{\mathbb{G}} \rho(t, x)u$, where $u = \overline{tx}$. This implies that $ux^{-1} =_{\mathbb{G}} \rho(t, x)^{-1}t$, and we define $\rho(u, x^{-1}) := \rho(t, x)^{-1}$.

We can now recursively extend the definition of ρ to words $w \in (X^{\pm})^*$, by defining $\rho(t, \epsilon) = \epsilon$ for all $t \in T$ and, for a word $v \in (X^{\pm})^*$ and $x \in X^{\pm}$, $\rho(t, vx) = \rho(t, v)\rho(u, x)$, where $\overline{tv} = u$. So we have $tw =_{\mathbb{G}} \rho(t, w)\overline{tw}$ for all $t \in T$ and $w \in (X^{\pm})^*$.

Note that, if w represents an element of $\mathbb{H}$, then $\overline{w} = \epsilon$, and so $w =_{\mathbb{G}} \rho(t, w)$, which proves:

Proposition 4.4. *Under the assumptions above, we have* $\mathbb{H} = \langle Y\rangle$.

Suppose that, for each $y \in Y$, we are given a word $\varphi(y) \in (X^{\pm})^*$ with $y =_{\mathbb{G}} \varphi(y)$. The following result, which we shall use in our hand calculations, is proved in [10, Theorem 2.63] and the remark that follows it. We note that we are following the common practice of abusing notation by using Y to denote both a subset of $\mathbb{H}$ and a set of generators in a presentation of $\mathbb{H}$.

Theorem 4.5. *Under the assumptions above,* $\langle Y \mid S_1 \cup S_2\rangle$ *is a presentation of* $\mathbb{H}$ *on the generating set* Y*, where* $S_1 = \{\rho(t, w) : t \in T, w \in R\}$*, and* $S_2 = \{\rho(\epsilon, \varphi(y))y^{-1} : y \in Y\}$.

In our application below, the relators R are given as a set R' of relations $w_1 = w_2$, and it is convenient to replace the relators in S_1 by the equivalent relations $\{\rho(t, w_1) = \rho(t, w_2) : t \in T, (w_1 = w_2) \in R'\}$.

4.3. The words $\rho(t, x)$ for the subgroup H of G.

Our aim in the remainder of this section is to apply Theorem 4.5 to the subgroups H of G and H_i of G_i that were defined in Subsection 4.1. We start by finding a generating set Y of H and a transversal T of H in G, and computing the words $\rho(t, a_i) \in (Y^{\pm})^*$ for $t \in T$ and $1 \le i \le n$.

Define the words

$$\xi_1 := a_1,\ \xi_i := a_{i+1}\ (2 \le i \le n-2),\ \xi_{n-1} := a_2^2,\ \xi_n := a_n^2,\ \xi_{n+1} := a_2 a_1 \prod_{i=3}^{n} a_i.$$

It is straightforward to check that the words $\xi_i \in (X^{\pm})^*$ represent elements of H for $1 \le i \le n+1$, and we denote the element of H that is represented by the word ξ_i by y_i. So, in the notation of the preceding subsection, we can define $\varphi(y_i) := \xi_i$. It follows from Proposition 4.6 below and Theorem 4.5 that $H = \langle y_i : 1 \le i \le n+1 \rangle$.

For $1 \le k < l \le n$, define

$$t_{k,l} := \prod_{i=2}^{l-1} a_i \prod_{i=1}^{k-1} a_i.$$

Then the element $\sigma(t_{k,l}) \in \mathrm{Sym}(n)$ maps 1 to k and 2 to l (note that we are composing permutations from left to right), so the elements $t_{k,l}$ form a right transversal of H in G.

Proposition 4.6. *For $1 \le k < l \le n$ and $1 \le m \le n$, let words $\rho(t_{k,l}, a_m) \in Y^*$ and pairs of integers k', l' be as specified by the following table.*

Then for all such k, l, m, the equation $t_{k,l} a_m =_G \rho(t_{k,l}, a_m) t_{k',l'}$ holds.

Case	$\rho(t_{k,l}, a_m)$	(k', l')
$l < m < n$	y_{m-1}	(k, l)
$l = m < n$	ϵ	$(k, l+1)$
$k = m = l-1$	y_1	(k, l)
$k = m < l-1$	ϵ	$(k+1, l)$
$k < m = l-1$	$y_{n-1}^{\prod_{i=2}^{l-2} y_i}$	$(k, l-1)$
$k < m < l-1$	y_m	(k, l)
$k = m+1$	$y_{n-1}^{\prod_{i=1}^{k-1} y_i}$	$(k-1, l)$
$k > m+1$	y_{m+1}	(k, l)
$k = 1, l = m = n$	$y_1^{y_{n+1}^{-1}}$	(k, l)
$k = 1, l < m = n$	$y_n y_{n+1}^{-1}$	(l, n)
$k > 1, l = m = n$	y_{n+1}	$(1, k)$
$k > 1, l < m = n$	$y_{n-2}^{y_{n+1}^{-1}}$	(k, l)

Proof. We shall go through the table entries line by line, proving the claimed result in each case.

When $l < m < n$, a_m commutes with $t_{k,l}$ and $a_m = y_{m-1}$.

When $l = m < n$, a_m commutes with $\prod_{j=1}^{k-1} a_i$, and so $t_{k',l'} =_G t_{k,l+1}$.

When $k = m = l - 1$, we need to prove that

$$\prod_{i=2}^{l-1} a_i \left(\prod_{j=1}^{l-2} a_j \right) a_{l-1} =_G a_1 \prod_{i=2}^{l-1} a_i \prod_{j=1}^{l-2} a_j,$$

which we do by induction on l. The base case $l = 2$ is trivial. Since a_{l-1} commutes with $a_1 \cdots a_{l-3}$ we have, using induction,

$$\begin{aligned} \prod_{i=2}^{l-1} a_i \left(\prod_{j=1}^{l-2} a_j \right) a_{l-1} \quad &=_G \quad \prod_{i=2}^{l-2} a_i \left(\prod_{j=1}^{l-3} a_j \right) a_{l-1} a_{l-2} a_{l-1} =_G \\ \prod_{i=2}^{l-2} a_i \left(\prod_{j=1}^{l-3} a_j \right) a_{l-2} a_{l-1} a_{l-2} \quad &=_G \quad a_1 \prod_{i=2}^{l-2} a_i \left(\prod_{j=1}^{l-3} a_j \right) a_{l-1} a_{l-2} =_G \\ & \quad\ a_1 \prod_{i=2}^{l-1} a_i \prod_{j=1}^{l-2} a_j. \end{aligned}$$

The result for $k = m < l - 1$ is immediate.

When $k < m = l - 1$, note first that the claimed value of $\rho(t_{k,l}, a_m)$ is

$$y_{n-1}^{\prod_{i=2}^{l-2} y_i} = (a_2^2)^{\prod_{i=3}^{l-1} a_i} =_G (a_2^{\prod_{i=3}^{l-1} a_i})^2$$

which, by Lemma 2.3, is equal in G to $(a_{l-1}^2)^{\left(\prod_{i=2}^{l-2} a_i\right)^{-1}}$ and so, after free cancellation, we have $y_{n-1}^{\prod_{i=2}^{l-2} y_i} t_{k,l-1} =_G \left(\prod_{i=2}^{l-2} a_i\right) a_{l-1}^2 \prod_{j=1}^{k-1} a_j$, and the claim follows from the fact that a_{l-1} commutes with $\prod_{j=1}^{k-1} a_j$.

When $k < m < l-1$, since a_m commutes with $\prod_{j=1}^{k-1} a_j$ and with $\prod_{i=m+2}^{l-1} a_i$, the proof reduces to showing that $\left(\prod_{i=2}^{m+1} a_i\right) a_m = a_{m+1} \prod_{i=2}^{m+1} a_i$, which follows from $\mathrm{BR}(a_m, a_{m+1})$ and the fact that a_{m+1} commutes with $\prod_{i=2}^{m-1} a_i$.

The case $k = m + 1$ seems to be the most difficult to prove. Note first that a_1 commuting with $\prod_{i=3}^{k} a_i$ followed by Lemma 2.3 give

$$y_{n-1}^{\prod_{i=1}^{k-1} y_i} =_G a_1^{-1} (a_2^2)^{\left(\prod_{i=3}^{k} a_i\right)} a_1 =_G a_1^{-1} (a_k^2)^{\left(\prod_{i=2}^{k-1} a_i\right)^{-1}} a_1,$$

so

$$y_{n-1}^{\prod_{i=1}^{k-1} y_i} t_{k-1,l} =_G a_1^{-1} (a_k^2)^{\left(\prod_{i=2}^{k-1} a_i\right)^{-1}} a_1 \prod_{i=2}^{l-1} a_i \prod_{j=1}^{k-2} a_j.$$

Now, the right hand side of this equality contains the subword $a_1^{\prod_{i=2}^{k-1} a_i}$ which, by Lemma 2.3 is equal in G to $a_{k-1}^{\left(\prod_{i=1}^{k-2} a_i\right)^{-1}}$, and this substitution results in the expression

$$a_1^{-1}\left(\prod_{i=2}^{k-1} a_i\right) a_k^2\, a_{k-1}^{\left(\prod_{i=1}^{k-2} a_i\right)^{-1}} \prod_{i=k}^{l-1} a_i \prod_{j=1}^{k-2} a_j$$

which, since $\prod_{i=1}^{k-2} a_i$ commutes with $\prod_{i=k}^{l-1} a_i$, reduces to $a_1^{-1}\left(\prod_{i=2}^{k-1} a_i\right) a_k^2 \prod_{i=1}^{l-1} a_i$. Now, by applying Lemma 4.7 below to a prefix of this expression followed by commutativity relations, we find that it is equal in G to

$$\prod_{i=2}^{k-1} a_i \left(\prod_{i=1}^{k-2} a_i\right) a_k a_{k-1}^2 \prod_{i=k+1}^{l-1} a_i =_G \prod_{i=2}^{l-1} a_i \left(\prod_{j=1}^{k-2} a_j\right) a_{k-1}^2 = t_{k,l} a_{k-1} = t_{k,l} a_m.$$

When $k > m+1$ we find, by using the commutativity relations and $\mathrm{BR}(a_m, a_{m+1})$, that $\left(\prod_{j=1}^{k-1} a_j\right) a_m =_G a_{m+1} \prod_{j=1}^{k-1} a_j$ and similarly, since $m+1 < l-1$, we have $\left(\prod_{j=2}^{l-1} a_j\right) a_{m+1} =_G a_{m+2} \prod_{j=2}^{l-1} a_j$. So in this case

$$t_{k,l} a_m =_G a_{m+2} t_{k,l} = y_{m+1} t_{k,l}$$

and the result follows.

For the case $k = 1$ and $l = m = n$, note that $y_1^{y_{n+1}^{-1}} = a_1^{\left(a_2 a_1 \prod_{i=3}^{n} a_i\right)^{-1}}$ and, since a_1 commutes with $\prod_{i=3}^{n-1} a_i$, this is equal in G to

$$(a_1 a_n a_1 a_n^{-1} a_1^{-1})^{\left(\prod_{i=2}^{n-1} a_i\right)^{-1}} =_G a_n^{\left(\prod_{i=2}^{n-1} a_i\right)^{-1}}$$

(using $\mathrm{BR}(a_1, a_n)$). It follows by free cancellation that

$$y_1^{y_{n+1}^{-1}} t_{1,n} = y_1^{y_{n+1}^{-1}} \prod_{i=2}^{n-1} a_i = \left(\prod_{i=2}^{n-1} a_i\right) a_n = t_{1,n} a_m.$$

When $k = 1$ and $l < m = n$, we apply commutativity to get

$$y_n y_{n+1}^{-1} t_{l,n} = a_n^2 \left(\prod_{i=3}^{n} a_i\right)^{-1} a_1^{-1} a_2^{-1} \prod_{i=2}^{n-1} a_i \prod_{j=1}^{l-1} a_j =_G a_n a_1^{-1} \left(\prod_{i=2}^{n-1} a_i\right)^{-1} \prod_{i=2}^{n-1} a_i \prod_{j=1}^{l-1} a_j.$$

The final product freely reduces to $a_n \prod_{i=2}^{l-1} a_i$ which, since a_n commutes with $\prod_{i=2}^{l-1} a_i$, is equal in G to $t_{1,l} a_n$.

For the case $k > 1$, $l = m = n$, we use first that a_n commutes with $\prod_{j=2}^{k-1} a_j$ and then that a_1 commutes with $\prod_{i=3}^{n-1} a_i$ to see that

$$\begin{aligned} t_{k,l} a_m &= t_{k,n} a_n = \prod_{i=2}^{n-1} a_i \left(\prod_{i=1}^{k-1} a_i\right) a_n =_G \left(\prod_{i=2}^{n-1} a_1\right) a_n \left(\prod_{i=2}^{k-1} a_i\right) \\ &= \left(a_2 a_1 \prod_{i=3}^{n} a_i\right) \prod_{i=2}^{k-1} a_i = y_{n+1} t_{1,k}. \end{aligned}$$

Finally, for the case $k > 1$, $l < m = n$, we need to verify

$$\left(\prod_{i=2}^{l-1} a_i \prod_{i=1}^{k-1} a_i\right) a_n =_G y_{n-2}^{y_{n+1}^{-1}} \left(\prod_{i=2}^{l-1} a_i \prod_{i=1}^{k-1} a_i\right)$$

We note that, since a_1 commutes with $\prod_{i=3}^{n-2} a_i$, we have

$$\begin{aligned} y_{n-2}^{y_{n+1}^{-1}} &=_G a_{n-1}^{\left(a_2 a_1 \prod_{i=3}^{n} a_i\right)^{-1}} =_G a_{n-1}^{\left(\left(\prod_{i=2}^{n-2} a_i\right) a_{n-1} a_1 a_n\right)^{-1}} \\ &=_G \left(a_{n-1}^{(a_{n-1} a_1 a_n)^{-1}}\right)^{\left(\prod_{i=2}^{n-2} a_i\right)^{-1}} = \left(a_{n-1} a_1 a_n a_{n-1} a_n^{-1} a_1^{-1} a_{n-1}^{-1}\right)^{\left(\prod_{i=2}^{n-2} a_i\right)^{-1}} \end{aligned}$$

Braid and commutator relations reduce that last product to

$$\left(a_1 a_n a_1^{-1}\right)^{\left(\prod_{i=2}^{n-2} a_i\right)^{-1}},$$

and then the commutator relators between a_i for $i = l, \ldots, n-2$ and both a_1 and a_n reduce it further to

$$(a_1 a_n a_1^{-1})^{\left(\prod_{i=2}^{l-1} a_i\right)^{-1}}.$$

So now, applying commutator relations, we deduce that

$$\begin{aligned} y_{n-2}^{y_{n+1}^{-1}} t_{k,l} &=_G \left(\prod_{i=2}^{l-1} a_i\right) \left(a_1 a_n a_1^{-1}\right) \left(\prod_{i=1}^{k-1} a_i\right) =_G \left(\prod_{i=2}^{l-1} a_i\right) (a_1 a_n) \left(\prod_{i=2}^{k-1} a_i\right) \\ &=_G \left(\prod_{i=2}^{l-1} a_i \prod_{i=1}^{k-1} a_i\right) a_n =_G t_{k,l} a_n \end{aligned}$$

□

Lemma 4.7. *Suppose that* $1 < k < n$. *Then*

$$a_1^{-1} \left(\prod_{i=2}^{k-1} a_i\right) a_k^2 \prod_{i=1}^{k} a_i =_G \prod_{i=2}^{k-1} a_i \left(\prod_{i=1}^{k-2} a_i\right) a_k a_{k-1}^2.$$

Proof. The proof is by induction on k. In the base case $k = 2$, the relation to be proved is $a_1^{-1} a_2^2 a_1 a_2 =_G a_2 a_1^2$, which follows from $\mathrm{BR}(a_1, a_2)$. So assume that $k > 2$.

Using the fact that a_1 commutes with a_k^2 and with $\prod_{i=3}^{k-1} a_i$, as well as $\mathrm{BR}(a_1, a_2)$, we deduce that the left hand side of the required relation is equal in G to

$$a_1^{-1} a_2 a_1 \left(\prod_{i=3}^{k-1} a_i\right) a_k^2 \prod_{i=2}^{k} a_i =_G a_2 a_1 a_2^{-1} \left(\prod_{i=3}^{k-1} a_i\right) a_k^2 \prod_{i=2}^{k} a_i$$

which, by applying the inductive hypothesis applied to the sequence $a_2, a_3, \ldots, a_k$ followed by commutativity relations, is equal in G first to

$$a_2 a_1 \prod_{i=3}^{k-1} a_i \left(\prod_{i=2}^{k-2} a_i \right) a_k a_{k-1}^2$$

and then (since a_1 commutes with each a_i with $i \geq 3$) to

$$\prod_{i=2}^{k-1} a_i \left(\prod_{i=1}^{k-2} a_i \right) a_k a_{k-1}^2.$$

□

4.4. The relations of $H/[H,H]$.

We shall now use Theorem 4.5 to calculate the set $S_1 \cup S_2$ of defining relators of H on the generators y_i, using in those calculations the values of $\rho(t_{k,l}, a_m)$ that were verified in Proposition 4.6. Since we want to calculate a presentation of $H/[H,H]$, we shall abelianise the relators of H immediately, and write them as words in the images z_i of y_i in $H/[H,H]$ using additive notation. Note that this means that each of the entries for $\rho(t_{k,l}, a_m)$ in the table within Proposition 4.6 that are conjugates y_i^w for some i and some $w \in (Y^{\pm})^*$ can be replaced by y_i for the purposes of these calculations, and we shall henceforth denote such a conjugate of y_i by $c(y_i)$. So apart from the entry $y_n y_{n+1}^{-1}$ in the case $k = 1, l < m = n$ (which will be replaced by $z_n - z_{n+1}$), these are all words of length at most 1.

We find that all the relators of H within the set $S_2 = \{\rho(\epsilon, \varphi(y_i)) y_i^{-1} : 1 \leq i \leq n+1\}$ are empty. For example, where $i = n+1$, $\varphi(y_{n+1}) = a_2 a_1 \prod_{i=3}^{n} a_i$, and we have $t_{1,2} a_2 = t_{1,3}$, $t_{1,3} a_1 = t_{2,3}$, $t_{2,l} a_l = t_{2,l+1}$ for $3 \leq l \leq n-1$, and $t_{2,n} a_n = y_{n+1} t_{1,2}$, so the resulting relator of H is $y_{n+1} y_{n+1}^{-1}$, freely reducing to the empty word.

So now we consider the relators of H within the set $S_1 = \{\rho(t_{k,l}, w) : t_{k,l} \in T, w \in R\}$. We find that each relator of $H/[H,H]$ is derived several times, for various values of $t_{k,l}, w$.

The calculations of the abelianisations of relations of H that are derived from relations $a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1}$ $(1 \leq i < n)$, $a_n a_1 a_n = a_1 a_n a_1$, or $a_i a_j = a_j a_i$ $(1 < i < j-1 \leq n-1)$ of G are routine. So we shall work out a couple of examples, and otherwise list the results.

We derive relations $z_j = z_{j+1}$ for $H/[H,H]$ for all i with $2 \leq j \leq n-3$ using the braid relations $w = a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1}$ of G with $1 \leq i \leq n-1$.

More precisely, exactly what is derived from the braid relation above depends on the associated transversal element $t = t_{k,l}$. Suppose first that $i < n-1$. If

$l < i$ then we derive $z_{i-1} = z_i$, while if $k < i < l-3$ then we derive $z_i = z_{i+1}$, and if $i+1 < k-1$ we derive $z_{i+1} = z_{i+2}$, but all other cases give an empty relation. Otherwise suppose that $i = n$. Then if $(k,l) = (1, n-1)$ we derive $z_{n-1} = z_n$, while if $1 < k < n-1$, $l = n$ we derive $z_{n-1} = z_n$, but all other cases give an empty relation.

The braid relation $a_n a_1 a_n = a_1 a_n a_1$ is more complicated, and yields the empty relation for $(k,l) = (1,2)$ and $(1,n)$; $z_n + z_2 = z_{n-2} + z_{n-1}$ when $1 = k$, $3 \le l < n$; $z_{n-2} = z_2$ when $2 \le k < l < n$; $z_n = z_{n-1}$ when $(k,l) = (2,n)$; and $z_{n-2} = z_2$ when $2 < k < l = n$.

For example, for the relation $a_n a_1 a_n = a_1 a_n a_1$ with $1 = k$, $3 \le l < n$, we get $t_{1,l} a_n = y_n y_{n+1}^{-1} t_{l,n}$, $t_{l,n} a_1 = y_2 t_{l,n}$, $t_{l,n} a_n = y_{n+1} t_{1,l}$; and $t_{1,l} a_1 = t_{2,l}$, $t_{2,l} a_n = c(y_{n-2}) t_{2,l}$, $t_{2,l} a_1 = c(y_{n-1}) y_{n-1} t_{1,l}$ which, as claimed, yields $z_n + z_2 = z_{n-2} + z_{n-1}$ on abelianisation.

For the relations $a_i a_j = a_j a_i$ ($1 \le i < j-1 \le n-1$, and (i,j)=$(1,n)$) of G, there is a large but finite number of different cases to consider depending on the relative values i, j, k, l. We claim first that they all yield the empty relation of $H/[H,H]$ when $j < n$.

To see this, assume that $j < n$, and note first that, if $k, l \notin \{i, i+1, j, j+1\}$, then $\overline{t_{k,l} a_i} = \overline{t_{k,l} a_j} = t_{k,l}$, and so the resulting relation $\rho(t_{k,l}, w_1) = \rho(t_{k,l}, w_2)$ is just $\rho(t_{k,l}, a_i)\rho(t_{k,l}, a_j) = \rho(t_{k,l}, a_j)\rho(t_{k,l}, a_i)$, which is trivial on abelianisation.

Suppose next that exactly one of k, l is in $\{i, i+1, j, j+1\}$, say $k \in \{i, i+1\}$ (the other three cases are similar). Then $\overline{t_{k,l} a_j} = t_{k,l}$ and $\overline{t_{k,l} a_i} = t_{k',l}$, with $k' = k \pm 1$, and then $\overline{t_{k',l} a_j} = t_{k',l}$. We find from the table in Proposition 4.6 that $\rho(t_{k,l}, a_j) = \rho(t_{k',l}, a_j) = y_{j-1}$ or y_j (depending on whether $l < j$ or $l > j$), so again the resulting relation of $H/[H,H]$ is trivial.

Finally (still assuming that $j < n$), suppose that $k, l \in \{i, i+1, j, j+1\}$, so $k \in \{i, i+1\}$ and $l \in \{j, j+1\}$. Then $\overline{t_{k,l} a_i} = t_{k',l}$, with $k' = k \pm 1$, $\overline{t_{k,l} a_j} = t_{k,l'}$, with $l' = l \pm 1$, and $\overline{t_{k,l'} a_i} = \overline{t_{k',l} a_j} = t_{k',l'}$. Again, by using the table of Proposition 4.6, we find that in each of the four possible cases for k and l, we get the trivial relation of $H/[H,H]$.

When $j = n$ we find, by similar calculations using the table of Proposition 4.6, that we get the relation $z_i = z_{i+1}$ of $H/[H,H]$ when $2 \le i \le n-2$ and ($k = 1, i < l-1 < n-1$ or $l = n, i < k-1$); and the relation $z_{i-1} = z_i$ when $3 \le i \le n-1$ and ($k = 1, i > l$ or $l = n, i > k$).

For example, if $2 \le i \le n-3$, $j = n$, $k = 1$, and $i < l-1 < n-1$, then we have $t_{1,l} a_i = y_i t_{1,l}$, $t_{1,l} a_n = y_n y_{n+1}^{-1} t_{l,n}$, and $t_{l,n} a_i = y_{i+1} t_{l,n}$, so we get the relation $z_i = z_{i+1}$, as claimed.

Now all of these relations taken together reduce to $z_2 = z_3 = \cdots = z_{n-2}$ and $z_n = z_{n-1}$, and hence $H/[H,H]$ is free abelian of rank 4 with free basis the images of z_1, z_2, z_n, z_{n+1}. This proves the first part of Proposition 4.3.

4.5. The cycle and twisted cycle commutator relations of $H_i/[H_i, H_i]$

The relations of H_0 and H_i for $1 \le t \le n-2$ consist of all of the relations of H together with those that are derived from the cycle and twisted cycle commutator relations of G_0 and G_t. These twisted cycle commutator relations have the form $a_1 w = w a_1$, where $w = v a_n v^{-1}$, and $v = a_2 a_3 \cdots a_{n-1}$ for the cycle commutator $\mathrm{CC}(a_1, a_2, \ldots, a_n)$, and $v = a_2^{-1} a_3^{-1} \cdots a_{t+1}^{-1} a_{t+2} \cdots a_{n-1}$ for the twisted cycle commutator $\mathrm{TC}(a_1, a_2, \ldots, a_n)_t$.

Since $\sigma(a_1)$ and $\sigma(w)$ both fix $\{1,2\}$ in all of the groups H_i, we see that the relations $\rho(t_{k,l}, a_1 w) = \rho(t_{k,l}, w a_1)$ of H_i are trivial when $k > 2$ and when $\{k,l\} = \{1,2\}$. So we need only consider the cases when $k = 1$ or 2 and $l > 2$.

In these cases, we have $t_{1,l} a_1 = t_{2,l}$ and $t_{2,l} a_1 = c(y_{n-1}) t_{1,l}$, whereas $t_{1,l} w = \alpha_{1l} t_{2,l}$ and $t_{2,l} w = \alpha_{2l} t_{1,l}$ for some words $\alpha_{1l}, \alpha_{2l} \in (Y^{\pm})^*$ (which may depend also on which group H_i we are considering). Then the relations of H_i in the cases $i = 1$ and 2 and $l > 2$ are $\alpha_{2l} = \alpha_{1l} y_{n-1}$ and $y_{n-1} \alpha_{1l} = \alpha_{2l}$, respectively, which have the same abelianisation, and hence we need only consider the cases $k = 1$, $3 \le l \le n$.

Before calculating the ensuing relations of $H_i/[H_i, H_i]$, it is helpful to calculate the images of $\{1,l\}$ and of $\{2,l\}$ under the image $\sigma(v)$ of the word v under σ. Since $\sigma(a_m) = \sigma(a_m^{-1})$ for all i, these are the same in all of the groups H_i.

We have $\{1,l\}^{\sigma(a_m)} = \{1,l\}$ for $2 \le m \le l-2$, $\{1,l\}^{\sigma(a_{l-1})} = \{1,l-1\}$, and $\{1,l-1\}^{\sigma(a_m)} = \{1,l-1\}$ for $l \le m \le n-1$, so $\{1,l\}^{\sigma(v)} = \{1,l-1\}$.

For the image of $\{2,l\}$, we have $\{m,l\}^{\sigma(a_m)} = \{m+1,l\}$ for $2 \le m \le l-2$, $\{l-1,l\}^{\sigma(a_{l-1})} = \{l-1,l\}$, and $\{l-1,m\}^{\sigma(a_m)} = \{l-1,m+1\}$ for $l \le m \le n-1$, so $\{2,l\}^{\sigma(v)} = \{l-1,n\}$.

In the case of H_0, the quotient by the normal closure of the cycle commutator, we have $t_{1,l} v = \beta_{1l} t_{1,l-1}$ and $t_{2,l} v = \beta_{2l} t_{l-1,n}$ for some words $\beta_{1l}, \beta_{2l} \in (Y^{\pm})^*$, and also $t_{1,l-1} a_n = y_n y_{n+1}^{-1} t_{l-1,n}$ and $t_{l-1,n} a_n = y_{n+1} t_{1,k-1}$ so, denoting the images of β_{1l}, β_{2l} in $H_0/[H_0, H_0]$ by γ_{1l}, γ_{2l}, the resulting relations of $H_0/[H_0, H_0]$ are $\gamma_{2l} + z_{n+1} - \gamma_{1l} = \gamma_{1l} + z_n - z_{n+1} - \gamma_{2l} + z_{n-1}$ or, equivalently,

$$2\gamma_{1l} - 2\gamma_{2l} + z_{n-1} + z_n - 2z_{n+1} = 0.$$

A routine calculation shows that, for all l with $3 \le l \le n$, we have $\gamma_{1l} = \sum_{i=2}^{n-1} z_i$ and $\gamma_{2l} = z_1$, so from the cycle commutator we get the single extra relation

$$-2z_1 + 2\sum_{i=2}^{n-2} z_i + 3z_{n-1} + z_n - 2z_{n+1}.$$

We saw above that z_n and z_{n-1} have equal images in $H/[H,H]$, so $H_0/[H_0,H_0] \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \mathbb{Z}^3$.

In the case of the quotients H_t of H by the normal closures of the twisted cycle commutators $\mathrm{TC}(a_1, a_2, \ldots, a_n)_t$, the corresponding words in $(Y^{\pm})^*$ and their images in $H_t/[H_t,H_t]$ depend also on t, and so we denote them by β_{ilt} and γ_{ilt} for $i = 1, 2$. In this case, the corresponding calculation shows that we get two extra relations when $1 \le t \le t-3$. (We observed at the beginning of this section that $G_0 \cong G_{t-2}$ and hence $H_0 \cong H_{t-2}$.)

When $t+1 \le l-2$ we have

$$\gamma_{1lt} = -\sum_{i=2}^{t+1} z_i + \sum_{i=t+2}^{n-2} z_i + z_{n-1}, \quad \gamma_{2lt} = -tz_{n-1} + z_1,$$

and when $t+1 \ge l-1$ we have

$$\gamma_{1lt} = -\sum_{i=2}^{t} z_i + \sum_{i=t+1}^{n-2} z_i, \quad \gamma_{2lt} = -(t-1)z_{n-1} - z_1.$$

Here is some more detail for the case $t+1 \ge l-1$. For $2 \le i \le l-2$ we get $t_{1,l}a_i^{-1} = y_i^{-1}t_{1,l}$, then $t_{1,l}a_{l-1}^{-1} = t_{1,l-1}$, then for $l \le i \le t+1$ we have $t_{1,l-1}a_i^{-1} = y_{i-1}^{-1}t_{1,l-1}$, and finally for $t+2 \le i \le n-1$ we have $t_{1,l-1}a_i = y_{i-1}t_{1,l-1}$, which results in the claimed value of γ_{1lt}.

For $2 \le i \le l-2$ we get $t_{i,l}a_i^{-1} = c(y_{n-1})^{-1}t_{i+1,l}$, then $t_{l-1,l}a_{l-1}^{-1} = y_1^{-1}t_{l-1,l}$, then for $l \le i \le t+1$ we have $t_{l-1,i}t_i^{-1} = c(y_{n-1})^{-1}t_{l-1,i+1}$, and finally for $t+2 \le i \le n-1$ we have $t_{l-1,i}a_i = t_{l-1,i+1}$, so this results in the claimed value for γ_{2lt}.

The above equations result in the two relations

$$\begin{aligned} -2\sum_{i=1}^{t+1} z_i + 2\sum_{i=t+2}^{n-2} z_i + (2t+3)z_{n-1} + z_n - 2z_{n+1} &= 0, \\ 2z_1 - 2\sum_{i=2}^{t} z_i + 2\sum_{i=t+1}^{n-2} z_i + (2t-1)z_{n-1} + z_n - 2z_{n+1} &= 0 \end{aligned}$$

of $H_t/[H_t,H_t]$. Subtracting the first of these from the second yields $4z_1 + 4z_{t+1} - 4z_{n-1} = 0$ and, again using the fact that z_{n-1} and z_n have equal images in $H/[H,H]$, we see that $H_t/[H_t,H_t] \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \mathbb{Z}^2$, for $1 \le t \le n$, which completes the proof of Proposition 4.3.

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