Please solve the following problems:

Qu.4, Qu.6 and Qu.7;

hand in your solutions on Tuesday the 24th of February by 16.00. Tutorial is on Thursday the 5th of February at 16.00 in TR2, Level 4, Herschel Building. **Z.A.Lykova**

MAS8220 Topology and Functional Analysis (2015)

Examples Sheet 2

Qu. 1. For $r > 0$ define the norm $\|\cdot\|_r$ on $\mathbb{C}$ by

$$\|\lambda\|_r = r|\lambda|, \quad \lambda \in \mathbb{C}.$$

For which r is $(\mathbb{C}, \|\cdot\|_r)$ a normed algebra?

Qu. 2. Show that an identity element e in a normed algebra satisfies $\|e\| \geq 1$. Can it happen that $\|e\| > 1$?

Qu. 3. Let X be a complex Banach space. Show that the normed algebra $\mathcal{B}(X)$ described in Ex. 1, Qu. 5 and 6, is complete, hence a Banach algebra.

Qu. 4. Let M be a set and let $\ell^\infty(M)$ denote the normed space of bounded $\mathbb{C}$-valued functions on M with the natural algebraic operations and "supremum norm":

$$\|f\|_\infty = \sup_{m \in M} |f(m)|.$$

Show that $\ell^\infty(M)$ is a Banach algebra. **25 marks**

Qu. 5. Let Ω be an open set in $\mathbb{C}$ and let $H^\infty(\Omega)$ be the set of functions bounded and analytic in Ω, with the usual addition and multiplication of functions and with supremum norm:

$$\|f\|_\infty = \sup_{z \in \Omega} |f(z)|.$$

Show that $H^\infty(\Omega)$ is a Banach algebra.

Qu. 6. Let A be a Banach algebra with indentity e. Show that, for $x, y \in A$, if $e - xy$ is a regular element then so is $e - yx$ [consider the element $u = e + y(e - xy)^{-1}x$]. Deduce that

$$\sigma(xy) \setminus \{0\} = \sigma(yx) \setminus \{0\}.$$

15 marks

Qu. 7. In a Banach algebra A with identity an element x has a "left inverse", that is, an element y such that $yx = e$. Must x be regular? [consider $A = \mathcal{B}(\ell^2)$, $x =$ a shift operator.] **10 marks**