Please solve the following problems:

Qu.1, Qu.2, Qu.3, Qu.5 and Qu.6;

hand in your solutions on Tuesday the 10th of February by 16.00. Tutorial is on Thursday the 29th of January at 16.00 in TR2, Level 4, Herschel Building. **Z.A.Lykova**

MAS8220 Topology and Functional Analysis (2015)

Examples Sheet 1

Qu. 1. Consider the following sets with their natural addition and multiplication

(i) $M_2(\mathbb{Z}_6)$, the 2×2 matrices over $\mathbb{Z}_6$;

(ii) $\mathbb{R}^+[z]$, the polynomials in z with non-negative real coefficients;

(iii) $D_3(\mathbb{C})$, the diagonal 3×3 matrices over $\mathbb{C}$;

(iv) $C[0,1]$, the continuous $\mathbb{C}$-valued functions on the unit interval $[0,1]$.

Say whether each of them is

(a) a ring,

(b) a ring with identity,

(c) a commutative ring,

(d) an algebra over an appropriate field.

30 marks

Qu. 2. Give an example of an algebra which does not have an identity. **10 marks**

Qu. 3. Show that a ring can have at most one identity. **10 marks**

Qu. 4. Let Ω be an open set in $\mathbb{C}$ and let $\mathcal{O}(\Omega)$ be the set of analytic functions on Ω. Show that $\mathcal{O}(\Omega)$ is an algebra over $\mathbb{C}$ with respect to the natural algebraic operations.

Qu. 5. Let X be a normed linear space over $\mathbb{C}$ and let $\mathcal{B}(X)$ denote the set of bounded linear operators on X. For $S, T \in \mathcal{B}(X)$ and $\lambda \in \mathbb{C}$ define operators $S+T$, ST and λS on X by

$$(S+T)x = Sx + Tx, \quad (ST)x = S(Tx) \quad \text{and} \quad (\lambda S)x = \lambda(Sx) \quad \text{for every } x \in X.$$

Show that

(i) $S+T$, ST and λS are in $\mathcal{B}(X)$;

(ii) $\mathcal{B}(X)$ is an algebra over $\mathbb{C}$ with respect to these operations. **30 marks**

Qu. 6. Let $\|\cdot\|$ be the operator norm on the algebra $\mathcal{B}(X)$ in Question 5, so that for $T \in \mathcal{B}(X)$

$$\|T\| = \sup_{\|x\|_X \le 1} \|Tx\|_X.$$

Show that $(\mathcal{B}(X), \|\cdot\|)$ is a normed algebra. **20 marks**