MAS8220: Topology and Functional Analysis

Dr. Zinaida Lykova

School of Mathematics and Statistics
Newcastle University
Room 3.13 in Herschel Building

UNIVERSITY OF
NEWCASTLE

This subject constitutes a synthesis of some of the main trends in analysis over the past century. One studies functions not individually, but as a collection which admits natural operations of addition and multiplication and has geometric structure. An algebra is a vector space with an associative multiplication. There is an abundance of natural examples, many of them having the structure of a Banach space. Examples are the spaces of $n \times n$ matrices and the continuous functions on the interval $[0, 1]$, with suitable norms. Putting together algebras and norms one is led to the idea of a Banach algebra. A rich and elegant theory of such objects was developed over the second half of the twentieth century. Several members of staff have research interests close to this area.

You can consult the following books.

1. F. F. Bonsall, J. Duncan, *"Complete Normed Algebras"*. Springer-Verlag, New York, 1973.
2. G. K. Pedersen, *"Analysis now"*, Springer-Verlag, 1989.
3. N. J. Young, *"An Introduction to Hilbert Space"*, Cambridge University Press, 1988.
4. W. Rudin, *"Functional analysis"*, 2nd Edition, McGraw-Hill, 1991.
5. S. Willard, *"General topology"*, Dover Publications, 2004.

Teaching Information - MAS8220 - 2015

Lectures:

Monday 15.00 – 17.00 HERB.4 TR2
Thursday 15.00 – 16.00 HERB.4 TR2

Tutorials:

Thursday 16.00 – 17.00 HERB.4 TR2

Office hours:

Monday 17.00 – 18.00 HERB.L3 Room 3.13.
Tuesday 14.00 – 15.00 HERB.L3 Room 3.13.

Revision Lectures:

Monday 11th May 2015, 15.00-17.00, HERB.L4 TR2.
Thursday 14th May 2015, 15.00-16.00, HERB.L4 TR2.

Thursday 14th May 2015, 16.00-17.00, HERB.L4 TR2 (Drop-in)

Assessment:

1. Assessment of course work (10%).

   Assignments:

   Assignments are to be handed in by 4pm on Tuesdays:
   10 February (no. 1), 24 February (no. 2, 3), 10 March (no. 4, 5), 21 April (no. 6, 7), 5 May 2015 (no. 8).

2. 2 hours 15 min examination at the end of Semester (90%).

Part I. Banach algebras and continuous homomorphisms

This part concerns the theory of Banach algebras and continuous homomorphisms between Banach algebras. We shall define these terms and study their properties.

Contents of Part I

1 Rings and algebras

Examples of rings: $\mathbb{Z}$, $\mathbb{C}$, $\mathbb{C}[z]$, $M_n(\mathbb{C})$.

Definition 1.1. *A* ring *is a triple* $(R, +, \cdot)$ *where* R *is a set and* $+, \cdot$ *are binary operations on* R *such that*

- $(R, +)$ *is an abelian group;*
- *multiplication* $\cdot$ *is an associative operation:* $\forall a, b, c \in R$,
$$a \cdot (b \cdot c) = (a \cdot b) \cdot c;$$
- *multiplication* $\cdot$ *is distributive over* $+$*:* $\forall a, b, c \in R$,
$$a \cdot (b + c) = (a \cdot b) + (a \cdot c),$$
$$(b + c) \cdot a = (b \cdot a) + (c \cdot a).$$

Notes 1.2.

1. We call $+, \cdot$ addition and multiplication respectively. We often write ab for $a \cdot b$.
2. Every field is a ring: thus $\mathbb{R}, \mathbb{C}, \mathbb{Z}_5$ with the usual addition and multiplication are rings.
3. The set $2\mathbb{Z}$ of even integers, with the usual addition and multiplication, is a ring. It does not have an identity element for multiplication (a "one").
4. In the ring $M_2(\mathbb{R})$ of 2×2 matrices over $\mathbb{R}$, with the usual addition and multiplication, $a \cdot b$ is typically not equal to $b \cdot a$. $M_2(\mathbb{R})$ is a non-commutative ring.

Definition 1.3. *We say that* $(R, +, \cdot)$ *is a ring with* identity *if there is an element* $e \in R$ *such that* $x \cdot e = e \cdot x = x$ *for all* $x \in R$.

Definition 1.4. $(R, +, \cdot)$ *is said to be a* commutative ring *if* $a \cdot b = b \cdot a$ *for all* $a, b \in R$.

Examples 1.5.

1. $\mathbb{C}[x, y]$, the set of polynomials in the two variables x, y with complex coefficients, is a ring with respect to the usual addition and multiplication of polynomials. It is commutative and has an identity, which is the constant polynomial 1.
2. $\mathbb{Z}[x]$, the set of polynomials in the variable x with integer coefficients, is a commutative ring with identity.

3. $\mathbb{Z}_6$, the set of residue classes of integers $\mod 6$, with the usual addition and multiplication, is a commutative ring with identity. It is not a field since [2] has no multiplicative inverse.

   Recall that a *field* is a commutative ring with identity in which every non-zero element has a multiplicative inverse.

4. T_n, the set of upper triangular complex $n \times n$ matrices with the usual matrix addition and multiplication is a non-commutative ring with identity (where $n > 1$).

In checking a statement like (4) we do not check every ring axiom. We observe that T_n is a subset of the standard ring $M_n(\mathbb{C})$, and use the following observation.

Lemma 1.6. *If R is a ring and S is a subset of R such that*

(i) $0 \in S$, *where* 0 *is the zero of* R*;*

(ii) *for each* $x \in S$*, the additive inverse of* x *is also in* S*.*

(iii) S *is closed under addition and multiplication,*

then S *is a ring with respect to the same algebraic operations as* R*.*

Among the above examples of rings, some are also vector spaces, some not.

Definition 1.7. *Let* k *be a field. An* algebra *over* k *is a vector space* A *over* k *equipped with a binary operation* $\cdot$ *such that*

(i) $(A, +, \cdot)$ *is a ring;*

(ii) $\lambda(x \cdot y) = (\lambda x) \cdot y = x \cdot (\lambda y)$

for all $\lambda \in k$ *and* $x, y \in A$*.*

Examples 1.8. 1. $M_n(\mathbb{C})$ is an algebra over $\mathbb{C}$.

2. $\mathbb{R}[x]$ is an algebra over $\mathbb{R}$.

3. $\mathbb{Z}$ is a ring, but is not an algebra over $\mathbb{R}$ or $\mathbb{C}$: there is no natural way to define $\lambda n \in \mathbb{Z}$ for general $\lambda \in \mathbb{R}$ and $n \in \mathbb{Z}$.

4. $\mathbb{Z}[x]$ is a ring, but not an algebra over $\mathbb{R}$ or $\mathbb{C}$.

2 Normed algebras

These are algebras over $\mathbb{R}$ or $\mathbb{C}$ which are also normed space.

Examples: $C[0,1]$, $M_n(\mathbb{C})$.

Recall that a *normed linear subspace* is a vector space V over $\mathbb{R}$ or $\mathbb{C}$ together with a mapping $\|\cdot\| : V \to \mathbb{R}^+$ such that, for any $x, y \in V$ and $\lambda \in \mathbb{R}$ or $\mathbb{C}$, the following conditions are satisfied

1. $\|x\| = 0$ if and only if $x = 0$;
2. $\|\lambda x\| = |\lambda| \|x\|$;
3. $\|x + y\| \leq \|x\| + \|y\|$.

Definition 2.1. *A* normed algebra *is a normed linear space* $(A, \|\cdot\|)$ *equipped with a binary operation* $(x, y) \mapsto xy$ *which makes* A *into an algebra and satisfies*

$$\|xy\| \leq \|x\| \|y\| \text{ for all } x, y \in A. \tag{2.1}$$

Property (2.1) is called the "submultiplicativity of the norm".

Examples 2.2.

1. $(\mathbb{C}, |.|)$ is a *normed algebra* over $\mathbb{C}$.
2. $(C[0,1], \|.\|_\infty)$ is a normed algebra over $\mathbb{C}$, where $C[0,1]$ is the set of continuous $\mathbb{C}-$valued functions on $[0,1]$, addition and multiplication are the usual "pointwise" operations,

$$(x + y)(t) = x(t) + y(t),\ (xy)(t) = x(t)y(t), \text{ for all } x, y \in C[0,1],\ t \in [0,1],$$

and the norm

$$\|x\|_\infty = \sup_{0 \leq t \leq 1} |x(t)|.$$

Check "submultiplicative property". Consider $x, y \in C[0,1]$.

$$\begin{aligned} \|xy\|_\infty &= \sup_{0 \leq t \leq 1} |(xy)(t)| \\ &= \sup_{0 \leq t \leq 1} |x(t)||y(t)| \\ &\leq \sup_{0 \leq t \leq 1} |x(t)| \sup_{0 \leq t \leq 1} |y(t)| \\ &= \|x\|_\infty \|y\|_\infty. \end{aligned}$$

3. $(\mathbb{C}[z], \|.\|_\infty)$ is a normed algebra over $\mathbb{C}$, where $\mathbb{C}[z]$ denotes the set of polynomials over $\mathbb{C}$ and addition, multiplication and $\|.\|_\infty$ are as in (2).

4. $(M_n(\mathbb{C}), \|.\|)$, the algebra of $n \times n$ matrices over $\mathbb{C}$ is a normed algebra over $\mathbb{C}$, where $\|.\|$ is the "operator norm":

$$\|T\| = \sup_{\|x\|_{\mathbb{C}^n} \leq 1} \|Tx\|_{\mathbb{C}^n}$$

Here $\|.\|_{\mathbb{C}^n}$ is the usual Euclidean norm on $\mathbb{C}^n$

$$\|x\|_{\mathbb{C}^n} = \left\{ \sum_{j=1}^{n} |x_j|^2 \right\}^{\frac{1}{2}}.$$

Check submultiplictivity. Consider $S, T \in M_n(\mathbb{C})$.

$$\begin{aligned} \|ST\| &= \sup_{\|x\|_{\mathbb{C}^n} \leq 1} \|STx\|_{\mathbb{C}_n} \\ &\leq \sup_{\|x\|_{\mathbb{C}^n} \leq 1} \|S\|\|Tx\|_{\mathbb{C}_n} \\ &= \|S\| \sup_{\|x\|_{\mathbb{C}^n} \leq 1} \|Tx\|_{\mathbb{C}_n} \\ &= \|S\|\|T\|. \end{aligned}$$

Recall that a normed linear space is said to be *complete* if every Cauchy sequence converges in the space.

A *Cauchy sequence* in $(X, \|.\|)$ is a sequence (x_n) in X such that, for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\|x_n - x_m\| < \epsilon \text{ whenever } m, n \geq N.$$

A *Banach space* is a complete normed linear space.

Definition 2.3. *A Banach algebra is a normed algebra over* $\mathbb{R}$ *or* $\mathbb{C}$ *that is complete as a normed linear space.*

We shall only study complex Banach algebras, i.e. Banach algebras over $\mathbb{C}$.

Examples 2.4. 1. $(C[0,1], \|.\|_\infty)$ is a Banach space and a normed algebra, hence it is a Banach algebra. (cf. Example 2.2(2).)

2. $(\mathbb{C}[z], \|.\|_\infty)$ is a subalgebra of $C[0,1]$. It is a normed algebra, but is incomplete, so not a Banach algebra.

   Claim: $(\mathbb{C}[z], \|.\|_\infty)$ **is incomplete.** Consider the sequence (x_n) in $\mathbb{C}[z]$, where
$$x_n(z) = 1 + z + \frac{z^2}{2!} + ... + \frac{z^n}{n!}.$$
   If $m > n$ then
$$\|x_m - x_n\|_\infty = \sup_{0 \le z \le 1} \left| \frac{z^{n+1}}{(n+1)!} + ... + \frac{z^m}{m!} \right|$$
$$\le \frac{1}{(n+1)!} + ... + \frac{1}{m!}.$$
   It is easy to see that (x_n) is Cauchy for $\|.\|_\infty$, but
$$x_n \to \exp \quad \text{as } n \to \infty$$
   in $(C[0,1], \|.\|_\infty)$. Therefore x_n cannot tend to a polynomial, i.e. (x_n) is a non-convergent Cauchy sequence in $(\mathbb{C}[z], \|.\|_\infty)$. Hence $(\mathbb{C}[z], \|.\|_\infty)$ is incomplete.

3. $M_n(\mathbb{C})$ is a finite-dimensional normed algebra. It is well-known that every finite-dimensional normed linear space over $\mathbb{C}$ or $\mathbb{R}$ is complete. Thus $M_n(\mathbb{C})$ is complete, so is a Banach algebra.

4. "Zero multiplication algebras". Any Banach space E becomes a Banach algebra if we define xy to be 0 for all $x, y \in E$.

Multiplication in a normed algebra A is a mapping from $A \times A$ to A. We can make $A \times A$ into a normed linear space by defining, for $x_1, y_1, x_2, y_2 \in A$ and $\lambda \in \mathbb{C}$,
$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
$$\lambda(x_1, y_1) = (\lambda x_1, \lambda y_1)$$
$$\|(x_1, y_1)\|_{A \times A} = \max\{\|x_1\|_A, \|y_1\|_A\}.$$

Similarly $\mathbb{C} \times A$ is a normed linear space with respect to

$$\|(\lambda, x)\|_{\mathbb{C}\times A} = \max\{|\lambda|, \|x\|_A\}$$

for $\lambda \in \mathbb{C}, x \in A$.

Theorem 2.5. *In any normed algebra A, multiplication is a continuous mapping from $A \times A$ to A; scalar multiplication is a continuous mapping from $\mathbb{C} \times A$ to A.*

Proof. Consider $(x_0, y_0) \in A \times A$. Let $\epsilon > 0$. Observe that, for $x, y \in A$,

$$\begin{aligned}\|xy - x_0y_0\| &= \|x(y - y_0) + (x - x_0)y_0\| \\ &\leq \|x\|\|y - y_0\| + \|x - x_0\|\|y_0\|.\end{aligned}$$

Let $\epsilon > 0$, choose

$$\begin{aligned}\delta_1 &= \min\left\{1, \frac{\epsilon}{2(1 + \|y_0\|)}\right\} \\ \delta_2 &= \min\left\{1, \frac{\epsilon}{2(1 + \|x_0\|)}\right\} \\ \delta &= \min\{\delta_1, \delta_2\}.\end{aligned}$$

Suppose $\|(x, y) - (x_0, y_0)\|_{A\times A} < \delta$, that is, $\max\{\|x - x_0\|_A, \|y - y_0\|_A\} < \delta$. Then

$$\|x - x_0\| < \delta_1 \leq 1,$$

so that

$$\|x\| < 1 + \|x_0\|.$$

Furthermore

$$\begin{aligned}\|x - x_0\| &< \frac{\epsilon}{2(1 + \|y_0\|)}, \\ \|y - y_0\| &< \frac{\epsilon}{2(1 + \|x_0\|)}.\end{aligned}$$

Hence

$$\begin{aligned}\|xy - x_0y_0\| &\leq \|x\|\|y - y_0\| + \|x - x_0\|\|y_0\| \\ &< (1 + \|x_0\|)\frac{\epsilon}{2(1 + \|x_0\|)} + (1 + \|y_0\|)\frac{\epsilon}{2(1 + \|y_0\|)} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon.\end{aligned}$$

Hence multiplication is continuous at (x_0, y_0).

Scalar multiplication - exercise. $\square$

3 Spectra and regular elements

The spectrum of an $n \times n$ matrix T is the set of its eigenvalues, that is, the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - T$ is singular. The notion extends fruitfully to Banach algebras.

In this section we consider a Banach algebra A with identity element e. We assume $e \neq 0$. *(Think of* $C[0,1]$ *and* $M_n(\mathbb{C})$*.)*

Definition 3.1. *An element* $x \in A$ *is said to be a* regular *element (or* invertible *element) if there exists* $y \in A$ *such that* $xy = e$ *and* $yx = e$*.*

An element $x \in A$ *is* singular *if it is not regular. Any such* y *is called an* inverse of x.

Note that x can have at most one inverse, for if y_1 and y_2 are inverses of x then

$$y_1 = y_1 e = y_1(xy_2) = (y_1 x)y_2 = ey_2 = y_2.$$

We may speak of "the" inverse of a regular element x; we denote it by x^{-1}.

Examples 3.2. 1. The element

$$T_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

is regular in $M_2(\mathbb{C})$; the element

$$T_2 = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

is singular in $M_2(\mathbb{C})$.

2. In $C[0,1]$,

$f : [0,1] \to \mathbb{C} : t \mapsto 1 + t$ is regular, with the inverse $t \mapsto \frac{1}{1+t}$.

$g : [0,1] \to \mathbb{C} : t \mapsto t$ is singular.

f is regular in $C[0,1]$ if and only if $0 \notin \mathrm{Range} f$.

Engineers call the following statement the "small gain theorem".

Lemma 3.3. *Let A be a Banach algebra with the identity e. If $x \in A$ and $\|e-x\| < 1$ then x is a regular element of A and*

$$\|x^{-1} - e\| \leq \left(\frac{1}{1 - \|e - x\|}\right) \|e - x\|.$$

Proof. Let $u = e - x$, so that

$$x = e - u \text{ and } \|u\| < 1.$$

We shall show that

$$x^{-1} = (e - u)^{-1} = e + u + u^2 + ... \tag{3.1}$$

First show that the RHS of (3.1) is meaningful, that is, defines an element of A.

For $n \in \mathbb{N}$, let

$$y_n = e + u + u^2 + ... + u^n.$$

Claim: (y_n) **is a Cauchy sequence in** A**.**

For $m > n$,

$$y_m - y_n = u^{n+1} + ... + u^m,$$

and so

$$\begin{aligned} \|y_m - y_n\| &\leq \|u\|^{n+1} + ... + \|u\|^m \\ &\leq \frac{\|u\|^{n+1}}{1 - \|u\|} \qquad \text{(since } \|u\| < 1). \end{aligned}$$

Thus $\|y_m - y_n\| \to 0$ as $m, n \to \infty$. Therefore (y_n) converges to a limit $y \in A$. Furthermore

$$\begin{aligned} xy_n &= (e - u)y_n = y_n - uy_n \\ &= e + u + ... + u^n - (u + u^2 + ... + u^{n+1}) \\ &= e - u^{n+1} \end{aligned}$$

$$\therefore \|xy_n - e\| = \|u^{n+1}\| \leq \|u\|^{n+1}$$

and so, since $\|u\| < 1$,

$$xy_n \to e \;\; as \;\; n \to \infty.$$

Since $y_n \to y$ and multiplication is continuous,

$$xy = x \lim_n y_n = \lim_n xy_n = e.$$

Similarly $yx = e$. Thus x is a regular element. Furthermore $x^{-1} = y$, and

$$\begin{aligned} x^{-1} - e = y - e &= \lim_n (y_n - e) \\ &= \lim_n (u + u^2 + ... + u^n). \end{aligned}$$

$$\begin{aligned} \therefore \ \|x^{-1} - e\| &= \|\lim_n (u + u^2 + ... + u^n)\| \\ &= \lim_n \|u + u^2 + ... + u^n\| \\ &\leq \lim \|u\| + \|u\|^2 + ... + \|u\|^n \\ &= \sum_{j=1}^{\infty} \|u\|^j \\ &= \frac{\|u\|}{1 - \|u\|} \qquad \text{since } \|u\| < 1 \\ &= \frac{\|e - x\|}{1 - \|e - x\|}. \end{aligned}$$

□

Definition 3.4. *Let X be a normed linear space. For $x \in X$ and $\epsilon > 0$ the open ball of centre x, radius ϵ is the set*

$$\{y \in X : \|x - y\| < \epsilon\}.$$

and is denoted by $B_\epsilon(x)$.
A subset G of X is said to be open *if, for every $x \in G$, there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset G$. A closed set is a set whose complement is open.*

Theorem 3.5. *Let A be a Banach algebra with the identity e and let $G(A)$ be the set of regular elements in A. Then*

1. *$G(A)$ is a group under multiplication;*
2. *$G(A)$ is an open set in A;*
3. *the inversion map*

   $$x \mapsto x^{-1}$$

   is a continuous map from $G(A)$ to $G(A)$.

Proof. 1. If $x, y \in G(A)$ then x^{-1}, y^{-1} exist and $y^{-1}x^{-1}$ satisfies

$$(xy)(y^{-1}x^{-1}) = e = (y^{-1}x^{-1})(xy),$$

so that $xy \in G(A)$ and $(xy)^{-1} = y^{-1}x^{-1}$.

The element e is clearly a multiplicative identity for $G(A)$.

If $x \in G(A)$ then x^{-1} has inverse x in A, so $x^{-1} \in G(A)$.

Multiplication is associative. Hence $G(A)$ is a group with respect to multiplication.

2. Consider any $x \in G(A)$. To prove that $G(A)$ is open in A, we must find $\epsilon > 0$ such that $B_\epsilon(x) \subset G(A)$. Let $\epsilon = \|x^{-1}\|^{-1} > 0$. Consider any $y \in B_\epsilon(x)$, then $\|x - y\| < \|x^{-1}\|^{-1}$. Thus we have

$$\|e - x^{-1}y\| = \|x^{-1}(x-y)\| \leq \|x^{-1}\|\|x - y\| < \|x^{-1}\|\|x^{-1}\|^{-1} = 1.$$

By Lemma 3.3, we have $x^{-1}y \in G(A)$.

$\therefore x(x^{-1}y) \in G(A)$.

$\therefore y \in G(A)$.

$\therefore B_\epsilon(x) \subset G(A)$.

Hence $G(A)$ is open.

3. *Inversion is continuous*
Let

$$f : G(A) \to G(A) : x \to x^{-1}.$$

We want to show that if $x_n \to x$ in $G(A)$ then $f(x_n) \to f(x)$.
(i). First consider a sequence (x_n) in $G(A)$ such that $x_n \to e$. By Lemma 3.3, if $\|e - x_n\| < 1$,

$$\|f(x_n) - e\| = \|x_n^{-1} - e\|$$

$$\leq \frac{\|e - x_n\|}{1 - \|e - x_n\|}$$

$$\to 0 \quad \text{as} \;\; x_n \to e.$$

Now suppose $x_n \to x$ in $G(A)$. Then $x^{-1}x_n \to e$, by continuity of multiplication.

By Part (i),

$$(x^{-1}x_n)^{-1} \to e$$

as $n \to \infty$, that is, $x_n^{-1}x \to e$ as $n \to \infty$,

$$\therefore \quad x_n^{-1} = x_n^{-1}xx^{-1} \to ex^{-1} = x^{-1},$$

again by continuity of multiplication. Hence $f : x \mapsto x^{-1}$ is continuous.

$\square$

Definition 3.6. *The* spectrum *of an element $x \in A$ is the set*

$$\{\lambda \in \mathbb{C} : \lambda e - x \text{ is singular in } A\}.$$

It is denoted by $\sigma(x)$. The resolvent *set of x, denoted by $\rho(x)$, is the complement of $\sigma(x)$:*

$$\rho(x) = \{\lambda \in \mathbb{C} : \lambda e - x \text{ is regular in } A\}.$$

Examples 3.7. (i) For $T \in M_n(\mathbb{C})$, $\sigma(T)$ is the set of eigenvalues of T.

(ii) For $f \in C[0,1], \sigma(f) = \text{Range}\, f$.

Theorem 3.8. *Let A be a Banach algebra with the identity e. For any $x \in A$,*

1. $\lambda \in \sigma(x) \Rightarrow |\lambda| \leq \|x\|$;
2. *$\rho(x)$ is open in $\mathbb{C}$;*
3. *$\sigma(x)$ is a compact subset $\mathbb{C}$.*

Proof. 1. We show that $|\lambda| > \|x\| \Rightarrow \lambda \notin \sigma(x)$, that is, $\lambda \in \rho(x)$.

Suppose $\lambda \in \mathbb{C}$ and $|\lambda| > \|x\|$. Then $\lambda \neq 0$ and $\|x/\lambda\| < 1$, and so, by Lemma 3.3, $e - x/\lambda$ is regular .

$\therefore\ \lambda(e - x/\lambda)$ is regular.

$\therefore\ \lambda e - x$ is regular.

$\therefore\ \lambda \in \rho(x)$.
Hence $\lambda \in \sigma(x) \Rightarrow |\lambda| \leq \|x\|$. Thus $\sigma(x)$ is a bounded set in $\mathbb{C}$.

2. Let $x \in A$ and define

$$F : \mathbb{C} \to A : \lambda \mapsto \lambda e - x.$$

For $\lambda, \mu \in \mathbb{C}$, we have

$$\|F(\lambda) - F(\mu)\| = \|\lambda e - \mu e\| = |\lambda - \mu|\|e\|,$$

and hence F is continuous.
By definition,

$$\rho(x) = \{\lambda \in \mathbb{C} : F(\lambda) \text{ is regular in A}\} = F^{-1}(G(A)).$$

Since $G(A)$ is open and F is continuous, $\rho(x)$ is open.

3. Since $\rho(x)$ is open and $\sigma(x) = \mathbb{C} \setminus \rho(x)$, $\sigma(x)$ is closed. The spectrum $\sigma(x)$ is also bounded. Thus, by the Bolzano-Weierstrass theorem, $\sigma(x)$ is compact.

$\square$

4 Analytic vector functions and the resolvent

Recall that, for an open set $\Omega \subset \mathbb{C}$, a function $f : \Omega \to \mathbb{C}$ is analytic if f is *complex differentiable* at every point of Ω, that is, if the limit

$$\lim_{h\to 0,\, h\in\mathbb{C}} \frac{f(z+h)-f(z)}{h} \quad \text{exists for each } z \in \Omega.$$

In other words, if $\forall z \in \Omega$ there exists $f'(z) \in \mathbb{C}$ such that $\forall \epsilon > 0\ \exists \delta > 0$ such that

$$\left| \frac{f(z+h)-f(z)}{h} - f'(z) \right| < \epsilon$$

whenever $|h| < \delta$ and $z + h \in \Omega$.

We can adapt there notions with minimal modifications to vector-valued functions. Let X be a Banach space and consider a function

$$f : \Omega \to X.$$

For example, we might think of $T \in M_n(\mathbb{C})$ and

$$f : \rho(T) \to M_n(\mathbb{C}) : \lambda \mapsto (\lambda I - T)^{-1}.$$

Definition 4.1. *We say that $f : \Omega \to X$ is* differentiable *at $z \in \Omega$ if*

$$\lim_{h\to 0,\, h\in\mathbb{C}} \frac{f(z+h)-f(z)}{h} \quad \textit{exists},$$

or in other words, if there exists $f'(z) \in X$ such that for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left\| \frac{f(z+h)-f(z)}{h} - f'(z) \right\|_X < \epsilon$$

whenever $|h| < \delta$ and $z + h \in \Omega$.

We say that f is analytic *or* holomorphic *in Ω if f is differentiable at every point of Ω.*

Theorem 4.2. *Let A be a Banach algebra with identity e and let $x \in A$. For $\lambda \in \rho(x)$, let*

$$x(\lambda) = (\lambda e - x)^{-1}.$$

Then the function

$$x(\cdot) : \rho(x) \to A$$

is analytic in $\rho(x)$ and, for all $\lambda, \mu \in \rho(x)$,

$$x(\mu) - x(\lambda) = (\lambda - \mu) x(\lambda) x(\mu). \tag{4.1}$$

Proof. By Theorem 3.5(3) and Theorem 3.8(2), $x(\cdot) : \rho(x) \to A$ is a continuous function, being the composition of two known continuous functions

$$\lambda \mapsto \lambda e - x \mapsto (\lambda e - x)^{-1}.$$

$$\rho(x) \to G(A) \to G(A).$$

For $\lambda, \mu \in \rho(x)$,

$$\begin{aligned} x(\lambda)^{-1} - x(\mu)^{-1} &= \lambda e - x - (\mu e - x) \\ &= (\lambda - \mu)e. \end{aligned}$$

Therefore

$$x(\lambda)(x(\lambda)^{-1} - x(\mu)^{-1})x(\mu) = x(\lambda)(\lambda - \mu)ex(\mu)$$

$$\therefore \;\; x(\mu) - x(\lambda) = (\lambda - \mu)x(\lambda)x(\mu).$$

Thus (4.1) holds. It is called the *resolvent identity*.

Next we show that $x(\cdot)$ is differentiable at $\lambda \in \rho(x)$. Put $\mu = \lambda + h \in \rho(x)$. By (4.1),

$$x(\lambda + h) - x(\lambda) = -hx(\lambda)x(\lambda + h).$$

$$\therefore \;\; \frac{x(\lambda + h) - x(\lambda)}{h} - (-x(\lambda)^2) = x(\lambda)(x(\lambda) - x(\lambda + h))$$

$$\therefore \;\; \left\| \frac{x(\lambda + h) - x(\lambda)}{h} - (-x(\lambda)^2) \right\|_A \leq \|x(\lambda)\|_A \|x(\lambda) - x(\lambda + h)\|_A.$$

RHS $\to 0$ as $h \to 0$. Hence

$$\lim_{h \to 0} \frac{x(\lambda + h) - x(\lambda)}{h} = -x(\lambda)^2.$$

Thus $x'(\lambda)$ exists and equals $-x(\lambda)^2$ for any $\lambda \in \rho(x)$. Thus $x(\cdot)$ is analytic on $\rho(x)$. $\square$

Definition 4.3. *The function* $x(\cdot) : \rho(x) \to A$ *is called the* resolvent *of* x*.*

4.1 Cauchy's integral formula

If X is a Banach space and f is an analytic X-valued function on an open set $U \subset \mathbb{C}$ then we may define contour integrals

$$\int_\gamma f(z)dz,$$

for a contour $\gamma : [a,b] \to U$, just as for scalar-valued functions. We assume that the contour $\gamma(t) = u(t) + iv(t)$, $t \in [a,b]$, has continuously differentiable u and v on (a,b).

We consider a subdivision

$$a = t_0 < \xi_0 < t_1 < \xi_1 < t_2 < ... < \xi_{n-1} < t_n = b$$

of $[a,b]$ and form the corresponding "Riemann sum"

$$\sum_{j=0}^{n-1} f(\gamma(\xi_j))[\gamma(t_{j+1}) - \gamma(t_j)] \in X.$$

It can be proved that as the partition is refined, this sum tends to a limit in X this limit is defined to be $\int_\gamma f(z)dz$.

Let X be a Banach space, let U be a starlike open set in $\mathbb{C}$ and let $f : U \to X$ be analytic. For any close contour γ in U and any point $a \in U$ not on γ,

$$\int_\gamma \frac{f(z)dz}{z-a} = 2\pi i\, n(\gamma; a) f(a) \tag{4.2}$$

and

$$\int_\gamma \frac{f(z)dz}{(z-a)^2} = 2\pi i\, n(\gamma; a) f'(a). \tag{4.3}$$

Lemma 4.4. *Let X be a Banach space, let $R > 0$ and let f be an X-valued function analytic on the disc*

$$\triangle(a,R) = \{z \in \mathbb{C} : |z-a| \leq R\}.$$

Suppose

$$\|f(z)\|_X \leq M \quad \forall z \in \triangle(a,R).$$

Then

$$\|f'(a)\|_X \leq \frac{M}{R}.$$

Proof. In Cauchy's Integral Formula (4.3), take γ_R to be the anti-clockwise oriented circle that bounds $\triangle(a, R)$:

$$\gamma_R(t) = a + Re^{it}, \qquad 0 \le t \le 2\pi.$$

Then $n(\gamma_R; a) = 1$, and so

$$f'(a) = \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(z)dz}{(z-a)^2}.$$

Hence

$$\|f'(a)\|_X = \frac{1}{2\pi} \left\| \int_{\gamma_R} \frac{f(z)dz}{(z-a)^2} \right\|_X . \tag{4.4}$$

The integral on the RHS is the limit of Riemann sums of the form

$$S_\tau = \sum_{j=0}^{n-1} \frac{f \circ \gamma(\xi_j)}{(\gamma(\xi_j) - a)^2} (\gamma(t_{j+1}) - \gamma(t_j))$$

where

$$\tau: \ 0 = t_0 < \xi_0 < t_1 < \xi_1 < t_2 < ... < \xi_{n-1} < t_n = 2\pi.$$

By the triangle inequality,

$$\begin{aligned} \|S\| &\le \sum_{j=0}^{n-1} \frac{\|f \circ \gamma(\xi_j)\|_X}{|\gamma(\xi_j) - a|^2} |\gamma(t_{j+1}) - \gamma(t_j)| \\ &\le \frac{M}{R^2} 2\pi R = 2\pi \frac{M}{R}. \end{aligned}$$

On combining this inequality with (4.4) we have

$$\|f'(a)\|_X \le \frac{M}{R}.$$

$\square$

Theorem 4.5. Liouville's Theorem. *A function that is analytic and bounded in the entire complex plane is constant.*

Proof. Let $f : \mathbb{C} \to X$ be analytic and bounded by M, so that

$$\|f(z)\|_X \le M$$

for all $z \in \mathbb{C}$.

Consider any $a \in \mathbb{C}$ and $R > 0$. The function f is analytic on $\triangle(a, R)$, and so, by Lemma 4.4,

$$\|f'(a)\| \le \frac{M}{R}.$$

Since this relation holds $\forall R > 0$, $\|f'(a)\|_X = 0$, and so $f'(a) = 0$. Thus $f'(z) = 0$ for all $z \in \mathbb{C}$. It follows that f is constant on $\mathbb{C}$. $\square$

Theorem 4.6. *For any element x in a Banach algebra A with identity $e \neq 0$, the spectrum $\sigma(x)$ of x is a non-empty closed bounded set in $\mathbb{C}$.*

Proof. By Theorem 3.8, $\sigma(x)$ is closed and bounded in $\mathbb{C}$. We must show $\sigma(x) \neq \emptyset$. Suppose $\sigma(x) = \emptyset$, so that $\rho(x) = \mathbb{C}$. By Theorem 4.2,

$$x(\cdot) : \mathbb{C} \to A : \lambda \mapsto (\lambda e - x)^{-1}$$

is analytic on the entire complex plane. Note that $x \neq 0$ since $0 \notin \sigma(0)$.

We claim that $x(\cdot)$ is a bounded function on $\mathbb{C}$. If $|\lambda| > 2\|x\|$,

$$\begin{aligned}
\|(\lambda e - x)^{-1})\| &= \|\lambda^{-1}\left(e - \frac{x}{\lambda}\right)^{-1}\| \\
&= |\lambda|^{-1}\|e + \frac{x}{\lambda} + \frac{x^2}{\lambda^2} + ...\| \quad \text{(see Lemma 3.3)} \\
&\leq \frac{1}{2\|x\|}\left(\|e\| + \sum_{n=1}^{\infty}\left\|\frac{x}{\lambda}\right\|^n\right) \\
&\leq \frac{1}{2\|x\|}(\|e\| + 1).
\end{aligned}$$

On the other hand $\|x(\cdot)\|$ is continuous on the closed bounded set

$$\{z \in \mathbb{C} : |z| \leq 2\|x\|\},$$

hence is bounded above by some $M_1 > 0$.
Let

$$M = \max\left\{M_1, \frac{\|e\| + 1}{2\|x\|}\right\}.$$

Then

$$\|x(\lambda)\| \leq M \qquad \forall \lambda \in \mathbb{C}.$$

By Liouville's Theorem 4.5, $x(\cdot)$ is constant

$$\therefore \quad (\lambda e - x)^{-1} = x(0) = -x^{-1} \quad \forall \lambda \in \mathbb{C}$$

$$\therefore \quad \lambda e - x = -x \qquad \forall \lambda \in \mathbb{C}$$

$$\therefore \quad \lambda e = 0 \qquad \forall \lambda \in \mathbb{C},$$

a contradiction to $e \neq 0$. Hence $\sigma(x) \neq \emptyset$. □

5 The Gelfand-Mazur Theorem

Definition 5.1. *A* division algebra *is a non-zero algebra with identity (over an arbitrary field) in which every non-zero element has a multiplicative inverse [is regular].*

Example 5.2. Every field is a division algebra over itself. The quaternions $\mathbb{H}$ form a (non-commutative) division algebra over $\mathbb{R}$.

Definition 5.3. *We say that two algebras A, B over $\mathbb{C}$ are* isomorphic *if there is a bijective mapping $F : A \to B$ such that $\forall x, y \in A, \lambda \in \mathbb{C}$,*

$$F(x + y) = F(x) + F(y),$$

$$F(xy) = F(x)F(y),$$

$$F(\lambda x) = \lambda F(x).$$

Definition 5.4. *We say that two Banach algebras A, B over $\mathbb{C}$ are* isomorphic *if there is a bounded linear operator $F : A \to B$ such that F is invertible and $\forall x, y \in A$,*

$$F(xy) = F(x)F(y).$$

Theorem 5.5. Gelfand-Mazur Theorem. *Every Banach division algebra over $\mathbb{C}$ is isomorphic the Banach algebra $\mathbb{C}$.*

Proof. Let A be a Banach algebra with identity $e \neq 0$, and suppose A is a division algebra over $\mathbb{C}$.

Consider any $x \in A$. By Theorem 4.6, $\sigma(x) \neq \emptyset$, so $\exists \lambda \in \sigma(x)$. Hence $\lambda e - x$ is singular. Since A is a division algebra, only 0 is singular.

$$\therefore \lambda e - x = 0,$$

$$\therefore x = \lambda e.$$

Define $F : \mathbb{C} \to A$ by $F(\lambda) = \lambda e$. We have just shown that F is surjective. Clearly F is injective, for if $\lambda \neq \mu$ and $F(\lambda) = F(\mu)$ then $F(\lambda) - F(\mu) = 0$.

$$\therefore \quad (\lambda - \mu)e = 0,$$

$$\therefore \quad (\lambda - \mu)^{-1}(\lambda - \mu)e = 0,$$

$$\therefore \quad e = 0.$$

This is contradiction. Hence F is bijective. For all $\lambda, \mu \in \mathbb{C}$,

$$F(\lambda + \mu) = (\lambda + \mu)e = \lambda e + \mu e = F(\lambda) + F(\mu),$$

$$F(\lambda\mu) = (\lambda\mu)e = (\lambda e)(\mu e) = F(\lambda)F(\mu),$$

$$F(\mu\lambda) = (\mu\lambda)e = \mu(\lambda e) = \mu F(\lambda).$$

Therefore F is an isomorphism of algebras over $\mathbb{C}$.

Note that F is bounded, since

$$\|F\| = \sup_{\lambda \in \mathbb{C},\ |\lambda| \leq 1} \|\lambda e\| = \|e\|.$$

The inverse map is bounded too, since

$$F^{-1} : A \to \mathbb{C} : x \mapsto \lambda$$

where $\lambda \in \sigma(x)$ and, by Theorem 3.8, $|\lambda| \leq \|x\|$. Hence $\|F^{-1}\| \leq 1$. Therefore F is an isomorphism of Banach algebras A and $\mathbb{C}$. $\square$

6 Convolution algebras

The main examples of Banach algebras encountered so far are algebras of continuous functions like $C[0,1]$ and $H^\infty(\Omega)$, and algebras of operators like $M_n(\mathbb{C})$ and $B(X)$. Another important class comprises convolution algebras.

Example 6.1. The algebra $\ell^1(\mathbb{Z}_+)$.
Let $\mathbb{Z}_+$ be the additive semigroup of non-negative integers, and let $\ell^1(\mathbb{Z}_+)$ be the Banach space of sequences $(x_n)_{n=0}^\infty$, with $x_n \in \mathbb{C}$ for all $n \geq 0$, such that

$$\sum_{n=0}^\infty |x_n| < \infty,$$

with co-ordinatewise addition and scalar multiplication and norm

$$\|(x_n)\|_1 = \sum_{n=0}^\infty |x_n|.$$

We could make $\ell^1(\mathbb{Z}_+)$ into a Banach algebra by introducing co-ordinatewise multiplication, but there is another natural multiplication which is more useful. We can identify a sequence $(x_n)_{n\geq 0}$ with the power series $\sum x_n z^n$, which we can think of as a formal power series or as the expansion of an analytic function on the unit disc $\triangle(0,1)$. Note that

$$\begin{aligned}
&(x_0z^0 + x_1z^1 + x_2z^2 + ...)(y_0z^0 + y_1z^1 + y_2z^2 + ...) \\
&= x_0y_0z^0 + (x_0y_1 + x_1y_0)z^1 + (x_0y_2 + x_1y_1 + x_2y_0)z^2 \\
&\quad + ... + (x_0y_n + x_1y_{n-1} + ... + x_ny_0)z^n + ...
\end{aligned}$$

This suggests the multiplication

$$\begin{aligned}
(x_n)_{n\geq 0} * (y_n)_{n\geq 0} &= (x_0y_n + x_1y_{n-1} + ... + x_ny_0)_{n\geq 0} \\
&= (\sum_{j=0}^n x_jy_{n-j})_{n\geq 0}
\end{aligned}$$

This is called the *convolution multiplication.*

If $x = (x_n)_{n\geq 0},\ y = (y_n)_{n\geq 0} \in \ell^1(\mathbb{Z}_+)$, then

$$\begin{aligned}
\|x * y\|_1 &= \|(\sum_{j=0}^{n} x_j y_{n-j})\|_1 \\
&= \sum_{n=0}^{\infty} \left| \sum_{j=0}^{n} x_j y_{n-j} \right| \\
&\leq \sum_{n=0}^{\infty} \sum_{j=0}^{n} |x_j||y_{n-j}| \\
&\leq \sum_{j,k=0}^{\infty} |x_j||y_k| \\
&= \|x\|_1 \|y\|_1.
\end{aligned}$$

Thus $\|.\|_1$ is submultiplicative on $(\ell^1(\mathbb{Z}_+), +, *)$ and $\ell^1(\mathbb{Z}_+)$ is a Banach algebra with respect to $*$.

Example 6.2. The algebra $\ell^1(\mathbb{Z})$.
Let $\ell^1(\mathbb{Z})$ be the Banach space of sequences $(x_n)_{n\in\mathbb{Z}}$, $x_n \in \mathbb{C}$ for all $n \in \mathbb{Z}$, such that

$$\sum_{n=-\infty}^{\infty} |x_n| < \infty$$

with co-ordinatewise addition and scalar multiplication and norm

$$\|(x_n)_{n\in\mathbb{Z}}\|_1 = \sum_{n=-\infty}^{\infty} |x_n|.$$

The sequence $(x_n)_{n\in\mathbb{Z}}$ can be identified with the Laurent series $\sum_{n=-\infty}^{\infty} x_n z^n$. This power series does converge for $z \in \mathbb{T}$, the unit circle, though we may think of it as a formal power series. Formally,

$$\begin{aligned}
&\left(... + x_{-2}z^{-2} + x_{-1}z^{-1} + x_0 z^0 + x_1 z^1 + ...\right)\left(... + y_{-2}z^{-2} + y_{-1}z^{-1} + y_0 z^0 + y_1 z^1 + ...\right) \\
&= \left(\qquad\qquad + \left(\sum_{k=-\infty}^{\infty} x_k y_{-k}\right) z^0 + \left(\sum_{k=-\infty}^{\infty} x_k y_{1-k}\right) z^1 + ... \right) \\
&= \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x_k y_{n-k}\right) z^n.
\end{aligned}$$

We define convolution multiplication $*$ on $\ell^1(\mathbb{Z})$ by

$$(x_n)_{n\in\mathbb{Z}} * (y_n)_{n\in\mathbb{Z}} = \left(\sum_{k=-\infty}^{\infty} x_k y_{n-k}\right)_{n\in\mathbb{Z}}.$$

As in the previous example, we find, for $x, y \in \ell^1(\mathbb{Z})$,

$$\|x * y\|_1 \leq \|x\|_1 \|y\|_1.$$

Hence $\ell^1(\mathbb{Z})$ is a commutative Banach algebra with respect to $*$. Since $\ell^1(\mathbb{Z})$ can be thought of as an algebra of power series $\sum_{-\infty}^{\infty} x_n z^n$ with pointwise multiplication, it is clear that it has an identity

$$e = (...0, 0, 1, 0, 0, ...),$$

where 1 is in the "zeroth co-ordinate", corresponds to the constant function 1.

Example 6.3. Group algebras $\mathbb{C}_G$.
Let G be a finite group. The *group algebra* $\mathbb{C}_G$ is the algebra of formal sums $\sum_{g\in G} a_g g$ where $a_g \in \mathbb{C}$, with the natural addition and scalar multiplication, and multiplication $*$ given by

$$\left(\sum_{g\in G} a_g g\right) * \left(\sum_{g\in G} b_g g\right) = \left(\sum_{g\in G} \left(\sum_{kh=g} a_k b_h\right) g\right).$$

The formal sum $\sum_{g\in G} a_g g$ is really "the same thing" as the function $a : G \to \mathbb{C}$ given by $a(g) = a_g$. With this notation, $a * b$ is given by

$$\begin{aligned} a * b : G \to \mathbb{C} : g \mapsto & \sum_{k,h\in G,\ kh=g} a(k)b(h) \\ &= \sum_{k\in G} a(k)b(k^{-1}g) \\ &= \sum_{h\in G} a(gh^{-1})b(h). \end{aligned}$$

Hence $\mathbb{C}_G$ is a finite dimensional algebra with identity $1e_G$, where e_G is the identity of G. Note that $\mathbb{C}_G$ is commutative $\iff$ G is abelian. The algebra $\mathbb{C}_G$ is a Banach algebra with respect to the norm

$$\|\sum_{g\in G} a_g g\|_1 = \sum_{g\in G} |a_g|.$$

Example 6.4. The algebra $L^1(\mathbb{R})$.
Let $L^1(\mathbb{R})$ be the Banach space of Lebesgue measurable functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$\int_{-\infty}^{\infty} |f(t)|dt < \infty$$

with pointwise addition and scalar multiplication and norm

$$\|f\|_1 = \int_{-\infty}^{\infty} |f(t)|dt.$$

In this space we regard two functions f and g as equal if they agree "almost everywhere", that is, everywhere except a countable sequence of points. One of the most significant properties of the Lebesgue integral is that $L^1(\mathbb{R})$ is complete.

For $f, g \in L^1(\mathbb{R})$ we define $f * g : \mathbb{R} \to \mathbb{C}$ by

$$f * g(t) = \int_{-\infty}^{\infty} f(s)g(t-s)ds.$$

We shall show that $f * g \in L^1(\mathbb{R})$ using two properties of the Lebesgue integral.

1. *Translation invariance.* For $h \in L^1(\mathbb{R}),\ s \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} h(t-s)dt = \int_{-\infty}^{\infty} h(t)dt.$$

2. *The Fubini-Tonnelli theorem.* For an integrable Lebesgue measurable function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, if $F(s,t) \geq 0$ for all $s, t \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(s,t)dtds = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(s,t)dsdt$$

("reversing the order of integration").

We have

$$\begin{aligned}
\int_{-\infty}^{\infty} |(f*g)(t)| dt &= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f(s)g(t-s)ds \right| dt \\
&\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(s)||g(t-s)|dsdt \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(s)||g(t-s)|dtds \quad \text{(the Fubini-Tonnelli theorem)} \\
&= \int_{-\infty}^{\infty} |f(s)| \int_{-\infty}^{\infty} |g(t-s)|dt\, ds \\
&= \int_{-\infty}^{\infty} |f(s)| \int_{-\infty}^{\infty} |g(t)|dt\, ds \quad \text{(translation invariance)} \\
&= \int_{-\infty}^{\infty} |f(s)| \|g\|_1 ds \\
&= \|f\|_1 \|g\|_1
\end{aligned}$$

Hence $f * g \in L^1(\mathbb{R})$, and, moreover, $\|.\|_1$ is submultiplicative.

Let us show that $*$ is associative. For $f, g, h \in L^1(\mathbb{R})$ and $t \in \mathbb{R}$,

$$\begin{aligned}
((f*g)*h)(t) &= \int_{-\infty}^{\infty} (f*g)(s)\; h(t-s)\; ds \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u)g(s-u)du\; h(t-s)ds \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u)g(s-u)h(t-s)\; ds\; du \quad \text{(the Fubini-Tonnelli theorem)} \\
&= \int_{-\infty}^{\infty} f(u) \int_{-\infty}^{\infty} g(s-u)h(t-s)\; ds\; du \\
&= \int_{-\infty}^{\infty} f(u) \int_{-\infty}^{\infty} g(s)h(t-u-s)ds\; du \quad \text{(translation invariance)} \\
&= \int_{-\infty}^{\infty} f(u)\; (g*h)(t-u)\; du \\
&= (f*(g*h))(t).
\end{aligned}$$

Hence $*$ is associative. Thus $L^1(\mathbb{R})$ is a Banach algebra. It is commutative, but has no identity.

Note the *probabilistic interpretation of* $*$: if f, g are probability density functions of random variables X and Y then $f * g$ is the probability density function of $X + Y$.

7 Homomorphisms and ideals

Definition 7.1. *Let A, B be Banach algebras. A homomorphism from A to B is a bounded linear operator*

$$\phi : A \to B$$

such that, $\forall x, y \in A$,

$$\phi(xy) = \phi(x)\phi(y).$$

If A, B have identities e_A, e_B respectively and a homomorphism $\phi : A \to B$ satisfies

$$\phi(e_A) = e_B$$

then ϕ is said to be a unital homomorphism.

Examples 7.2.

1. For any Banach algebras A and B, the map $\phi : A \to B : x \mapsto 0$ is a homomorphism, called the zero homomorphism.

2. For $\alpha \in [0, 1]$, let
$$v_\alpha : C[0, 1] \to \mathbb{C} : x \mapsto x(\alpha).$$
The map v_α is "evaluation at α". For $x, y \in C[0, 1]$, and $\lambda \in \mathbb{C}$,
$v_\alpha(x + y) = (x + y)(\alpha) = x(\alpha) + y(\alpha) = v_\alpha(x) + v_\alpha(y)$,
$v_\alpha(\lambda x) = (\lambda x)(\alpha) = \lambda x(\alpha) = \lambda v_\alpha(x)$,
$v_\alpha(xy) = (xy)(\alpha) = x(\alpha)y(\alpha) = v_\alpha(x)v_\alpha(y)$.
The operator norm
$$\|v_\alpha\| = \sup_{\|x\|_\infty \leq 1} |v_\alpha(x)| = \sup_{\|x\|_\infty \leq 1} |x(\alpha)| = 1.$$
Recall that $C[0, 1]$ has an identity e given by $e(t) = 1 \quad \forall t \in [0, 1]$. Thus $v_\alpha(e) = e(\alpha) = 1$ the identity of $\mathbb{C}$.
Hence v_α is a unital homomorphism of Banach algebras.

3. Notation. If X, Y are sets, $M \subset X$ and $f : X \to Y$ is a map then $f|_M$ is the restriction of f to M, that is, the map
$$f|_M : M \to Y : x \mapsto f(x).$$
Define
$$\phi : C[0, 1] \to C\left[0, \frac{1}{2}\right] : f \mapsto f|_{[0,\frac{1}{2}]}.$$
It is easy to check that
$$\|\phi\| \leq 1$$
and ϕ is a homomorphism.

4. $\phi : M_2(\mathbb{C}) \to M_4(\mathbb{C}) : A \mapsto \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$ is a homomorphism.

5. For $\alpha \in \triangle(0,1)$ define

$$v_\alpha : \ell^1(\mathbb{Z}_+) \to \mathbb{C} : (x_n)_{n\in\mathbb{Z}_+} \mapsto \sum_{n=0}^{\infty} x_n \alpha^n.$$

   Then v_α is a homomorphism. Linearity is easy.

$$\begin{aligned} v_\alpha(x * y) &= \sum_{n=0}^{\infty} (x*y)_n \alpha^n \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} x_k y_{n-k} \alpha^n \\ &= \sum_{n,m=0}^{\infty} x_n y_m \alpha^{n+m} \\ &= \sum_{n=0}^{\infty} x_n \alpha^n \sum_{m=0}^{\infty} y_m \alpha^m \\ &= v_\alpha(x) v_\alpha(y). \end{aligned}$$

   See Example 6.1.

6. Let $C(\triangle(0,1))$ be the Banach algebra of continuous $\mathbb{C}$-valued functions on $\triangle(0,1)$ with sup norm. Define

$$\phi : \ell^1(\mathbb{Z}_+) \to C(\triangle(0,1))$$

   by if $x = (x_n)_{n\geq 0} \in \ell^1(\mathbb{Z}_+)$, $\phi(x)$ is the function

$$z \mapsto \sum_{0}^{\infty} x_n z^n \quad \text{in} \quad C(\triangle(0,1)).$$

   The proof that $\phi(x*y) = \phi(x)\phi(y)$ is the same calculation as in (5).

All except (1) are unital homomorphisms.
None of them are isomorphisms, i.e. none is bijective.
All of them are continuous operators.

Definition 7.3. *A character of a commutative Banach algebra A is a non-zero homomorphism from A to $\mathbb{C}$.*

Examples 7.2 (2) and (5) above are examples of characters.

Remark 7.4. *$\phi(A)$ is a subspace of $\mathbb{C}$ and is not $\{0\}$, hence is $\mathbb{C}$. Thus ϕ is surjective.*

Theorem 7.5. *Let ϕ be a character of a commutative Banach algebra A with identity $e \neq 0$. Then $\phi(e) = 1$ and, for any $x \in A$,*

$$\phi(x) \in \sigma(x),$$

and so $|\phi(x)| \leq \|x\|$.

Proof. Since $\phi(A) = \mathbb{C}$ there exists $a \in A$ such that $\phi(a) = 1$. Then

$$1 = \phi(a) = \phi(ae) = \phi(a)\phi(e) = \phi(e).$$

Hence ϕ is unital homomorphism.

Suppose $\phi(x) \notin \sigma(x)$, that is, $\phi(x)e - x$ is a regular element in A. Hence it has an inverse $u \in A$:

$$(\phi(x)e - x)u = e.$$

Since ϕ is multiplicative,

$$\phi((\phi(x)e - x)u) = \phi(e) = 1,$$

$$\therefore (\phi(x)\phi(e) - \phi(x))\phi(u) = 1,$$

$$\therefore 0 = 1 \text{ - contradiction.}$$

Hence $\phi(x) \in \sigma(x)$. By Theorem 3.8,

$$|\phi(x)| \leq \|x\|.$$

□

Corollary 7.6. *Let A be a commutative Banach algebra with identity $e \neq 0$. Characters of A are bounded linear functionals on A, and so belong to the dual space A^* of A and have norm ≤ 1.*

Characters are also called *multiplicative linear functionals.*

Definition 7.7. *The set of characters of a commutative Banach algebra A is called* the character space of *A and is denoted by $\hat{A}$.*

Examples 7.8. (1) $\widehat{C[0,1]} \supset \{v_\alpha : 0 \leq \alpha \leq 1\}$.

(2) $\widehat{\ell^1(\mathbb{Z}_+)} \supset \{\phi_\alpha : \alpha \in \triangle(0,1)\}$.
Does equality hold? To be answered later.

Remark 7.9. Suppose $\phi : A \to B$ is a homomorphism of algebras over $\mathbb{C}$. Recall that
$$\ker \phi = \{x \in A : \phi(x) = 0\}.$$
Kerϕ is also called the *null space of* ϕ.

Since ϕ is a linear mapping, Ker ϕ is a linear subspace of A.

If $x \in \ker \phi$ and $a \in A$ then
$$\phi(ax) = \phi(a)\phi(x) = \phi(a)0 = 0,$$
and similarly $\phi(xa) = 0$.

Hence if $x \in \ker \phi$, then $ax \in \ker \phi$ and $xa \in \ker \phi$ for all $a \in A$.

Definition 7.10. *An* ideal *in an algebra A is a linear subspace I of A such that, for any $x \in I$ and $a \in A$, $ax \in I$ and $xa \in I$.*
An closed ideal *in a Banach algebra A is a closed linear subspace I of A such that, for any $x \in I$ and $a \in A$, $ax \in I$ and $xa \in I$.*

An ideal I is proper *if $I \neq A$.*

Examples 7.11.

1. In any algebra A, $\{0\}$ and A are ideals.
2. In $C[0,1]$, for any set $E \subset [0,1]$,
$$\{x \in C[0,1] : x|E = 0\}$$
is an ideal.

   In particular, for $\alpha \in [0,1]$,
$$\ker v_\alpha = \{x \in C[0,1] : x(\alpha) = 0\}$$
is an ideal.
3. In $\ell^1(\mathbb{Z}_+)$, for $\alpha \in \triangle(0,1)$,
$$\ker \phi_\alpha = \{(x_n) \in \ell^1(\mathbb{Z}_+) : \sum_{n=0}^{\infty} x_n \alpha^n = 0\}$$
is an ideal.

4. In $\mathbb{C}[z]$,
$$\{f : f(3-i) = f(4+2i) = 0\}$$
is an ideal.

5. c_0 is an ideal in ℓ^∞.

Theorem 7.12. *In an algebra A with identity $e \neq 0$, let I be an ideal. The following are equivalent:*

1. I is proper;

2. $e \notin I$;

3. I does not contain any regular element.

Proof. (3) $\Rightarrow$ (2) $\Rightarrow$ (1) is obvious.

To prove (1) $\Rightarrow$ (3), suppose I is proper and I contains a regular element x. Then x has an inverse $x^{-1} \in A$, and so for any $a \in A$ we have
$$(ax^{-1})x \in I.$$
Therefore, $a \in I$. Hence $A = I$, this is a contradiction. Thus (1) $\Rightarrow$ (3).

Consequently (1)-(3) are equivalent. $\square$

Definition 7.13. *An ideal I in an algebra A is* maximal *if it is proper and the only ideals of A which contain I are I and A.*

Examples 7.14. 1. $\{0\}$ is a maximal ideal in $\mathbb{C}$.

2. In $C[0,1]$, $I = \{f : f(0) = 0 = f(1)\}$ is an ideal, but is not maximal since the ideal
$$J = \{f : f(0) = 0\}$$
is an ideal, $J \supset I$ and J is not equal to I or $C[0,1]$.

Theorem 7.15. *If ϕ is a character of a commutative Banach algebra A then* $\ker\phi$ *is a maximal ideal of A.*

Proof. Let $M = \ker\phi$. Clearly M is an ideal of A, and M is proper, else ϕ is a zero homomorphism contrary to the definition of character.

Let us show that M is maximal. Suppose $M \subset I \subset A$, where I is an ideal of A and $M \neq I$. We wish to show $I = A$. Observe that $\phi(I)$ is a linear subspace of

$\mathbb{C}$, and hence either $\phi(I) = \{0\}$ or $\phi(I) = \mathbb{C}$.

If $\phi(I) = \{0\}$ then $I \subset \ker \phi = M$, hence $I = M$, contrary to hypothesis. Hence $\phi(I) = \mathbb{C}$.

Pick $x \in I$ such that $\phi(x) = 1$. For any $a \in A$ we have

$$\phi(a - \phi(a)x) = \phi(a) - \phi(a) = 0,$$

and so

$$a - \phi(a)x \in M.$$

Hence

$$a \in \phi(a)x + M \in I + M = I.$$

That is, $a \in I$. Hence $I = A$. Thus M is a maximal ideal. $\square$

Example 7.16. $\ker v_{\frac{1}{2}i} = \{f \in H^\infty(\mathbb{D}) : f(\frac{1}{2}i) = 0\}$ is a maximum ideal in $H^\infty(\mathbb{D})$.

Theorem 7.17. *In an algebra A with identity every proper ideal is contained in a maximal ideal.*

Proof. Let I be an ideal of A and let e be the identity element of A.

Let $\mathcal{J}$ be the set of proper ideals of A that contain I. $\mathcal{J}$ is a partially ordered set under inclusion:

$$\mathcal{J}_1 \leq \mathcal{J}_2 \quad \text{means} \quad \mathcal{J}_1 \subset \mathcal{J}_2.$$

If $\mathcal{C}$ is a chain in $\mathcal{J}$ then $\bigcup_{J \in \mathcal{C}} J$ is an ideal of A containing I, and $\bigcup_{J \in \mathcal{C}} J$ is proper since it does not contain e. That is, $\bigcup_{J \in \mathcal{C}} J \in \mathcal{J}$, and so every chain in $\mathcal{J}$ has an upper bound in $\mathcal{J}$. By Zorn's Lemma, $\mathcal{J}$ contains a maximal element M. Thus M is a proper ideal of A containing I. If J is an ideal of A containing M then either $J = A$ or $J \in \mathcal{J}$, and hence, since M is a maximal element of $\mathcal{J}$, $J = M$. Therefore M is a maximal ideal of A, and $I \subset M$.

$\square$

Corollary 7.18. *Let A be a commutative algebra with identity. For an element $x \in A$, x is singular if and only if x belongs to some maximal ideal of A.*

Proof. $\Leftarrow$ If x is not singular then, by Theorem 7.12, x does not belong to any proper ideal of A, and hence x does not belong to any maximal ideal. Therefore, x is in some maximal ideal $\Rightarrow$ that x is singular.

$\Rightarrow$ Suppose x is singular. Let

$$xA = \{xa : a \in A\}.$$

The linear subspace xA is an ideal of A and, since x is singular, $e \notin xA$. Hence, by Theorem 7.12, xA is a proper ideal, and $x = xe \in xA$. By Theorem 7.17, there is a maximal ideal M of A such that $xA \subset M$. Hence x belongs to some maximal ideal of A. $\square$

8 The algebra C(K).

Definition 8.1. *Let K be a compact Hausdorff space. $C(K)$ denotes the Banach algebra of continuous $\mathbb{C}$-valued functions on K with pointwise operations and supremum norm $\|f\|_\infty = \sup_{t\in K} |f(t)|$. Thus, for $f, g \in C(K)$ and $\lambda \in \mathbb{C}$, we define*

$$\begin{aligned}
(f+g)(t) &= f(t) + g(t), && t \in K,\\
(\lambda f)(t) &= \lambda f(t), && t \in K,\\
(fg)(t) &= f(t)g(t), && t \in K.
\end{aligned}$$

Theorem 8.2. *Let K be a compact Hausdorff space. Then $(C(K), \|\cdot\|_\infty)$ is a Banach algebra.*

Proof. For $f \in C(K)$,

$$\sup_{t\in K} |f(t)| < \infty$$

since continuous $\mathbb{R}$-valued functions on compact sets are bounded. Hence $\|\cdot\|_\infty$ is well defined on $C(K)$.

The linear space $C(K)$ is closed under the stated algebraic operations. For example, if $f, g \in C(K)$ then $f + g$ is continuous, being the composition of two continuous maps:

$$\begin{array}{ccc}
K \to & \mathbb{C} \times \mathbb{C} \to & \mathbb{C}: \\
t \mapsto & (f(t), g(t)) \mapsto & f(t) + g(t).
\end{array}$$

It is straightforward to check that $C(K)$ is a normed algebra over $\mathbb{C}$. For all $f, g \in C(K)$,

$$\|fg\|_\infty = \sup_{t\in K} |f(t)g(t)| = \sup_{t\in K} |f(t)||g(t)| \le \sup_{t\in K} |f(t)| \sup_{t\in K} |g(t)| = \|f\|_\infty \|g\|_\infty.$$

Completeness of $(C(K), \|\cdot\|_\infty)$. Let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $C(K)$. We must show $(f_n)_{n=1}^\infty$ tends to a limit f in $(C(K), \|\cdot\|_\infty)$.

For any $t \in K$, $(f_n(t))_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{C}$ since

$$|f_n(t) - f_m(t)| \le \|f_n - f_m\|_\infty.$$

The normed space $(\mathbb{C}, |\cdot|)$ is complete, and so there exists $f(t) \in \mathbb{C}$ such that

$$\lim_{n\to\infty} f_n(t) = f(t).$$

We will show below that f is continuous on K and that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. □

Digression on limits of sequences of continuous functions.

It is not true in general that if f_n is continuous on K for every n and $f_n(t) \to f(t)$ as $n \to \infty$ for all $t \in K$ then f is continuous on K.

Example 8.3. *Let*

$$f_n(t) = \begin{cases} 1 - nt & \textit{if } 0 \leq t \leq 1/n, \\ 0 & \textit{if } t > 1/n. \end{cases}$$

Then $f_n(t) \to f(t)$ *as* $n \to \infty$, *where* $f(t) = \begin{cases} 1 & \textit{if } t = 0, \\ 0 & \textit{if } t > 0 \end{cases}$ *is not continuous on* $[0, 1]$.

Definition 8.4. *Let* f_n, $n \in \mathbb{N}$, *and* g *be* $\mathbb{C}$*-valued functions on a set* E. *We say that* $f_n \to g$

(i) pointwise *on* E *as* $n \to \infty$ *if* $\lim_{n\to\infty} f_n(t) = g(t)$ *for every* $t \in E$;

(ii) uniformly *on* E *as* $n \to \infty$ *if, for every* $\epsilon > 0$, *there exists* $N \in \mathbb{N}$ *such that* $n \geq N$ *implies*

$$|f_n(t) - g(t)| < \epsilon \textit{ for all } t \in E.$$

In the above example $f_n \to f$ as $n \to \infty$ pointwise but not uniformly. The example shows that a pointwise limit of continuous functions on $[0, 1]$ need not be continuous.

Theorem 8.5. *If* $(f_n)_{n=1}^{\infty}$ *is a sequence of continuous* $\mathbb{C}$*-valued maps on a topological space* E *and* $f_n \to g$ *uniformly on* E *as* $n \to \infty$ *then* g *is continuous on* E.

Proof. This is an "$\frac{\epsilon}{3}$ argument". Consider $t \in E$. We want to show that g is continuous at t.

Let $\epsilon > 0$. Since $f_n \to g$ uniformly there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|f_n(s) - g(s)| < \frac{\epsilon}{3}$ for all $s \in E$.

Since f_N is continuous at t there is a neighbourhood U of t in E such that

$$|f_N(t) - f_N(s)| < \frac{\epsilon}{3} \text{ for all } s \in U.$$

Then, for any $s \in U$,

$$\begin{aligned}|g(t) - g(s)| &= |g(t) - f_N(t) + f_N(t) - f_N(s) + f_N(s) - g(s)| \\ &\leq |g(t) - f_N(t)| + |f_N(t) - f_N(s)| + |f_N(s) - g(s)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon.\end{aligned}$$

Hence g is continuous at $t \in E$. $\square$

Return to the proof that $C(K)$ is complete (Theorem 8.2).

Recall that $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence and $f_n(t) \to f(t)$ as $n \to \infty$ for every $t \in K$.

Let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that $m, n \geq N$ implies

$$\|f_m - f_n\|_\infty < \epsilon,$$

and hence, for any $t \in K$,

$$|f_m(t) - f_n(t)| < \epsilon.$$

Therefore

$$\lim_{m\to\infty} |f_m(t) - f_n(t)| \leq \epsilon$$

and

$$|f_n(t) - f(t)| \leq \epsilon \ \ \forall t \in K \text{ and } \ n \geq N.$$

Thus $f_n \to f$ uniformly on K as $n \to \infty$. By Theorem 8.5, f is continuous on K. Moreover, for any $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$\|f_n - f\|_\infty \leq \epsilon \ \text{ whenever } \ n \geq N,$$

by above. Thus $f_n \to f$ as $n \to \infty$ in $(C(K), \|\cdot\|_\infty)$.

Hence $C(K)$ is complete with respect to $\|\cdot\|_\infty$. Thus $C(K)$ is a Banach algebra. Note that $C(K)$ is a commutative Banach algebra with identity e where $e(t) = 1 \ \forall t \in K$.

9 Characters of $C(K)$

Definition 9.1. *For $t \in K$, we call the functional*

$$v_t : C(K) \to \mathbb{C} : \ f \mapsto f(t)$$

an evaluation functional *on* $C(K)$.

Evaluation functionals are characters of $C(K)$.

Theorem 9.2. *Let K be a compact Hausdorff space. Let ϕ be a character of $C(K)$. There exists $t_0 \in K$ such that*

$$\phi(f) = f(t_0) \qquad\qquad \forall f \in C(K)$$

Thus the characters of $C(K)$ are precisely the evaluation functionals.

Proof. Let $M = \ker \phi$. Since ϕ is non-zero, M is a proper ideal in $C(K)$.

Suppose there is no $t_0 \in K$ such that every element of M vanishes at t_0. Then, for each $t \in K$, there exists $f_t \in M$ such that $f_t(t) \neq 0$.

Let $g_t = \bar{f}_t f_t$: then $g_t \in M$ and $g_t \geq 0$ on K and $g_t(t) > 0$. Let

$$U_t = g_t^{-1}(0, \infty) = \{\tau \in K : g_t(\tau) > 0\}.$$

Then U_t is an open neighbourhood of t, and so $\{U_t : t \in K\}$ is an open cover of K. Since K is compact, there exists a finite subcover $\{U_{t_1}, ..., U_{t_n}\}$ for some $t_1, ..., t_n \in K$. Let

$$g = g_{t_1} + g_{t_2} + ... + g_{t_n}.$$

Then $g \in M$ and $g > 0$ on K. Hence $\frac{1}{g} \in C(K)$, that is, g is a regular element of $C(K)$. This contradicts the fact that M is a proper ideal of $C(K)$ (see Theorem 7.12). Hence $\exists t_0 \in K$ such that

$$f(t_0) = 0 \qquad\qquad \forall f \in M.$$

In other words,

$$M \subset \{f \in C(K) : f(t_0) = 0\}.$$

By Theorem 7.15, M is a maximal ideal of $C(K)$. Thus, by Theorem 7.17, we must have

$$M = \{f \in C(K) : f(t_0) = 0\}.$$

Since ϕ is non-zero, we have $\phi(e) \neq 0$. However, since $e^2 = e$, we have $\phi(e)^2 = \phi(e)$, and so $\phi(e) = 1$. For any $h \in C(K)$, we can write

$$h = (h - h(t_0)e) + h(t_0)e$$

where $h - h(t_0)e$ vanishes at t_0 hence belongs to M. Thus

$$\phi(h) = 0 + h(t_0)\phi(e) = h(t_0).$$

Consequently ϕ is evaluation at t_0. □

We can state this result:

$$C(K)^\Lambda = \{v_t : t \in K\}$$

where v_t is evaluation at t, that is, $v_t(f) = f(t)$ for all $f \in C(K)$.

Part II. Topics in Topology

This part concerns the theory of pseudometrics on sets, weak*-topologies on dual Banach spaces and product topologies. We shall define these objects and study their properties.

Contents of Part II

10 Metric spaces

Definition 10.1. *A metric space is a pair* $(M,\ d)$ *where* M *is a set and a metric*

$$d : M \times M \to \mathbb{R}^+$$

satisfies

1. $d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$ (symmetry)
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

for all $x, y, z \in M$.

Examples 10.2.

1. Any normed linear space X is a metric space with respect to the metric
$$d(x, y) = \|x - y\| \text{ for all } \ x, y \in X.$$
2. Any subset of $(X,\ d)$ of a metric space $(M,\ d)$ is a metric space with respect to the restriction of d, since the restriction of a metric is a metric.
3. For any set M, define
$$d : M \times M \to \mathbb{R}^+$$
by
$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$
This is the "small world" metric.

Definition 10.3. *In a metric space* (M, d) *we define, for* $\epsilon > 0$, *the* open ball of centre x, radius ϵ, *to be*

$$B_\epsilon(x) = \{y \in M : \ d(x, y) < \epsilon\} \subset M.$$

For any set $U \subset M$ *we say that* $x \in U$ *is an* interior point *of* U *if there exists* $\epsilon > 0$ *such that* $B_\epsilon(x) \subset U$.

We say that U *is an* d-open set *in* M *if every point of* U *is an interior point of* U.

Thus U is an open set in (M, d) if for all $x \in U$ there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset U$. If we need to emphasize the metric we write $B_\epsilon(x, d)$.

Examples 10.4. 1. For (M, d) as in the discrete metric, $B_{\frac{1}{2}}(x) = x$ for any $x \in M$. Thus every subset of M is open.

2. Let $M = \mathbb{R}^2$ and let d_1, d_2, d_∞ be the metrics induced by the norms $\|.\|_1, \|.\|_2$ and $\|.\|_\infty$ respectively. Thus

$$\begin{aligned} \|(x_1, x_2)\|_1 &= |x_1| + |x_2| \\ \|(x_1, x_2)\|_2 &= \{|x_1|^2 + |x_2|^2\}^{\frac{1}{2}} \\ \|(x_1, x_2)\|_\infty &= \max\{|x_1|, |x_2|\}, \end{aligned}$$

and for $x, y \in \mathbb{C}^2$,

$$\begin{aligned} d_1(x, y) &= \|x - y\|_1 \\ d_2(x, y) &= \|x - y\|_2 \\ d_\infty(x, y) &= \|x - y\|_\infty. \end{aligned}$$

The set $U = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$ is open with respect to all 3 metrics. Choose $\epsilon = \min\{x_1, x_2\} > 0$. Then

$$B_\epsilon(x, d_1) \subset U, \quad B_\epsilon(x, d_2) \subset U, \quad B_\epsilon(x, d_\infty) \subset U.$$

11 Topologies of metric spaces

Definition 11.1. *Two metrics on a set M are said to be topologically equivalent if they give rise to the same open sets.*

Note that d_2 and d_∞ are topologically equivalent metrics on $\mathbb{R}^2$, while d_2 and d defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y, \end{cases}$$

are topologically inequivalent, since the singleton set $\{0\}$ is d-open, but not d_2-open.

Definition 11.2. *The topology $\mathcal{J}$ of a metric space (M, d) is the collection of d-open sets in M.*

Theorem 11.3. *The topology $\mathcal{J}$ of a metric space (M, d) satisfies the following conditions:*

T1. $\emptyset \in \mathcal{J}$ and $M \in \mathcal{J}$;

T2. if $U, V \in \mathcal{J}$ then $U \cap V \in \mathcal{J}$;

T3. if $\{U_\alpha : \alpha \in A\}$ is a collection of members of $\mathcal{J}$, for any index set A, then

$$\bigcup_{\alpha \in A} U_\alpha \in \mathcal{J}.$$

Proof. T1. It is immediate.

T2. Let $U, V \in \mathcal{J}$ and let $x \in U \cap V$. Choose ϵ_1, ϵ_2 such that $B_{\epsilon_1}(x) \subset U$, $B_{\epsilon_2}(x) \subset V$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Then $B_\epsilon(x) \subset U \cap V$. Hence x is interior to $U \cap V$. Thus $U \cap V \in \mathcal{J}$.

T3. Let $U_\alpha \in \mathcal{J}$ for $\alpha \in A$ and let $U = \bigcup_\alpha U_\alpha$. Consider $x \in U$. Then $x \in U_\beta$ for some $\beta \in A$, and hence there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset U_\beta \subset U$. Hence x is interior to U. Thus $U \in \mathcal{J}$.

□

Remark 11.4. It follows from (T2) that the topology $\mathcal{J}$ of a metric space is closed under finite intersections: if $U_1, U_2, ..., U_n$ are open sets then $U_1 \cap ... \cap U_n$ is open. However $\mathcal{J}$ is not in general closed under infinite intersections. In $\mathbb{R}$, with its standard metric ($d(x, y) = |x - y|$), take

$$U_n = (-\frac{1}{n}, \frac{1}{n}) = B_{\frac{1}{n}}(0).$$

$U_1 \cap U_2 \cap ... = \{0\}$ which is not open.

12 Topological spaces

In analysis we often need a more subtle notion of "nearness" or continuity than that provided by a metric. The idea of a topology provides such a notion.

Definition 12.1. *A* topology *on a set X is a family $\mathcal{J}$ of subsets of X with the properties:*

T1. $\emptyset \in \mathcal{J}$, $X \in \mathcal{J}$;

T2. $\mathcal{J}$ *is closed under finite intersections;*

T3. $\mathcal{J}$ *is closed under arbitrary unions.*

A topological space *is a pair $(X, \mathcal{J})$ where X is a set and $\mathcal{J}$ is a topology on X. Elements of $\mathcal{J}$ are called* open sets or $\mathcal{J}$-open sets.

By Theorem 11.3, every metric space is naturally a topological space.

Not all topological spaces are obtainable from metrics.

Examples 12.2.

1. Let (M, d) is a metric space and let $\mathcal{J}$ be the collection of d-open sets. Then $(M, \mathcal{J})$ is a topological space.

2. Let X be a set. The *indiscrete or trivial topology* on X is $\mathcal{J} = \{\emptyset, X\}$. Thus an indiscrete topological space is
$$(X, \mathcal{J}) \text{ where } \mathcal{J} = \{\emptyset, \mathrm{X}\}.$$

3. The *discrete topology*: $(X, \mathcal{J})$ where $\mathcal{J}$ is the collection of all subsets of X.

   The topology is induced by the "small world metric": $d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$

4. The *cofinite topology*: $(\mathbb{N}, \mathcal{J})$ where $\mathcal{J}$ consists of the empty set and all sets with finite complement in $\mathbb{N}$.

5. $\mathbb{R}^n, \mathbb{C}^n$ have a natural or standard topology, induced by the Euclidean norm.

Remark 12.3. Note that in a metric space (M, d), for any $x \in M$ the set $U = M \setminus \{x\}$ is an open set. For $y \in U$ such that $d(x, y) > 0$, we have $B_\epsilon(y) \subset U$ where $\epsilon = \frac{1}{2}d(x, y)$.

In Example 2 with X having more than one point, the complements of singletons do not belong to $\mathcal{J}$. Hence there is no metric d on X such that the topology corresponding to d is $\mathcal{J}$.

Thus the notion of a topological space is more general than the notion of a metric space.

13 Pseudometrics

Definition 13.1. *A pseudometric on a set X is a function*

$$d : X \times X \to \mathbb{R}^+$$

which satisfies, for all $x, y, z \in X$,

1. $d(x, y) = d(y, x)$;
2. $d(x, z) \leq d(x, y) + d(y, z)$.

Note: it can happen that $d(x, y) = 0$ even though $x \neq y$.

Example 13.2. On $C[0, 1]$, define d_t, for $0 \leq t \leq 1$, by

$$d_t(f, g) = |f(t) - g(t)|.$$

Note that d_t is a pseudometric for $t \in [0, 1]$.

Definition 13.3. *Let $\mathcal{D}$ be a non-empty set of pseudometrics on a set X. For any $x \in X, \epsilon > 0$ and any finite set $d_1, ..., d_n \in \mathcal{D}$, we define*

$$B_\epsilon(x; d_1, ..., d_n) = \{y \in X : d_j(x, y) < \epsilon \ \ \textit{for}\ j = 1, 2, ..., n\}.$$

For any subset U of X, we say that x is an interior point *of U with respect to $\mathcal{D}$ if there exist $\epsilon > 0$ and finitely many $d_1, ..., d_n \in \mathcal{D}$ such that*

$$B_\epsilon(x; d_1, ..., d_n) \subset U.$$

*A set $U \subset X$ is $\mathcal{D}$-*open *if every point of U is interior with respect to $\mathcal{D}$.*

Lemma 13.4. *$B_\epsilon(x, d_1, ..., d_n)$ is $\mathcal{D}$-open.*

Proof. Consider $y \in B_\epsilon(x, d_1, ..., d_n)$, so that

$$d_j(x, y) < \epsilon, \ \ \text{for all } j = 1, 2, \ldots, n.$$

Let

$$\eta = \min_{1 \leq j \leq n} (\epsilon - d_j(x, y)).$$

Then $\eta > 0$, and

$$B_\eta(y, d_1, ..., d_n) \subset B_\epsilon(x, d_1, ..., d_n).$$

Thus y is an interior point of $B_\epsilon(x, d_1, ..., d_n)$, and $B_\epsilon(x, d_1, ..., d_n)$ is $\mathcal{D}$-open. $\square$

Theorem 13.5. *Let $\mathcal{D}$ be a set of pseudometrics on a set X. The collection of $\mathcal{D}$-open sets is a topology on X.*

Proof. Let $\mathcal{J}$ be the collection of $\mathcal{D}$-open subsets of X.

T1. $\emptyset, X \in \mathcal{J}$ trivially.

T2. Suppose U, V are $\mathcal{D}$-open and consider $x \in U \cap V$. There exist $\epsilon_1, \epsilon_2 > 0$ and $d_1, ..., d_n, d_{n+1}, ..., d_m \in \mathcal{D}$ such that

$$B_{\epsilon_1}(x; d_1, ..., d_n) \subset U,\ B_{\epsilon_2}(x; d_{n+1}, ..., d_m) \subset V.$$

Then, if $\epsilon = \min\{\epsilon_1, \epsilon_2\}$,

$$B_{\epsilon}(x; d_1, ..., d_n, ..., d_m) \subset U \cap V.$$

Thus every point of $U \cap V$ is $\mathcal{D}$-interior to $U \cap V$, that is, $U \cap V \in \mathcal{J}$.

T3. Suppose $U_\alpha \in \mathcal{J}\ \forall \alpha \in \Lambda$, and consider

$$x \in U = \bigcup_{\alpha \in \Lambda} U_\alpha.$$

We have $x \in U_\beta$ for some $\beta \in \Lambda$, and there exist $\epsilon > 0$, $d_1, ..., d_n \in \mathcal{D}$ such that $B = B_\epsilon(x; d_1, ..., d_n) \subset U_\beta$. Clearly $B \subset U$, and so x is $\mathcal{D}$-interior to U. Thus U is $\mathcal{D}$-open.

□

Definition 13.6. *The collection $\mathcal{J}$ of $\mathcal{D}$-open sets is called the topology induced by the set of the set $\mathcal{D}$ of pseudometrics.*

Examples 13.7.

1. On $C[0, 1]$, let

$$\mathcal{D} = \{d_t : 0 \leq t \leq 1\}$$

where

$$d_t(f, g) = |f(t) - g(t)|.$$

For $t_1, ..., t_n \in [0, 1]$,

$$B_\epsilon(f, d_{t_1}, ..., d_{t_n}) = \{g \in C[0, 1] : |f(t_j) - g(t_j)| < \epsilon, 1 \leq j \leq n\}.$$

The topology induced by $\mathcal{D}$ is called the *topology of pointwise convergence on* $C[0, 1]$.

2. Let E be a Banach space and E^* its dual. For any $x \in E$, define a pseudometric d_x on E^* by

$$d_x(F, G) = |F(x) - G(x)|, \quad F, G \in E^*.$$

The set $\{d_x : x \in E\}$ of pseudometrics induces a topology on E^* called the w^*-topology.

14 Hausdorff spaces

Definition 14.1. *A topological space X is* Hausdorff *if, for any pair x, y of distinct points in X, there exists disjoint open sets U, V in X containing x, y respectively.*

Examples 14.2.

1. Metric space are Hausdorff.
2. In $\mathbb{N}$ with the cofinite topology, any pair of non-empty open sets have (infinitely many) points in common and so the space is not Hausdorff.
3. If E is a Banach space and E^* is its dual space, then E^* is a Hausdorff space in the w^*-topology.

   Consider distinct elements $F, G \in E^*$. Since $F \neq G$, there exists $x \in E$ such that $F(x) \neq G(x)$.

   Let
   $$\epsilon = \frac{1}{2}|F(x) - G(x)| = \frac{1}{2}d_x(F, G).$$
   Then $\epsilon > 0$, and
   $$B_\epsilon(F; d_x),\ B_\epsilon(G; d_x)$$
   are disjoint open sets containing F, G respectively. Therefore E^* is Hausdorff.

Note: Since $\mathbb{N}$ with the cofinite topology is not Hausdorff, it's topology does not arise from a metric.

Note: Not all Hausdorff topologies arise from metrics. In fact E^* with the w^*-topology (with $\dim E = \infty$) does not.

Definition 14.3. *For any $(x_n)_{n=1}^\infty$ in a topological space $(X, \mathcal{J})$ we say $x_n \to a$ as $n \to \infty$, or*
$$\lim_{n\to\infty} x_n = a,$$
if, for every neighbourhood U of a, there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.

In $C[0, 1]$, with the topology of pointwise convergence,
$$f_n \to g \iff f_n(t) \to g(t) \ \ \forall t \in [0, 1].$$

15 Compactness

Definition 15.1. *Let $(X, \mathcal{J})$ be a topological space. A* cover *of a subset Y of X is a collection of subsets of X whose union contains Y.*

Examples 15.2.

1. In $\mathbb{R}$, a cover of $[0,1]$ is $\{(-1, \frac{3}{4}), (\frac{1}{4}, 2)\}$. This is a finite cover.

2. In $\mathbb{R}$, a cover of $(0,1)$ is the collection of sets $\{U_x : x \in (0,1)\}$ where

$$U_x = (\frac{x}{2}, \frac{1+x}{2}).$$

Note that no finite subcollection of the U_x covers $(0,1)$: if $x_1 < x_2 < ... < x_n$, then

$$U_{x_1} \cup ... \cup U_{x_n} \subset (\frac{x_1}{2}, \frac{1+x_n}{2})$$

and so $U_{x_1} \cup ... \cup U_{x_n}$ is a proper subset of $(0,1)$.

Definition 15.3. *Let $(X, \mathcal{J})$ be a topological space. An* open cover *of a subset Y of X is a cover of Y consisting of open sets. A* subcover *of a cover $\mathcal{U}$ of Y is a subcollection of $\mathcal{U}$ that covers Y.*

Definition 15.4. *Let $(X, \mathcal{J})$ be a topological space. A subset Y of X is* compact *if every open cover of Y has a finite subcover.*

Theorem 15.5. *Any bounded closed interval in $\mathbb{R}$ is compact.*

Theorem 15.6. *Let $(X, \mathcal{J})$ be a compact topological space. Any continuous function $f : X \to \mathbb{R}$ is bounded and attains its supremum and infimum on X.*

Proof. Let $U_n = f^{-1}(-n, n)$, since f is continuous, U_n is open, and $\{U_n : n \in \mathbb{N}\}$ is an open cover of X. It therefore has a finite subcover $\{U_{n_1}, ..., U_{n_k}\}$, where $n_1 < ... < n_k$. Thus $|f(x)| < n_k \; \forall x \in X$, and so f is bounded.
Let

$$M = \sup_{x \in X} f(x) < \infty.$$

Suppose f does not attain its supremum M on X: this means that $f(x) < M$ $\forall x \in X$. For each $x \in X$ pick y_x such that

$$f(x) < y_x < M,$$

and let

$$U_x = f^{-1}(-\infty, y_x).$$

U_x is an open neighbourhood of x, since f is continuous. The family $\{U_x : x \in X\}$ is an open cover of X, and so by compactness it has an open subcover:

$$X \subset U_{x_1} \cup ... \cup U_{x_n}$$

for some $x_1, ..., x_n \in X$.

We have $f(x) \leq y_{x_j}$ on U_{x_j}, and so

$$\sup_{x \in X} f(x) \leq \max_{1 \leq j \leq n} y_{x_j} < M,$$

contrary to choice of M as the least upper bound of f. Hence f attains its supremum on X.

Similarly f attains its infimum on X. $\square$

Theorem 15.7. *A closed subset of a compact topological space is compact.*

Proof. Let $(X, \mathcal{J})$ be compact and let Y be closed in X. Then $X \setminus Y$ is open. Consider any open cover $\mathcal{U}$ of Y. Then $\mathcal{U}' = \{U : U \in \mathcal{U}$ and $X \setminus Y\}$ is an open cover of X. Since X is compact, $\mathcal{U}'$ has a finite subcollection that covers X:

$$X = (X \setminus Y) \cup U_1 \cup ... \cup U_n,$$

some $U_1, ..., U_n \in \mathcal{U}$. Thus

$$Y \subset U_1 \cup ... \cup U_n.$$

Hence $\mathcal{U}$ has a finite subcover. Therefore Y is compact. $\square$

Theorem 15.8. *A compact subset of a Hausdorff space is closed.*

Proof. Let $(X, \mathcal{J})$ be a Hausdorff space and let Y be a compact subset of X. Consider any $x \in X \setminus Y$. For each $y \in Y$, by the Hausdorff condition, there exist disjoint open neighbourhoods U_y of y, V_y of x.

The family $\{U_y : y \in Y\}$ is an open cover of the compact Y, hence has a finite subcover $\{U_1, ..., U_{y_n}\}$ with $y_j \in Y$. Let $V(x) = V_{y_1} \cap ... \cap V_{y_n}$, then $V(x)$ is an open neighbourhood of x. $V(x)$ is disjoint from $U_{y_1} \cup ... \cup U_{y_n}$, hence from Y, so that $V(x) \subset X \setminus Y$.

We have

$$X \setminus Y = \bigcup_{x \in Y \setminus X} V(x),$$

and since each $V(x)$ is open in X, it follows that $X \setminus Y$ is open in X. Hence Y is closed. $\square$

Examples 15.9. Non-compact spaces.

1. $(0,1)$ is non-compact. $\{(\frac{x}{2}, \frac{1+x}{2}) : x \in (0,1)\}$ is an open cover having no finite subcover.

2. $\mathbb{R}$ is non-compact. $\{(-n,n) : n \in \mathbb{N}\}$ is an open cover having no finite subcover.

3. The closed unit ball in ℓ^2 is non-compact in the norm topology.

Theorem 15.10 (Heine-Borel)**.** *Any closed bounded set in $\mathbb{R}^n$ is compact.*

Theorem 15.11. *For any normed linear space E, the closed unit ball of E^* is compact in the $w*$-topology. That is, $\{F \in E^* : \|F\| \leq 1\}$ is compact in the topology defined by the pseudometrics $\{d_x : x \in E\}$, where*

$$d_x(F,G) = |F(x) - G(x)|.$$

One can find a proof of this theorem in Walter Rudin's book "Functional Analysis", 1973, Theorem 3.15.

16 Topological subspaces

Definition 16.1. *Let $(X, \mathcal{J})$ be a topological space and let Y be a subset of X. The relative topology of Y in X, or the induced topology of Y, is the topology*

$$\mathcal{J}_Y = \{U \cap Y : U \in \mathcal{J}\}.$$

Check that $\mathcal{J}_Y$ is a topology on Y.

There are two natural ways to specifiy a topology on $\mathbb{Q}$: by the natural metric $d(x, y) = |x - y|$, $x, y \in \mathbb{Q}$, or by giving $\mathbb{Q}$ the relative topology as a subset of $\mathbb{R}$ (with its natural metric). In fact they coincide.

Theorem 16.2. *Let $\mathcal{D}$ be a set of pseudometrics which defines a topology $\mathcal{J}$ on a set X, and let Y be a subset of X. The induced topology $\mathcal{J}_Y$ coincides with the topology determined by the pseudometrics*

$$\mathcal{D}_Y = \{d|_{Y \times Y} : d \in \mathcal{D}\}.$$

Proof. Let V be $\mathcal{J}_Y$-open, so that $V = U \cap Y$ for some $U \in \mathcal{J}$. Consider any $x \in V$. Then $x \in U$, and so, since U is $\mathcal{D}$-open, $\exists \epsilon > 0$ and $d_1, ..., d_n \in \mathcal{D}$ such that

$$B_\epsilon(x; d_1, ..., d_n) \subset U.$$

Let $d'_j = d_j|_{Y \times Y} \in \mathcal{D}_Y$. Then

$$B_\epsilon(x; d'_1, ..., d'_n) = Y \cap B_\epsilon(x; d_1, ..., d_n) \subset Y \cap U = V.$$

Hence x is $\mathcal{D}_Y$-interior to V.

$\therefore$ V is $\mathcal{D}_Y$-open.

$\therefore$ Every $\mathcal{J}_Y$-open set is $\mathcal{D}_Y$-open.

Conversely, let V be $\mathcal{D}_Y$-open. For each $x \in V$ we can pick a pseudoball

$$B_\epsilon(x; d'_1, ..., d'_n) \subset V,$$

where $d'_j = d_j|_{Y \times Y} \in \mathcal{D}_Y$. Then $U(x) = B_\epsilon(x; d_1, ..., d_n)$ is $\mathcal{J}$-open in X, and

$$U(x) \cap Y = B_\epsilon(x, d'_1, ..., d'_n).$$

Let $U = \cup_{x \in V} U(x)$. Then $U \in \mathcal{J}$ and $U \cap Y = V$. Hence $V \in \mathcal{J}_Y$.

$\therefore$ Every $\mathcal{D}_Y$-open set is $\mathcal{J}_Y$-open. $\square$

Theorem 16.3. *Let $(X, \mathcal{J})$ be a topological space and let $Y \subset X$. The following statements are equivalent.*

1. *Y is a compact subset of X;*

2. *$(Y, \mathcal{J}_Y)$ is a compact topological space.*

Proof. (1) $\Rightarrow$ (2). Suppose Y is a compact subset of X, and so every cover of Y by open subsets of X has a finite subcover. Consider any cover of Y by $\mathcal{J}_Y$-open sets, say,

$$\mathcal{U} = \{U_\alpha : \alpha \in A\}$$

where $U_\alpha = V_\alpha \cap Y$, $V_\alpha \in \mathcal{J}$. Then $\{V_\alpha : \alpha \in A\}$ is a cover of Y by $\mathcal{J}$-open sets. Hence there is a finite subcover

$$\{V_{\alpha_1}, ..., V_{\alpha_n}\}$$

for some $\alpha_1, ..., \alpha_n \in A$. Then $\{U_{\alpha_1}, ..., U_{\alpha_n}\}$ is a finite subcover of $\mathcal{U}$. Hence $(Y, \mathcal{J}_Y)$ is compact.

(2) $\Rightarrow$ (1) Suppose $(Y, \mathcal{J}_Y)$ is compact. Consider any cover $\mathcal{U}$ of Y by $\mathcal{J}$-open sets. Then

$$\{U \cap Y : U \in \mathcal{U}\}$$

is a cover of Y by $\mathcal{J}_Y$-open sets, hence it has a finite subcover

$$\{U_{\alpha_1} \cap Y, ..., U_{\alpha_n} \cap Y\},$$

for some $U_{\alpha_1}, ..., U_{\alpha_n} \in \mathcal{U}$. Clearly $\{U_{\alpha_1}, ..., U_{\alpha_n}\}$ is a finite subcover of $\mathcal{U}$.
$\therefore$ Y is a compact subset of X. $\square$

17 Product topologies

Let $(X_1, \mathcal{J}_1)$ and $(X_2, \mathcal{J}_2)$ be topological spaces. We can define a topology on $X_1 \times X_2$ in a natural way. Consider a subset S of $X_1 \times X_2$ and a point $x = (x_1, x_2) \in S$.

We say that S is a $\mathcal{J}_1 \times \mathcal{J}_2$*-neighbourhood of* x if there exists a $\mathcal{J}_1$-neighbourhood U_1 of x_1 and a $\mathcal{J}_2$-neighbourhood U_2 of x_2 such that

$$U_1 \times U_2 \subset S.$$

We say that a subset S of $X_1 \times X_2$ is $\mathcal{J}_1 \times \mathcal{J}_2$*-open* if S is a $\mathcal{J}_1 \times \mathcal{J}_2$-neighbourhood of each of its points.

Theorem 17.1. *The collection of $\mathcal{J}_1 \times \mathcal{J}_2$-open sets is a topology on $X_1 \times X_2$.*

Check the statement.

This topology is called the *product topology* of $\mathcal{J}_1$ and $\mathcal{J}_2$, and is denoted by $\mathcal{J}_1 \times \mathcal{J}_2$.

[Strictly speaking, this is an abuse of notation, since $\mathcal{J}_1 \times \mathcal{J}_2$ "ought to" mean $\{(U_1, U_2) : U_1 \in \mathcal{J}_1, U_2 \in \mathcal{J}_2\}$.]

Example 17.2. Let $(X_1, \mathcal{J}_1) = (X_2, \mathcal{J}_2) = \mathbb{R}$ with the natural topology. Then $\mathcal{J}_1 \times \mathcal{J}_2$ coincides with the natural topology of $\mathbb{R}^2$, i.e. the topology induced by the Euclidean metric on $\mathbb{R}^2$.

Let d denote the Euclidean metric on $\mathbb{R}^2$. Consider any d-open set $U \subset \mathbb{R}^2$ and any $x \in U$. Since U is open there exists $\epsilon > 0$ such that $B_\epsilon(x; d) \subset U$. Then $(x_j - \frac{\epsilon}{\sqrt{2}}, x_j + \frac{\epsilon}{\sqrt{2}})$ is open in $\mathbb{R}$ for $j = 1, 2$, and

$$(x_1 - \frac{\epsilon}{\sqrt{2}}, x_1 + \frac{\epsilon}{\sqrt{2}}) \times (x_2 - \frac{\epsilon}{\sqrt{2}}, x_2 + \frac{\epsilon}{\sqrt{2}}) \subset B_\epsilon(x; d) \subset U.$$

Hence U is a $\mathcal{J}_1 \times \mathcal{J}_2$-neighbourhood of each of its points, i.e. $U \in \mathcal{J}_1 \times \mathcal{J}_2$.

Converse – exercise.

Theorem 17.3. *The product topology $\mathcal{J}_1 \times \mathcal{J}_2$ comprises the collection of all unions of families of products $U_\lambda \times V_\lambda$ where λ runs over an arbitrary index Λ and $U_\lambda \in \mathcal{J}_1, V_\lambda \in \mathcal{J}_2$ for all $\lambda \in \Lambda$.*

Proof. Let G be a $\mathcal{J}_1 \times \mathcal{J}_2$-open set. For each point $\lambda \in G$, since G is a $\mathcal{J}_1 \times \mathcal{J}_2$-neighbourhood of λ, there exist $\mathcal{J}_1$- and $\mathcal{J}_2$-open sets respectively such that $U_\lambda \times V_\lambda \subset G$. Then

$$\bigcup_{\lambda \in G} U_\lambda \times V_\lambda = G$$

and so G is a union of a family of sets $U_\lambda \times V_\lambda$ of the stated type.

Coversely, suppose

$$G = \bigcup_{\lambda \in \Lambda} U_\lambda \times V_\lambda$$

for some index set Λ and some families $\{U_\lambda\}_{\lambda \in \Lambda}, \{V_\lambda\}_{\lambda \in \Lambda}$ of $\mathcal{J}_1$- and $\mathcal{J}_2$- open sets respectively. Clearly, for any $\lambda \in \Lambda$, $U_\lambda \times V_\lambda$ is a $\mathcal{J}_1 \times \mathcal{J}_2$-neighbourhood of each of its points, i.e. it is $\mathcal{J}_1 \times \mathcal{J}_2$-open. Since $\mathcal{J}_1 \times \mathcal{J}_2$ is a topology, $\bigcup_{\lambda \in \Lambda} U_\lambda \times V_\lambda$ is $\mathcal{J}_1 \times \mathcal{J}_2$-open, and hence G is $\mathcal{J}_1 \times \mathcal{J}_2$-open.

$\square$

Theorem 17.4. *If $(X, \mathcal{T})$ and $(Y, \mathcal{J})$ are compact topological spaces then $X \times Y$ is compact with respect to the product topology $\mathcal{T} \times \mathcal{J}$.*

Proof. Let $\{G_\lambda\}_{\lambda \in \Lambda}$ be an open cover of $X \times Y$ for the product topology $\mathcal{T} \times \mathcal{J}$. We shall construct a finite subcover for $X \times Y$. To begin with hold $x \in X$ fixed an consider the points $\{(x, y) : y \in Y\}$.

For each $y \in Y$ there is a $\lambda_{(x,y)} \in \Lambda$ such that $(x, y) \in G_{\lambda_{(x,y)}}$ and since each G_λ is open there are open neighbourhoods $U_x(y)$, $V_x(y)$ of x, y in X, Y respectively such that

$$U_x(y) \times V_x(y) \subset G_{\lambda_{(x,y)}}.$$

Since $\{V_x(y) : y \in Y\}$ is an open cover of Y and Y is compact, there is a positive integer $N(x)$ and points $y_1, ..., y_{N(x)} \in Y$ such that

$$V_x(y_1) \cup ... \cup V_x(y_{N(x)}) = Y.$$

Let

$$U_x = U_x(y_1) \cap ... \cap U_x(y_{N(x)}).$$

Then $\{U_x : x \in X\}$ is an open cover of X. Hence, since X is compact, there is a positive integer M and points $x_1, ..., x_M \in X$ such that

$$U_{x_1} \cup ... \cup U_{x_M} = X.$$

We claim that $\{U_{x_i} \times V_{x_i}(y_j) : 1 \leq i \leq M, 1 \leq j \leq N(x_i)\}$ is an open cover of $X \times Y$. Since

$$U_{x_i} \times V_{x_i}(y_i) \subset G_{\lambda(x_i, y_j)},$$

it will follow that $\{G_\lambda\}_{\lambda \in \Lambda}$ has a finite subcover.

Consider any point $(x, y) \in X \times Y$. Since the $(U_{x_i})_{i=1}^{M}$ cover X, there exists $i \in \{1, ..., M\}$ such that $x \in U_{x_i}$. Since, for each i, the sets $(V_{x_i}(y_j))_{j=1}^{N(x_i)}$ cover Y, there exists $j \in \{1, 2, ..., N(x_i)\}$ such that $y \in V_{x_i}(y_j)$. Thus

$$(x, y) \in U_{x_i} \times V_{x_i}(y_j)$$

as required.

We have shown that every open subcover of $X \times Y$ has a finite subcover. Hence $X \times Y$ is compact. $\square$

18 Infinite products

The construction in the previous sections extends fairly easily to arbitary products.

Let Λ be an index set and suppose that, for each $\lambda \in \Lambda$, we have a topological space $(X_\lambda, \mathcal{J}_\lambda)$. Recall that the *Cartesian product* of the family $\{X_\lambda\}_{\lambda\in\Lambda}$ is defined to be the set of functions

$$x : \Lambda \to \bigcup_{\lambda\in\Lambda} X_\lambda$$

such that $x(\lambda) \in X_\lambda$ for all $\lambda \in \Lambda$.

We often write x_λ instead of $x(\lambda)$, and think of x_λ as the "λth co-ordinate" of x. Thus we can write $x = (x_\lambda)_{\lambda\in\Lambda}$.

We denote the Cartesian product of the family $\{X_\lambda\}_{\lambda\in\Lambda}$ by $\prod_{\lambda\in\Lambda} X_\lambda$.

In the case that all the X_λ are equal, say to X, the product $\prod_{\lambda\in\Lambda} X_\lambda$ is also written X^Λ.

We wish to define a "product topology" on $\prod_{\lambda\in\Lambda} X_\lambda$. For any $\lambda_1 \in \Lambda$ and $U_1 \in \mathcal{J}_{\lambda_1}$, let

$$\mathcal{N}(\lambda_1, U_1) = \{x \in \prod_{\lambda\in\Lambda} X_\lambda : x_{\lambda_1} \in U_1\}.$$

For any positive integer n, any $\lambda_1, ..., \lambda_n \in \Lambda$ and

$$U_{\lambda_j} \in \mathcal{J}_{\lambda_j} \quad \text{for} \quad j = 1, 2, ..., n,$$

let

$$\begin{aligned}\mathcal{N}(\lambda_1, ..., \lambda_n, U_1, ..., U_n) &= \mathcal{N}(\lambda_1, U_1) \cap ... \cap \mathcal{N}(\lambda_n, U_n)\\ &= \{x \in \prod_{\lambda\in\Lambda} X_\lambda : x_{\lambda_j} \in U_j, 1 \le j \le n\}.\end{aligned}$$

Consider a subset S of $\prod_{\lambda\in\Lambda} X_\lambda$ and a point $x \in S$. We define a topology

$$\mathcal{J} = \prod_{\lambda\in\Lambda} \mathcal{J}_\lambda$$

on $\prod_{\lambda\in\Lambda} X_\lambda$ by saying that S is a $\mathcal{J}$*-neighbourhood* of x if there exist a positive integer n, points $\lambda_1, ..., \lambda_n \in \Lambda$ and neighbourhoods $U_j \in \mathcal{J}_{\lambda_j}$ of x_{λ_j} for $j = 1, ..., n$ such that

$$\mathcal{N}(\lambda_1, ..., \lambda_n, U_1, ..., U_n) \subset S.$$

We say that $S \subset \prod_{\lambda\in\Lambda} X_\lambda$ is $\mathcal{J}$*-open* if S is a $\mathcal{J}$-neighbourhood of each of its points.

We define $\mathcal{J}$ to be the collection of all $\mathcal{J}$-open sets.

Theorem 18.1. *$\mathcal{J}$ is a topology on $\prod_{\lambda\in\Lambda} X_\lambda$.*

Note that in the case that $\Lambda = \{1, 2\}$ the topology $\mathcal{J}$ coincides with the product topology $\mathcal{J}_1 \times \mathcal{J}_2$ of Section 17.

Example 18.2. Let $\Lambda = \mathbb{N}$, $X_\lambda = \mathbb{R}$ for all $\lambda \in \mathbb{N}$. Here $\prod_{\lambda\in\Lambda} X_\lambda = \mathbb{R}^{\mathbb{N}}$ the set of infinite sequences $x = (x_n)_{n\in\mathbb{N}}$ of real numbers. Any open non-empty set in $\mathbb{R}^{\mathbb{N}}$ for the product topology is big.

Consider, for example, any open neighbourhood U of the zero sequence $(0)_{n\in\mathbb{N}}$. For some positive integer m and open neighbourhoods $U_1, ..., U_m$ of 0 in $\mathbb{R}$ we have

$$U \supset \mathcal{N}(1, 2, ..., m, U_1, ..., U_m)$$

$$= \{(x_n)_{n\in\mathbb{N}} : x_1 \in U_1, ..., x_m \in U_m\}.$$

Note that only finitely many co-ordinates of elements of $\mathcal{N}(1, ..., U_m)$ are restricted: if $(x_n)_{n\in\mathbb{N}} \in \mathcal{N}$ then also $(x_1, x_2, ..., x_m, x'_{m+1}, x'_{m+2}, ...) \in \mathcal{N}$ for any choice of $x'_{m+1}, x'_{m+2}, ... \in \mathbb{R}$.

Example 18.3. Let $\Lambda = \mathbb{N}$, $X_\lambda = [0, 1]$ for all $\lambda \in \mathbb{N}$. Here $X = \prod_{\lambda\in\Lambda} X_\lambda = [0, 1]^{\mathbb{N}}$. We can define a metric d on X by

$$d((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \sup_n |x_n - y_n|.$$

It is not the case that d induces the product topology on $[0, 1]^{\mathbb{N}}$. Consider the sequence $x^1, x^2, x^3, ...$ of elements of $[0, 1]^{\mathbb{N}}$, where

$$x^k = (0, 0, ..., 0, 1, 1, 1, ...)$$

x^k tends to the zero sequence as k tends to ∞ when $[0, 1]^{\mathbb{N}}$ has the product topology. However $d(x^k, x^l) = 1$ whenever $k \neq l$, and so the sequence $(x^k)_{k\in\mathbb{N}}$ does not converge in the topological space $([0, 1]^{\mathbb{N}}, d)$.

Remarkably enough the space $[0, 1]^{\mathbb{N}}$ is compact with respect to the product topology.

Theorem 18.4. (Tychonov's Theorem) *If* Λ *is a set and* $(X_\lambda, \mathcal{J}_\lambda)$ *is a compact topological space for every* $\lambda \in \Lambda$ *then* $\prod_{\lambda \in \Lambda} X_\lambda$ *is compact with respect to the product topology.*

This theorem requires one of the higher axioms of set theory. In fact it is equivalent to the axiom of choice. Thus it is not true in Zermelo-Fraenkel set theory without the axiom of choice. There are models of set theory in which Tychonov's Theorem is false. However almost all professional mathematicians work within the framework of Zermelo-Fraenkel set theory with the axiom of choice.

Part III. The Gelfand representation theorem

The main aim of this part of lectures is to prove the Gelfand representation theorem for a commutative Banach algebra A. To start with we shall define the character space $\hat{A}$ of a commutative Banach algebra A and the Gelfand topology on $\hat{A}$. We shall show that there is a bijective mapping between the character space $\hat{A}$ and the set of maximal ideals of A. We shall introduce the radical of A and a notion of semisimplicity of A.

Contents of Part III

19 The character space

Let A be a commutative Banach algebra. Recall that a character of A is a non-zero algebra homomorphism from A to $\mathbb{C}$; the character space $\hat{A}$ of A is the set of all characters of A.

Lemma 19.1. *Let A be a commutative Banach algebra (with or without e). For any character ϕ of A and any $x \in A$,*

$$|\phi(x)| \leq \|x\|.$$

Proof. Let $\lambda = \phi(x)$ and suppose $|\lambda| > \|x\|$. Then $\|x/\lambda\| < 1$, and so

$$y = \lim_{n\to\infty} \left(\frac{x}{\lambda} + \frac{x^2}{\lambda^2} + \ldots + \frac{x^n}{\lambda^n} \right)$$

exists; see a proof of Lemma 3.3.
We have

$$\begin{aligned}
\lambda y - xy &= \lambda \lim_{n\to\infty} \left(\frac{x}{\lambda} + \frac{x^2}{\lambda^2} + \ldots + \frac{x^n}{\lambda^n} \right) - x \lim_{n\to\infty} \left(\frac{x}{\lambda} + \frac{x^2}{\lambda^2} + \ldots + \frac{x^n}{\lambda^n} \right) \\
&= \lim_{n\to\infty} \left(x + \frac{x^2}{\lambda} + \ldots + \frac{x^n}{\lambda^{n-1}} - \frac{x^2}{\lambda} - \ldots - \frac{x^n}{\lambda^{n-1}} \right) \\
&= x.
\end{aligned}$$

Apply the character ϕ:

$$\begin{aligned}
\lambda = \phi(x) &= \phi(\lambda y - xy) = \lambda\phi(y) - \phi(x)\phi(y) \\
&= \lambda\phi(y) - \lambda\phi(y) = 0.
\end{aligned}$$

Therefore, $\lambda = 0$. This is a contradiction. Hence $|\phi(x)| \leq \|x\|$. □

Corollary 19.2. *Let A be a commutative Banach algebra. The character space $\hat{A}$ of A is a subset of the closed unit ball of A^*.*

Recall that, for $\phi \in A^*$,

$$\|\phi\| = \sup_{\|x\|\leq 1} |\phi(x)|.$$

Definition 19.3. *Let A be a commutative Banach algebra. For $x \in A$, the Gelfand transform of x is the function*

$$\hat{x} : \hat{A} \to \mathbb{C} : \phi \mapsto \phi(x).$$

Thus

$$\hat{x}(\phi) = \phi(x)$$

for every $x \in A$ and every character ϕ of A.

Example 19.4. If $A = C(K)$, then $\hat{A} = \{v_t : t \in K\}$. For $x \in C(K)$, $\hat{x} : \hat{A} \to \mathbb{C}$ satisfies

$$\hat{x}(v_t) = v_t(x) = x(t), \quad t \in K.$$

Thus, if v_t is identified with t, $\hat{x}$ is the same as x.

Example 19.5. If $A = \ell^1(\mathbb{Z}_+)$, then $\hat{A}$ contains the characters ϕ_α for $\alpha \in \triangle(0,1)$ where

$$\phi_\alpha((x_n)_{n=0}^\infty) = \sum_{n=0}^{\infty} x_n \alpha^n.$$

Then, if $x = (x_n)_{n=0}^\infty$,

$$\hat{x}(\phi_\alpha) = \phi_\alpha(x) = \sum_{n=0}^{\infty} x_n \alpha^n.$$

Definition 19.6. *Let A be a commutative Banach algebra. The Gelfand topology on $\hat{A}$ is the coarsest topology for which the functions $\{\hat{x} : x \in A\}$ are continuous.*

Recall that if σ, τ are topologies on a set E then σ is *coarser* than τ is $\sigma \subset \tau$. One also says that τ is *finer* than σ.

Lemma 19.7. *If $\mathcal{F}$ is a collection of $\mathbb{C}-$valued functions on a set E then there exists a coarsest topology κ for which the members of $\mathcal{F}$ are continuous. A base for κ consists of the sets*

$$V(f_1, ..., f_n; U_1, ..., U_n) = \{t \in E : f_j(t) \in U_j, 1 \leq j \leq n\}$$

where $n \in \mathbb{N}, f_1, ..., f_n \in \mathcal{F}$ and $U_1, ..., U_n$ are open in $\mathbb{C}$.

Proof. One can readily check that the sets $V(f_1, ..., f_n; U_1, ..., U_n)$ do comprise the base for a topology κ on E.

For any $f \in \mathcal{F}$ and open $U \subset \mathbb{C}$ we have

$$f^{-1}(U) = V(f; U),$$

a basic κ-open set. Hence the members of $\mathcal{F}$ are continuous with respect to k.

Let τ be a topology on E such that all members of $\mathcal{F}$ are continuous with respect to τ. Consider any $f_1, f_2, ..., f_n \in \mathcal{F}$ and open $U_1, ..., U_n$ in $\mathbb{C}$. Then

$$f_j^{-1}(U_j) \in \tau,$$

and hence

$$V(f_1, ..., f_n; , U_1, ..., U_n) = f_1^{-1}(U_1) \cap ... \cap f_n^{-1}(U_n) \in \tau.$$

Thus $\kappa \subset \tau$. Therefore κ is the coarsest topology for which the members of $\mathcal{F}$ are continuous. $\square$

Thus the definition of the Gelfand topology in Definition 19.6 makes sense, and we have:

Theorem 19.8. *Let A be a commutative Banach algebra. A base for the Gelfand topology on $\hat{A}$ is given by the sets*

$$V(x_1, ..., x_n; , U_1, ..., U_n) = \{\phi \in \hat{A} : \hat{x}_j(\phi) \in U_j, 1 \le j \le n\}$$

$$= \{\phi \in \hat{A} : \phi(x_j) \in U_j, 1 \le j \le n\}$$

where $n \in \mathbb{N}$, $x_1, ..., x_n \in A$, $U_1, ..., U_n$ are open in $\mathbb{C}$.

Note that, by Lemma 19.1, any character ϕ of A satisfies

$$\phi(x) \in \triangle(0, \|x\|) = \{z \in \mathbb{C} : |z| \le \|x\|\}$$

for $x \in A$. Hence

$$\phi \in \prod_{x \in A} \triangle(0, \|x\|).$$

Corollary 19.9. *Let A be a commutative Banach algebra. The Gelfand topology of $\hat{A}$ coincides with the relative topology $\hat{A}$ enjoys as a subset of $\prod_{x\in A} \triangle(0, \|x\|)$ with the product topology.*

Proof. The sets $V(x_1, ..., x_n; U_1, ..., U_n)$ in Theorem 19.8 constitute a base for both topologies. ☐

Theorem 19.10. *If the commutative Banach algebra A has an identity then $\hat{A}$ is a compact Hausdorff space with respect to the Gelfand topology.*

Proof. Let A have identity e. Let H be the set of homomorphisms from A to $\mathbb{C}$. Note that $H = \hat{A} \cup \{0\}$.

Let $Y = \prod_{x\in A} \triangle(0, \|x\|)$. By Tychonov's Theorem 18.4, Y is compact and Hausdorff in the product topology.

Claim: H is a closed subset of Y. Let $x \in A$ and

$$v_x : Y \to \mathbb{C} : \phi \mapsto \phi(x).$$

For U open in $\mathbb{C}$,

$$v_x^{-1}(U) = \{\phi \in \prod_{y\in A} \triangle(0, \|y\|) : \phi(x) \in U\}$$

is a basic open set in Y, and so v_x is continuous.

Hence, for any $x, y \in A$,

$$v_{xy} - v_x v_y : Y \to \mathbb{C}$$

is continuous. Therefore

$$\bigcap_{x,y \in A} (v_{xy} - v_x v_y)^{-1}(\{0\})$$

is closed in Y. That is, the set of multiplicative functions in Y is closed in Y. Similarly for the other algebraic operations.

Thus H is a closed subset of the compact Hausdorff space Y. By Theorem 15.7, H is a compact Hausdorff space.

For $\phi \in H$ we have

$$\phi \in \hat{A} \iff \phi(e) = 1,$$

so that

$$\hat{A} = H \cap v_e^{-1}(\{1\}).$$

Hence $\hat{A}$ is a closed subspace of H, and so $\hat{A}$ is a compact Hausdorff space. □

Theorem 19.11. *For any compact Hausdorff space K, the character space $C(K)^\Lambda$ of $C(K)$ in the Gelfand topology is homeomorphic to K.*

Proof. By Theorem 9.2, the map

$$v : K \to C(K)^\Lambda : t \mapsto v_t$$

is surjective. A theorem in topology (Urysohn's Lemma) asserts that if $t_1 \neq t_2$ in K then there exists $f \in C(K)$ such that $f(t_1) \neq f(t_2)$, and hence $v_{t_1} \neq v_{t_2}$. Thus v is bijective.

For any $x \in C(K)$, we have, for $t \in K$,

$$\hat{x}(v_t) = x(t) = x \circ v^{-1}(v_t),$$

so that

$$\hat{x} = x \circ v^{-1}.$$

Let τ be the "topology of K transferred to $C(K)^\Lambda$": more precisely,

$$\tau = \{v(U) : U \text{ open in } K\}.$$

Claim: for any $x \in C(K)$,

$$\hat{x} : (C(K)^\Lambda, \tau) \to \mathbb{C}$$

is continuous. For any open set $W \subset \mathbb{C}$ we have

$$\hat{x}^{-1}(W) = (x \circ v^{-1})^{-1}(W) = v(x^{-1}(W)) \in \tau$$

since $x^{-1}(W)$ is open in K. Hence $\hat{x}$ is continuous with respect to τ. Since the Gelfand topology is the coarsest topology for which all the $\hat{x}$ are continuous, τ is finer than the Gelfand topology. That is, for each Gelfand-open set V in $C(K)^\Lambda$, $v^{-1}(V)$ is open in K. Hence $v : K \to C(K)^\Lambda$ is continuous.

Every bijective continuous map from a compact space to a Hausdorff space is a homeomorphism, Hence K is homeomorphic to $C(K)^\Lambda$. $\square$

This theorem shows that we can "recover" K from the structure of the algebra $C(K)$.

20 The Gelfand representation theorem

To what extent is a general commutative Banach algebra like $C(K)$ for some Hausdorff space K?

Theorem 20.1. *Let A be a commutative Banach algebra with identity. The Gelfand transformation*

$$\Gamma : A \to C(\hat{A}) : x \mapsto \hat{x}$$

is an algebra homomorphism from A to the algebra of continuous $\mathbb{C}$-valued functions on the compact Hausdorff space $\hat{A}$ in its Gelfand topology, with pointwise operations. Moreover $\|\Gamma\| \leq 1$.

Proof. By Theorem 19.10, $\hat{A}$ is a compact Hausdorf space. For $x \in A$, $\hat{x}$ is continuous on $\hat{A}$ by Definition 19.6, that is, $\hat{x} \in C(\hat{A})$. Thus Γ does map A into $C(\hat{A})$.

Note that Γ is linear. Consider $x, y \in A$ and $\lambda \in \mathbb{C}$. For any $\phi \in \hat{A}$, we have

$$\widehat{(x+y)}(\phi) = \phi(x+y) = \phi(x) + \phi(y) = \hat{x}(\phi) + \hat{y}(\phi).$$

$$\therefore \ \widehat{(x+y)} = \hat{x} + \hat{y}.$$

$$\therefore \quad \Gamma(x+y) = (\Gamma x) + (\Gamma y)$$

for all $x, y \in A$. Therefore Γ is additive.

$$\widehat{\lambda x}\phi = \phi(\lambda x) = \lambda \hat{x}(\phi)$$

$$\therefore \quad \Gamma(\lambda x) = \lambda \Gamma(x)$$

for all $x \in A$ and $\lambda \in \mathbb{C}$. Thus Γ is multiplicative.

For $\phi \in \hat{A}$,

$$\widehat{(xy)}(\phi) = \phi(xy) = \phi(x)\phi(y) = \hat{x}(\phi)\hat{y}(\phi) = (\hat{x}\hat{y})(\phi).$$

$$\therefore \quad \Gamma(xy) = (\Gamma x)(\Gamma y)$$

for all $x, y \in A$. Hence Γ is a homomorphism of algebras.

For $x \in A$, we have

$$\begin{aligned}\|\Gamma x\|_{C(\hat{A})} &= \sup_{\phi \in \hat{A}} |\Gamma x(\phi)| \\ &= \sup_{\phi \in \hat{A}} |\hat{x}(\phi)| = \sup_{\phi \in \hat{A}} |\phi(x)| \\ &\leq \|x\|_A \quad \text{(by Lemma 19.1)}.\end{aligned}$$

Hence $\|\Gamma\| \leq 1$. $\square$

As we have seen in Theorem 19.11, if $A = C(K)$ for some compact Hausdorff space K then $\hat{A}$ can be identified with K and Γ is the identity mapping $A \to C(\hat{A})$. More commonly Range Γ is a proper subalgebra of $C(\hat{A})$.

Example 20.2. Let $A = C^1[0,1]$, the Banach algebra of continuously differentiable $\mathbb{C}$-valued functions on $[0,1]$ with pointwise operations and norm

$$\|f\|_{C^1} = \sup_{0\leq t\leq 1} |f(t)| + \sup_{0\leq t\leq 1} |f'(t)|.$$

Then $\hat{A}$ consists of evaluation functionals

$$v_t : A \to \mathbb{C} : f \mapsto f(t)$$

for $0 \leq t \leq 1$. The character space $\hat{A}$ in its Gelfand topology can be identified with $[0,1]$ in its standard topology, and $\Gamma : C^1[0,1] \to C[0,1]$ is the natural injection mapping.

We shall shortly give an example in which Γ is not an obvious injection.

20.1 Singly generated algebras.

Definition 20.3. *Let A be a Banach algebra with identity e. We say that A is* generated by an element $x \in A$ *if the smallest closed subalgebra of A containing e and x is A; equivalently, if the closed linear span of $\{e, x, x^2, ...\}$ in A is A.*

Theorem 20.4. *Let A be a commutative Banach algebra with identity e and suppose that A is generated by x. The character space $\hat{A}$ can be identified with the spectrum $\sigma(x)$ of x, and the Gelfand topology of $\hat{A}$ agrees with the natural topology of $\sigma(x) \subset \mathbb{C}$. More precisely,*

$$\hat{x} : \hat{A} \to \sigma(x) \subset \mathbb{C} : \phi \mapsto \hat{x}(\phi) = \phi(x)$$

is a homeomorphism.

Proof. Proof to follow on Page 76. □

Example 20.5. Consider $\ell^1(\mathbb{Z}_+)$ with convolution multiplication (Example 6.1). What is the Gelfand representation of A?

Note that A is singly generated. Let $e_n = (0, ..., 0, 1, 0, ...)$ (1 in the nth place for $n \in \mathbb{Z}_+$.) The element e_0 is an identity for A and e_1 is a generator. In fact $e_1 * e_1 * ... * e_1 = e_n$, and

$$(x_0, x_1, ..., x_n, 0, 0, ...) = x_0e_0 + x_1e_1 + x_2e_1 * e_1 + ... + x_ne_1 * ... * e_1.$$

Hence the smallest closed subalgebra of A containing e_0 and e_1 is A. It follows that $\hat{A}$ is homeomorphic to $\sigma(e_1)$. Since $\|e_1\| = 1$, we have $\sigma(e_1) \subset \triangle(0,1)$.

Claim: each $\phi \in \hat{A}$ has a form ϕ_α for some $\alpha \in \triangle(0,1)$. Consider any $\phi \in \hat{A}$ and suppose $\phi(e_1) = \alpha \in \triangle(0,1)$. For $x = (x_0, x_1, ..., x_n, 0, 0, ...) \in \ell^1(\mathbb{Z}_+)$, we have

$$x = x_0 e_0 + x_1 e_1 + x_2 e_1 * e_1 + ... + x_n e_1 * ... * e_1$$

and so

$$\begin{aligned} \phi(x) &= x_0\phi(e_0) + x_1\phi(e_1) + x_2\phi(e_1)^2 + ... + x_n\phi(e_1)^n \\ &= \sum_{j=0}^{n} x_j \alpha^j \\ &= \phi_\alpha(x) \text{ in the notation of Example 7.2(5).} \end{aligned}$$

Since $\{x \in A : \phi(x) = \phi_\alpha(x)\}$ is a closed subalgebra of A that contains e_0 and e_1 it equals A; that is, $\phi = \phi_\alpha$. Hence

$$\hat{e_1} : \hat{A} \to \sigma(e_1) = \triangle(0,1) : \phi_\alpha \mapsto \phi_\alpha(e_1) = \alpha$$

is a homeomorphism, and if we identity $\hat{A}$ with $\triangle(0,1)$ via $\phi \mapsto \alpha$ we have the Gelfand representation

$$\Gamma : \ell^1(\mathbb{Z}_+) \to C(\triangle(0,1)) : x \mapsto \hat{x}$$

where

$$\hat{x}(\alpha) = \phi_\alpha(x) = \sum_{n=0}^{\infty} x_n \alpha^n.$$

Notice that if $(x_n)_{n=0}^{\infty} \in \ell^1(\mathbb{Z}_+)$ then

$$\hat{x}(z) = \sum_{n=0}^{\infty} x_n z^n$$

is an absolutely convergent Taylor series in $\triangle(0,1)$.

Thus "Γ maps $\ell^1(\mathbb{Z}_+)$ to the algebra of analytic functions in $\mathbb{D}$ with absolutely convergent Taylor series in $\triangle(0,1)$."

21 Closed ideals

Let A be a Banach algebra with identity e. Ideals of A are not necessarily closed. For example, in ℓ^∞ the linear subspace c_F of finitely non-zero sequences is an ideal, but its closure is c_0.

Lemma 21.1. *Let A be a Banach algebra with identity e.*
(i) *The closure of an ideal in A is an ideal.*
(ii) *The closure of a proper ideal is a proper ideal.*

Proof. (i). Let I be an ideal, we denote the closure of I by $Cl\ I$. Consider $x, y \in Cl\, I$. Suppose $x+y \notin Cl\, I$. There is a neighbourhood U of $x+y$ which does not meet I. Since addition

$$A \times A \to A$$

is continuous at (x, y) there are neighbourhood V, W of x, y respectively such that $V + W \subset U$. Since $x, y \in Cl\ I$ there exist points $\xi \in V \cap I, \eta \in W \cap I$. Then $\xi + \eta \in V + W \subset U$ and $\xi + \eta \in I$, contradicting the fact that $U \cap I = \emptyset$. Hence $x + y \in Cl\ I$.

Likewise, for $a \in A$ and $x \in Cl\ I$, $xa \in Cl\ I$ and $ax \in Cl\ I$, and $\lambda x \in Cl\ I$ for $\lambda \in \mathbb{C}$. Hence $Cl\ I$ is an ideal.

(ii). Let I be a proper ideal of A. By Theorem 7.12, I does not contain any regular element of A. By Lemma 3.3, $B_1(e)$ is a neighbourhood of e consisting of regular elements, hence is disjoint from I. Thus $e \notin Cl\ I$, and so $Cl\ I$ is a proper ideal. □

Theorem 21.2. *Let A be a Banach algebra with identity e. Maximal ideals of A are closed.*

Proof. Let I be a maximal ideal of A. By definition I is proper, and so, by Lemma 21.1, $Cl\ I$ is a proper ideal of A. We have

$$I \subset Cl\ I \subsetneq A,$$

and so, by maximality of I, $I = Cl\, I$. Thus I is closed. □

22 Quotient algebras

Recall that the kernel of an algebra homomorphism is an ideal, see Remark 7.9. We shall show the converse: every ideal is the kernel of a homomorphism. Given an ideal I in an algebra A we shall construct an algebra A/I ("A mod I") and a homomorphism $\phi : A \to A/I$ such that ker $\phi = I$.

Definition 22.1. *Let A be an algebra and let I be an ideal in A. For $x \in A$ the* coset of x mod I *is the set $x + I$, that is, $\{x + y : y \in I\}$.*

The quotient set A/I *is the set of cosets in A* (mod I)*:*

$$A/I = \{x + I : x \in A\}.$$

Note: if we write $x \equiv y \pmod{\mathrm{I}}$ to mean $x - y \in I$, then $\equiv \pmod{\mathrm{I}}$ is an equivalence relation, and A/I consists of the corresponding equivalence classes.

Example 22.2. Let $A = C[0, 1]$ and let

$$I = \{f \in A : f(t) = 0 \ \text{ for } t \in \left[0, \frac{1}{2}\right]\}.$$

I is an ideal of A. For any $g \in A$, $g + I$ consists of all $f \in A$ such that f agrees with g on $\left[0, \frac{1}{2}\right]$. A/I can be naturally identified with $C\left[0, \frac{1}{2}\right]$.

Theorem 22.3. *Let A be an algebra and let I be an ideal in A. The quotient linear space A/I is an algebra with respect to the operations defined for $x, y \in A$ and scalar $\lambda \in \mathbb{C}$ by*

$$\begin{aligned}(x + I) + (y + I) &= (x + y) + I,\\ (x + I)(y + I) &= (xy) + I,\\ \lambda(x + I) &= (\lambda x) + I.\end{aligned}$$

Furthermore the map

$$k : A \to A/I : x \mapsto x + I$$

is a homomorphism of algebras.

Proof. The above operations are well defined, for if $x+I = x'+I$ and $y+I = y'+I$ then there are $i_1, i_2 \in I$ such that

$$x - x' = i_1, \qquad y - y' = i_2,$$

and hence $x + y - (x' + y') = i_1 + i_2 \in I$, so that

$$(x + y) + I = (x' + y') + I.$$

Similarly

$$x'y' - xy = (x - i_1)(y - i_2) - xy = -i_1y - xi_2 + i_1i_2 \in I,$$

so that

$$x'y' + I = xy + I,$$

and

$$\lambda x - \lambda x' = \lambda i_1 \in I,$$

so that

$$\lambda x + I = \lambda x' + I.$$

The algebra axioms for A/I follow easily from those for A.

The fact that k is a homomorphism is immediate from the definition. □

Definition 22.4. $k : A \to A/I$ *is called the* quotient map *or the* canonical homomorphism.

Corollary 22.5. *Let A be an algebra. The ideals of A coincide with the kernels of homomorphisms of A.*

Proof. If $\phi : A \to B$ is a homomorphism then $\ker \phi$ is an ideal. If I is an ideal then

$$k : A \to A/I$$

is a homomorphism of algebras and

$$\begin{aligned} x \in \ker k &\iff k(x) = 0 \text{ in } A/I \\ &\iff x + I = 0 + I \\ &\iff x \in I. \end{aligned}$$

Thus $I = \ker k$, and so I is the kernel of a homomorphism. □

Now consider the case of normed algebras.

Definition 22.6. *Let E be a normed space and let F be a closed subspace of E. For $x \in E$ the* coset of x mod F *is the set $x + F$. The* quotient set E/F *is the set of cosets of E* mod F.

Addition and scalar multiplication are defined in E/F by

$$(x + F) + (y + F) = (x + y) + F, \; \lambda(x + F) = (\lambda x) + F. \tag{22.1}$$

Theorem 22.7. *Let E be a normed space and let F be a closed subspace of E. The quotient set E/F is a linear space under operations* (22.1). *The pair $(E/F, \|.\|_{E/F})$ is a norm space, where*

$$\|x+F\|_{E/F} = \inf_{y\in F} \|x+y\|_E.$$

If E is a Banach space then so is E/F.

$\|.\|_{E/F}$ is called the *quotient norm*.

Corollary 22.8. *If I is a closed ideal in a Banach algebra A then A/I is a Banach algebra under the quotient norm. If A has an identity, so does A/I.*

Proof. By Theorems 22.5 and 22.7, A/I is an algebra and a Banach space. For $x, y \in A$,

$$\begin{aligned}
\|(x+I)(y+I)\|_{A/I} &= \|xy+I\|_{A/I} \\
&= \inf_{z\in I} \|xy+z\|_A \\
&\leq \inf_{z_1,z_2\in I} \|(x+z_1)(y+z_2)\|_A \\
&\leq \inf_{z_1,z_2\in I} \|x+z_1\|_A\|y+z_2\|_A \\
&= \|x+I\|_{A/I}\|y+I\|_{A/I}.
\end{aligned}$$

Thus $\|.\|_{A/I}$ is submultiplicative. Hence A/I is a Banach algebra.
If e is an identity in A then $e+I$ is one in A/I. □

Example 22.9. Let $A = C[0,1]$ and let

$$I = \{f \in A : f(t) = 0 \text{ for } t \in \left[0, \frac{1}{2}\right]\}.$$

I is an ideal of A as in Example 22.2. Define

$$\phi : A/I \to C\left[0, \frac{1}{2}\right] : f + I \mapsto f_{|[0,\frac{1}{2}]}.$$

Then $\phi : A/I \to C\left[0, \frac{1}{2}\right]$ is an isometric isomorphism.

23 Maximal ideals and characters

Recall Theorem 7.15 that the kernel of a character of a commutative Banach algebra is a maximal ideal.

Theorem 23.1. *Every maximal ideal of a commutative Banach algebra with identity is the kernel of a character.*

Proof. Let A be commutative Banach algebra with identity e and let I be a maximal ideal of A. By Theorem 21.2, I is a closed ideal. By Corollary 22.8, A/I is a commutative Banach algebra with identity.

The zero element of A/I is $0 + I$, which we can write $\{I\}$.

Claim: $\{I\}$ **is a maximal ideal of** A/I. Let

$$k : A \to A/I : x \mapsto x + I$$

be the canonical homomorphism. Consider any proper ideal $\mathcal{J}$ of A/I. Then

$$k^{-1}(\mathcal{J}) = \{x \in A : x + I \in \mathcal{J}\}$$

is a proper ideal of A containing I. Since I is maximal we have $k^{-1}(J) = I$, that is,

$$x + I \in \mathcal{J} \iff x \in I.$$

Therefore, $\mathcal{J} = \{I\}$. Hence $\{I\}$ is a maximal ideal, as claimed.

Since A/I is a commutative Banach algebra with identity in which the only ideals are $\{I\}$ and A/I, it follows from Examples Sheet 5, Q7, that A/I is a field and so, by the Gelfand-Mazur Theorem 5.5, that A/I is isomorphic (as an algebra) to $\mathbb{C}$.

Hence $k : A \to A/I \approx \mathbb{C}$ is a character (note that $k(e)$ is the identity of A/I, so $k \neq 0$), and $I = \ker k$ is the kernel of a character. □

Corollary 23.2. *There is a bijective mapping $\phi \mapsto \ker \phi$ between the character space and the set of maximal ideals of a commutative Banach algebra with identity.*

We shall denote by M_A the set of maximal ideals of a commutative Banach algebra A.

Proof. By Theorem 7.15,

$$\hat{A} \to M_A : \phi \mapsto \ker \phi$$

is a well defined mapping, and, by Theorem 23.1, it is surjective.

It is also injective. Suppose $\phi, \psi \in \hat{A}$ and $\ker \phi = \ker \psi$. We have

$$\phi(e) = \psi(e) = 1$$

where e is the identity of A. For any $x \in A$, $x - \phi(x)e \in \ker \phi = \ker \psi$, and so

$$0 = \psi(x - \phi(x)e) = \psi(x) - \phi(x)\psi(e) = \psi(x) - \phi(x).$$

Hence $\phi = \psi$.

Thus $\hat{A} \to M_A : \phi \mapsto \ker \phi$ is a bijection. □

Remark 23.3. In view of Corollary 23.2 we can identify $\hat{A}$ with M_A. For any $x \in A$, we can regard the Gelfand transform $\hat{x}$ as a function

$$\hat{x} : M_A \to \mathbb{C},$$

given by

$$\hat{x}(M) = \phi(x), \qquad \text{where } M = \ker \phi \in M_A.$$

The Gelfand topology on M_A is the coarsest topology for which the functions $\hat{x}$, $x \in A$, are continuous.

Definition 23.4. *Let A be commutative Banach algebra with identity e. The set of maximal ideals M_A, with the Gelfand topology, is called the maximal ideal space of A.*

Corollary 23.5. *In any commutative Banach algebra A with identity, for any $x \in A$,*

$$\sigma(x) = \{\phi(x) : \phi \in \hat{A}\} = \text{Range}\, \hat{x}.$$

Proof. By Theorem 7.5,

$$\{\phi(x) : \phi \in \hat{A}\} \subset \sigma(x).$$

Consider $\lambda \in \sigma(x)$. The ideal $(\lambda e - x)A$ is proper, hence, by Theorem 7.17, $(\lambda e - x)A$ is contained in a maximal ideal M. By Theorem 23.1, there is a character ϕ of A such that $M = \ker \phi$. Then $\lambda e - x \in \ker \phi$, that is

$$\phi(\lambda e - x) = 0.$$

Hence

$$\phi(x) = \phi(\lambda e) = \lambda.$$

Thus

$$\sigma(x) \subset \{\phi(x) : \phi \in \hat{A}\}.$$

Hence

$$\begin{aligned} \sigma(x) &= \{\phi(x) : \phi \in \hat{A}\} \\ &= \{\hat{x}(\phi) : \phi \in \hat{A}\} \\ &= \text{Range}\, \hat{x}. \end{aligned}$$

□

Proof of Theorem 20.4: if a commutative Banach algebra with identity e is generated by an element x then

$$\hat{x} : \hat{A} \to \sigma(x)$$

is a homeomorphism.

Proof. By Corollary 23.5, $\hat{x} : \hat{A} \to \sigma(x)$ is a surjective mapping. It is also injective, for suppose $\phi, \psi \in \hat{A}$ and $\hat{x}(\phi) = \hat{x}(\psi)$. Then

$$\{a \in A : \phi(a) = \psi(a)\}$$

is a closed subalgebra of A containing e and x, hence it is all of A. That is, $\phi = \psi$. It follows that $\hat{x} : \hat{A} \to \sigma(x)$ is a bijective mapping. The function $\hat{x}$ is continuous, by definition of the Gelfand topology. The character space $\hat{A}$ is compact and $\sigma(x)$ is Hausdorff, hence $\hat{x}$ is a homeomorphism. $\square$

24 Semisimplicity and the radical

Return to the Question of Section 20: to what extent does a general commutative Banach algebra resemble $C(K)$? In the case that A has an identity we answered the question by constructing the Gelfand transformation

$$\Gamma : A \to C(\hat{A}),$$

so that A can be homomorphically mapped into $C(\hat{A})$. If Γ is injective, this means that A can be regarded as a subalgebra of $C(\hat{A})$. However, Γ need not be injective.

Examples 24.1. 1. Let E be a Banach space endowed with zero multiplication: $xy = 0 \ \ \forall x, y \in E$. Then E is a commutative Banach algebra without identity. Note that $\hat{E}$ is empty. For suppose ϕ is a character of E. For $x \in E$, we have

$$\phi(x^2) = \phi(xx) = \phi(0) = 0,$$

and so $\phi = 0$ - a contradiction. Thus Γ is not defined for E.

2. Let A be the algebra obtained by "adjoining an identity" to E. That is, $A = \mathbb{C} \oplus E$, with operations

$$(\lambda, x) + (\mu, y) = (\lambda + \mu, x + y),$$

$$(\lambda, x)(\mu, y) = (\lambda\mu, \mu x + \lambda y),$$

$$c(\lambda, x) = (c\lambda, cx),$$

for $\lambda, \mu, c \in \mathbb{C}$ and $x, y \in E$.

A is a Banach algebra with identity under the norm

$$\|(\lambda, x)\| = |\lambda| + \|x\|_E.$$

Its identity is $e = (1, 0)$.

We can define a character ϕ_0 on A by

$$\phi_0(\lambda, x) = \lambda.$$

If ψ is any character on A then $\psi|E$ is a homomorphism $E \to \mathbb{C}$, hence is 0. Thus

$$\psi(\lambda, x) = \psi(\lambda e + (0, x)) = \lambda\psi(e) = \lambda,$$

and so $\psi = \phi_0$. Thus $\hat{A} = \{\phi_0\}$. Here $C(\hat{A})$ is one-dimensional, and

$$\Gamma : A \to C(\hat{A})$$

is far from injective. In fact $\ker \Gamma = E$.

3. A more natural example than (2): $A = \mathbb{C} \oplus L^1(0,1)$, with convolution multiplication also has a unique character

$$(\lambda, x) \mapsto \lambda.$$

When $\ker \Gamma$ is large the Gelfand transformation tells us little about the structure of an algebra A. However, for many natural algebras, $\ker \Gamma = \{0\}$.

Definition 24.2. *Let A be a commutative algebra with identity. The* Jacobian radical $\mathrm{Rad}A$ *of A is defined to be the intersection of all the maximal ideals of A. It is an ideal of A.*

Note that if A is a commutative Banach algebra with identity then $\mathrm{Rad}A$ is a closed ideal of A, since maximal ideals are closed.

Theorem 24.3. *For any commutative Banach algebra A with identity the kernel of the Gelfand transformation Γ of A is* $\mathrm{Rad}A$.

Proof. Consider $x \in A$.

$$\begin{aligned}
x \in \ker\, \Gamma &\iff \Gamma x = 0 \\
&\iff \hat{x} = 0 \\
&\iff \hat{x}(\phi) = 0 \quad \forall \phi \in \hat{A} \\
&\iff \phi(x) = 0 \quad \forall \phi \in \hat{A} \\
&\iff x \in \ker \phi \quad \forall \phi \in \hat{A} \\
&\iff x \in M \quad \text{for all } M \in M_A \\
&\iff x \in \bigcap_{M \in M_A} M \\
&\iff x \in \mathrm{Rad}A.
\end{aligned}$$

Hence $\ker \Gamma = \mathrm{Rad}\, A$ ☐

Corollary 24.4. *The Gelfand transformation of a commutative Banach algebra A with identity is an injective homomorphism if and only if* $\mathrm{Rad}\, A = \{0\}$.

Definition 24.5. *A commutative Banach algebra with identity is* semisimple *if its Jacobian radical is* $\{0\}$.

Thus A is semisimple $\iff$ $\mathrm{Rad}\, A = \{0\}$ $\iff$ Γ is injective.

Intuitively, A can be thought of as an algebra of continuous functions with pointwise operations if and only if A is semisimple.

Examples 24.6. 1. Any algebra which is defined as an algebra of functions with pointwise operations is semisimple. Indeed, any function in the radical lies in the kernel of all point evaluations, hence is identically zero.

Thus ℓ^∞ is semisimple.

2. $A = \ell^1(\mathbb{Z}_+)$ is semisimple. Recall Example 20.5,

$$\hat{A} = \triangle(0,1) = \{z \in \mathbb{C} : |z| \leq 1\}.$$

Consider $x \in \mathrm{Rad}\, \ell^1(\mathbb{Z}_+)$, $x = (x_n)_{n=0}^\infty$. We have, for all $z \in \triangle(0,1)$,

$$\phi_z(x) = \hat{x}(z) = \sum_{n=0}^\infty x_n z^n = 0.$$

Notice that if $(x_n)_{n=0}^\infty \in \ell^1(\mathbb{Z}_+)$ then

$$\hat{x}(z) = \sum_{n=0}^\infty x_n z^n$$

is an absolutely convergent Taylor series in $\triangle(0,1)$. It follows that all $x_n = 0$, for example, from the relation

$$x_n = \frac{1}{2\pi} \int_{|z|=1} \frac{\hat{x}(z)}{z^{n+1}} dz.$$