Please solve the following problems:

Qu.1, Qu.2, Qu.5, Qu.8, Qu.10 and Qu.16;

hand in your solutions by 10.00am on Friday the 22nd of March. A tutorial is on Thursday the 14th of March at 17.00 in HERB.G LT2 and a drop-in session is on Thursday the 21st of March at 17.00 in HERB.G LT2.

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MAS3706 Topology (2019)

Examples Sheet 3

Qu.1 (i) Give the definition of a convergent sequence in a metric space (X, d).

(ii) Consider the metric space $(\mathbb{C}^3, d_\infty)$, where

$$d_\infty(\underline{\mathbf{v}}, \underline{\mathbf{w}}) = \max_{j=1,2,3} |v_j - w_j|, \text{ for } \underline{\mathbf{v}} = (v_1, v_2, v_3), \underline{\mathbf{w}} = (w_1, w_2, w_3) \in \mathbb{C}^3.$$

Show that the sequence $(\underline{\mathbf{v}}_n)_{n=1}^\infty$, where $\underline{\mathbf{v}}_n = (3i - \frac{1}{n^2}, 5, \frac{i}{3^n}) \in \mathbb{C}^3$, $i^2 = -1$, converges to $\underline{\mathbf{u}} = (3i, 5, 0)$ as $n \to \infty$ with respect to the metric d_∞.

(iii) Find $\lim_{n\to\infty} \underline{\mathbf{v}}_n$ in $(\mathbb{C}^3, d_1)$ and $\lim_{n\to\infty} \underline{\mathbf{v}}_n$ in $(\mathbb{C}^3, d_2)$, where d_1 and d_2 are given by

$$d_1(\mathbf{x}, \mathbf{y}) = \Sigma_{\mathbf{i=1}}^{\mathbf{3}} |\mathbf{x_i} - \mathbf{y_i}|,$$
$$d_2(\mathbf{x}, \mathbf{y}) = \sqrt{\Sigma_{\mathbf{i=1}}^{\mathbf{3}} |\mathbf{x_i} - \mathbf{y_i}|^{\mathbf{2}}}.$$

15 marks

Qu.2 (i) Give the definition of a convergent sequence in a topological space (X, τ).

(ii) Let $(\mathbb{R}, \tau_{s.i.})$ be the topological space with the topology $\tau_{s.i.}$, which consists of $\mathbb{R}$, $\emptyset$ and all the semi-infinite open intervals (a, ∞), $a \in \mathbb{R}$. Show that if a sequence $(x_n)_{n=1}^\infty$, $x_n \in X$, converges to $x \in X$ as $n \to \infty$, then it converges to every $y \in X$ such that $y \leq x$.

15 marks

Qu.3 A collection $\mathcal{B}$ of subsets of a set X is a *base* for a topology τ on X if each member of τ is a union of members of $\mathcal{B}$. Show that, in a metric space, the set of open balls is a basis for the metric topology. (Note that, for any $x \in X$, $B(x, 0) = \emptyset$.)

Qu.4 (i) Give the definition of a closed subset in a topological space (X, τ).

Let A and B be subsets of a topological space X. Show that

(ii) if A is open and B is closed then $B \setminus A$ is closed;

(iii) if A is closed and B is open then $B \setminus A$ is open.

Recall that, for $A \subseteq B$, the *complement* of A in B is

$$B \setminus A = \{x \in B : x \notin A\}.$$

Qu.5 Give the definition of a *closed subset* in $\mathbb{R}$. Which of the following subsets of $\mathbb{R}$ are closed, which are not? Justify your answer:

$$\text{(i) } [0, 1], \text{ (ii) } [-10, \infty), \text{ (iii) } (-\infty, -2) \bigcup (0, 7],$$

$$\text{(iv) } \bigcap_{n=1}^{\infty} (1, 1 + \frac{1}{n}), \text{ (v) } \bigcap_{n=1}^{100} (1, 1 + \frac{1}{n}], \text{ (vi) } \bigcup_{n=1}^{\infty} (-n, n).$$

20 marks

Qu.6 Show that in a Hausdorff space singletons are closed sets.

Qu.7 Consider the metric space $(C[0,1], d_\infty)$, where

$$C[0,1] = \{f\colon [0,1] \to \mathbb{C}\colon f \text{ is continuous on } [0,1]\}$$

and

$$d_\infty(f,g) = \sup_{0\le t\le 1} |f(t) - g(t)|.$$

Give the definition of a closed ball in the metric space $(C[0,1], d_\infty)$.
Are the following subsets of the metric space $(C[0,1], d_\infty)$ closed?

$$\text{(i) } \{f \in C[0,1]\colon \sup_{0\le t\le 1} |f(t) - 2007| \le 1\},$$

$$\text{(ii) } \bigcap_{p=1}^{\infty} \{f \in C[0,1]\colon \sup_{0\le t\le 1} |f(t) - \cos t| < \frac{1}{p}\}.$$

Justify your answer.

Qu.8 Prove that the following holds

$$T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0.$$

15 marks

Qu.9 Consider the space $X = \{a, b, c\}$. Define the topology

$$\tau = \{\emptyset, \{a\}, \{b, c\}, X\}.$$

Prove that (X, τ) is regular.
Hint: Consider all possible combinations for the closed subsets and the points which are not contained there. These closed subsets are open as well.

Qu.10 (i) Give the definition of the interior of a subset A in a topological space (X, τ).
(ii) Consider $\mathbb{R}$ with the usual topology. Find the interior of the following subsets of $\mathbb{R}$: (a) $A_1 = [2, 10] \setminus \{3, 5\}$, (b) $A_2 = \mathbb{Q}$ the set of rational numbers.
(iii) The **exterior** $\text{Ext}(A)$ of a set A is the interior of its complement

$$\text{Ext}(A) = (A^c)^\circ.$$

Consider $\mathbb{R}$ with the usual topology and $A = \mathbb{Q} \subseteq \mathbb{R}$. Find the exterior of A.

20 marks

Qu.11 Prove the following statements. Let (X, τ) be a topological space and let $A, B \subseteq X$. Then (i) A° is the union of all the open subsets of A.
(ii) A° is an open set.
(iii) $A^\circ \subseteq A$.
(iv) A° is the largest open subset of A.
(v) A is an open set $\iff A = A^\circ$.
(vi) $A \subseteq B \implies A^\circ \subseteq B^\circ$.
(vii) $A^{\circ\circ} = A^\circ$.
(viii) $(A \cap B)^\circ = A^\circ \cap B^\circ$.

Qu.12 Give the definition of the closure of a subset A in a topological space (X, τ).

(ii) Consider $\mathbb{R}$ with the usual topology. Find the closure of the following subsets of $\mathbb{R}$: (a) $A_1 = (2, 10) \setminus \{3, 5\}$, (b) $A_2 = \mathbb{R} \setminus \mathbb{Q}$ the set of irrational numbers.

Qu.13 Prove the following statements. Let (X, τ) be a topological space and let A, B be subsets of (X, τ). Then
(i) $\overline{A}$ is the intersection of all the closed supersets of A.
(ii) $\overline{A}$ is a closed set.
(iii) $A \subseteq \overline{A}$.
(iv) $\overline{A}$ is the smallest closed superset of A.
(v) A is closed $\iff A = \overline{A}$.
(vi) $A \subseteq B \implies \overline{A} \subseteq \overline{B}$.
(vii) $\overline{\overline{A}} = \overline{A}$.
(viii) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Qu.14 Prove the following statement. Let (X, τ) be a topological space and let A be a subset of (X, τ). Then every point of X belongs to precisely one of the following three sets:
(i) the interior of A;
(ii) the exterior of A;
(iii) the boundary of A.
Thus X is the disjoint union of A°, $\mathrm{Ext}(A)$ and ∂A.

Qu.15 Give an example of a subset A of $\mathbb{R}$ with the usual topology for which $\overline{A^\circ} \neq (\overline{A})^\circ$.

Qu.16 Give the definition of the boundary of a subset A in a topological space (X, τ).
(ii) Consider $\mathbb{R}$ with the usual topology. Find the boundary of the set of irrational numbers $A = \mathbb{R} \setminus \mathbb{Q}$.

15 marks

Qu.17 Show that the boundary of a set is empty if and only if the set is both open and closed.

Qu.18 Identify (i) A°, (ii) $\overline{A}$, (iii) ∂A and (iv) $\mathrm{Ext}(A)$ for the following subsets A of X. (No proofs required.)
1. $A = \mathbb{R}$, $X = \mathbb{R}$.
2. $A = \mathbb{Q}$, $X = \mathbb{R}$.
3. $A = \{a\}$, $X = \{a, b\}$ with the discrete topology.
4. $A = \{a\}$, $X = \{a, b\}$ with the indiscrete topology.