Please solve the following problems:

Qu.1, Qu.2, Qu.5, Qu.10, Qu.11, Qu. 13 and Qu.14;

hand in your solutions by 10.00am on Friday the 8th of March. A tutorial is on Thursday the 28th of February at 17.00 in HERB.G LT2 and a drop-in session is on Thursday the 7th of March at 17.00 in HERB.G LT2.

Z.A.Lykova

MAS3706 Topology (2019)
Examples Sheet 2

Qu.1 Consider the metric space $(\mathbb{C}^n, d_1)$, where

$$d_1(\underline{\mathbf{v}}, \underline{\mathbf{w}}) \;=\; \sum_{j=1}^{n} |v_j - w_j|\,, \text{ for } \underline{\mathbf{v}} = (v_1, \ldots, v_n), \underline{\mathbf{w}} = (w_1, \ldots, w_n) \in \mathbb{C}^n.$$

Give the definition of an *open ball* in the metric space $(\mathbb{C}^n, d_1)$.
Are the following subsets of the metric space $(\mathbb{C}^n, d_1)$ open?

$$\text{(i) } \{\underline{\mathbf{v}} = (v_1, \ldots, v_n) \in \mathbb{C}^n\colon \sum_{j=1}^{n} |v_j - j| < 5\},$$

$$\text{(ii) } \bigcup_{p=1}^{\infty} \{\underline{\mathbf{v}} \in \mathbb{C}^n\colon \sum_{j=1}^{n} |v_j| < p\}.$$

Justify your answer.

15 marks

Qu.2 Consider the metric space $(C[0,1], d_\infty)$, where

$$C[0,1] = \{f\colon [0,1] \to \mathbb{C}\colon f \text{ is continuous on } [0,1]\}$$

and

$$d_\infty(f, g) = \sup_{0 \le t \le 1} |f(t) - g(t)|.$$

Give the definition of an open ball in the metric space $(C[0,1], d_\infty)$.
Are the following subsets of the metric space $(C[0,1], d_\infty)$ open?

$$\text{(i) } \{f \in C[0,1]\colon \sup_{0 \le t \le 1} |f(t) - 2007| < 1\},$$

$$\text{(ii) } \bigcap_{p=1}^{\infty} \{f \in C[0,1]\colon \sup_{0 \le t \le 1} |f(t) - \cos t| < \frac{1}{p}\}.$$

Justify your answer.

15 marks

Qu.3 Consider the metric space $(\mathbb{C}^n, d_\infty)$, where

$$d_\infty(\underline{\mathbf{v}}, \underline{\mathbf{w}}) \;=\; \max_{j=1,\ldots,n} |v_j - w_j|\,, \text{ for } \underline{\mathbf{v}} = (v_1, \ldots, v_n), \underline{\mathbf{w}} = (w_1, \ldots, w_n) \in \mathbb{C}^n.$$

Show that $W = \{\underline{\mathbf{x}} = (x_n)_{n=1}^5 \in \mathbb{C}^5 : \max_{n=1,\ldots,5} |x_n| \le 1\}$ is not an open subspace of $(\mathbb{C}^5, d_\infty)$.

Qu.4 Let a be a point in a metric space (X, d), and let r be a positive real number. Show that the set

$$\{x \in X\colon\ d(x, a) > r\}$$

is open.

Qu.5 (i) Give the definition of a *topology* on a set X.
(ii) Give the definition of the *discrete topology* on a set X.
(iii) Is the usual topology on $\mathbb{R}$ discrete? Justify your answer.

10 marks

Qu.6 Find all topologies on the two-element set $X = \{a, b\}$.

Qu.7 Find all topologies on the three-element set $X = \{a, b, c\}$.

Qu.8 Prove the following statements, where $f\colon A \longrightarrow B$ and $T_\lambda,\ \lambda \in \Lambda$, are subsets of B.

(i) $f^{-1}(\bigcup_{\lambda\in\Lambda} T_\lambda) = \bigcup_{\lambda\in\Lambda} f^{-1}(T_\lambda)$;

(ii) $f^{-1}(\bigcap_{\lambda\in\Lambda} T_\lambda) = \bigcap_{\lambda\in\Lambda} f^{-1}(T_\lambda)$;

(iii) $f^{-1}(B \setminus T) = (A \setminus f^{-1}(T))$;

(iv) $f^{-1}(B) = A$ and $f^{-1}(\emptyset) = \emptyset$;

(v) If $g\colon B \longrightarrow C$ then, for each $W \subseteq C$, $(f \circ g)^{-1}(W) = g^{-1}(f^{-1}(W))$.

Recall that, for a subset T of B the **inverse image** of T under f is

$$f^{-1}(T) = \{a \in A\colon f(a) \in T\}.$$

Recall that, for $S \subseteq A$, the **complement** of S in A is $A \setminus S = \{x \in A\colon x \notin S\}$.

Qu.9 Let $f\colon X \longrightarrow Y$, where Y is a topological space. Show that

$$\{f^{-1}(U)\colon\ U \text{ open in } Y\}$$

is a topology on X.

Qu.10 Let X be any nonempty set and let $p \in X$. Show that

$$\tau_p = \{\emptyset\} \cup \{S \subseteq X\colon p \in S\}$$

is a topology. This is called a **particular point topology**. If $X = \{0, 1\}$, then it is called the **Sierpinski** topology.

10 marks

Qu.11 A topological space X is **Hausdorff** if for each pair of points x_1 and x_2 there are disjoint open sets U_1 and U_2 such that $x_1 \in U_1$ and $x_2 \in U_2$. Show that every metric space (X, d) is Hausdorff.

20 marks

Qu.12 Give the definition of *topologically equivalent* metrics on a set X. Consider the following metrics on $\mathbb{C}^3$:

$$d_1(\mathbf{x}, \mathbf{y}) = \Sigma_{\mathbf{i=1}}^{\mathbf{3}} |\mathbf{x_i} - \mathbf{y_i}|,$$

$$d_2(\mathbf{x}, \mathbf{y}) = \sqrt{\Sigma_{\mathbf{i=1}}^{\mathbf{3}} |\mathbf{x_i} - \mathbf{y_i}|^{\mathbf{2}}}$$

and

$$d_\infty(\mathbf{x}, \mathbf{y}) = \sup\{|\mathbf{x_i} - \mathbf{y_i}|\colon \mathbf{1} \leq \mathbf{i} \leq \mathbf{3}\}$$

for $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})$ and $\mathbf{y} = (\mathbf{y_1}, \mathbf{y_2}, \mathbf{y_3})$ from $\mathbb{C}^3$. Show that d_1, d_2 and d_∞ are topologically equivalent.

Qu.13 Let $C[1,2]$ be the set of all continuous complex-valued functions defined on the closed interval $[1,2]$, and let

$$d_\infty(f,g) = \sup\{|f(x) - g(x)| \colon 1 \le x \le 2\},$$

and

$$d_2(f,g) = \left\{\int_1^2 |f(x) - g(x)|^2 dx\right\}^{1/2}$$

be metrics on $C[1,2]$. Show that d_∞ and d_2 are not topologically equivalent.

20 marks

Qu.14 Show that the topology $\tau_{s.i.}$ on $\mathbb{R}$ which consists of $\mathbb{R}$, $\emptyset$, and all the semi-infinite open intervals (a, ∞), $a \in \mathbb{R}$, is not Hausdorff.

10 marks

Qu.15 How many Hausdorff topologies are there on a finite set? Justify your answer.