MAS3706 Topology

Revision Lectures, May 2019

Z. A. Lykova

It is essential that you read and try to understand the lecture notes from the beginning to the end.

Many questions from the exam paper will be similar to what you have been asked to do during the semester.

I do not answer e-mail enquiries as to what material will be in the exam.

Exam Information

Exam:
14.00 p.m. - 16.00 p.m. on Friday 31st May 2019 in St James' Park, Room A, Bamburgh Suite, (Main Venue)

Office Hour:
14.00-17.00 on Thursday 30th May 2019 in Herschel Building, Room 3.13

Exam papers MAS3209 and MAS3706 on Topology can be found on
http://www.ncl.ac.uk/exam.papers/

Marks in Exam paper

Section A: 5 questions - 40 marks

Define - 14 marks (bookwork)
Prove - 9 marks (similar to homework examples)
Answer - 17 marks (similar to homework/lecture examples)

Section B: 2 questions - 60 marks

Define/State - 11 marks (bookwork)
Prove - 40 marks (bookwork or similar to homework examples)
Answer - 9 marks (similar to homework/lecture examples)

Section 1 Open Subsets of $\mathbb{R}$ and Continuous Functions

You need to understand all main notions from this section to be able to follow futher material: open set, continuous function, inverse image, topology on $\mathbb{R}$ (Thm 1.19).

Examples Sheet 1: Qu.1

Section 2 Metric Spaces

Definitions: metric, metric space, open ball, open set, metric topology (Thm 2.21).
Theorem 2.22 (statement and a proof).
Examples of metric spaces 2.4-2.11.
Examples of open balls 2.13-2.18.
Examples of open sets 2.24-2.25.

Examples Sheet 1: Qu.4, Qu.9

Qu.

1. Give the definition of an *open subset* in $\mathbb{C}$ with the usual topology.
2. Which of the following subsets of $\mathbb{C}$ are open, which are not? Justify your answer:
   (a) $\bigcap_{n=1}^{\infty}\{z \in \mathbb{C} : |z-2i| < 1+\frac{1}{n}\}$;
   (b) $\bigcup_{n=1}^{\infty}\{z \in \mathbb{C} : |z-2i| \leq n\}$.

Solution

1. A subset U of the topological space $(\mathbb{C}, \tau)$ with the usual topology τ is called **open** if for each point $z \in U$ there is an open ball $B(z,r) \subset U$.
2. (a) The set $\bigcap_{n=1}^{\infty}\{z \in \mathbb{C} : |z-2i| < 1+\frac{1}{n}\} = \{z \in \mathbb{C} : |z-2i| \leq 1\}$ is not open in $(\mathbb{C}, \tau)$.
   (b) The set $\bigcup_{n=1}^{\infty}\{z \in \mathbb{C} : |z-2i| \leq n\} = \mathbb{C}$ is open in $(\mathbb{C}, \tau)$.

H/W 1, Qu. 9. ℓ^∞ denotes the set of all bounded sequences $\vec{x} = (x_n)_{n=1}^{\infty}$ of complex numbers. Verify that

$$d_\infty(\vec{x}, \vec{y}) = \sup_{n\in\mathbb{N}} |x_n - y_n|$$

is a metric on ℓ^∞.

Solution. For each $\vec{x} \in \ell^\infty$, there is a constant $M > 0$ such that $\sup_{n\in\mathbb{N}} |x_n| < M$. Hence the mapping

$$d_\infty : \ell^\infty \times \ell^\infty \to \mathbb{R} : (\vec{x}, \vec{y}) \mapsto \sup_{n\in\mathbb{N}} |x_n - y_n|$$

is well-defined.
Let us verify [M1]-[M3].

[M1] For all $\vec{v} = (v_n)_{n=1}^{\infty},\ \vec{w} = (w_n)_{n=1}^{\infty} \in \ell^{\infty}$,
$d_{\infty}(\vec{v}, \vec{w}) = \sup_{j\in\mathbb{N}} |v_j - w_j| > 0$ if $\vec{v} \neq \vec{w}$ and

$$d_{\infty}(\vec{v}, \vec{v}) = \sup_{j\in\mathbb{N}} |v_j - v_j| = 0.$$

[M2] For all $\vec{v} = (v_n)_{n=1}^{\infty},\ \vec{w} = (w_n)_{n=1}^{\infty} \in \ell^{\infty}$,

$$d_{\infty}(\vec{v}, \vec{w}) = \sup_{j\in\mathbb{N}} |v_j - w_j|$$

$$= \sup_{j\in\mathbb{N}} |w_j - v_j| = d_{\infty}(\vec{w}, \vec{v}).$$

[M3] For all $\vec{v} = (v_n)_{n=1}^{\infty},\ \vec{w} = (w_n)_{n=1}^{\infty},\ \vec{u} = (u_n)_{n=1}^{\infty} \in \ell^{\infty}$, observe that

$$\begin{aligned} d_{\infty}(\vec{v}, \vec{w}) &= \sup_{j\in\mathbb{N}} |v_j - w_j| \\ &= \sup_{j\in\mathbb{N}} |v_j - u_j + u_j - w_j| \\ &\leq \sup_{j\in\mathbb{N}} (|v_j - u_j| + |u_j - w_j|) \\ &\leq \sup_{j\in\mathbb{N}} |v_j - u_j| + \sup_{j\in\mathbb{N}} |u_j - w_j| \\ &= d_{\infty}(\vec{v}, \vec{u}) + d_{\infty}(\vec{u}, \vec{w}). \end{aligned}$$

Therefore d_{∞} is a metric and $(\ell^{\infty}, d_{\infty})$ is a metric space.

Section 3 Topological Spaces

Definitions: topology (3.1), topological space (3.2).
Examples of topological spaces 3.4-3.9.

Examples Sheet 2: Qu.1, Qu.2, Qu.4, Qu.5, Qu.10, Qu.14

Qu.

1. Give the definition of a *topology* on a set X.
2. Let $\tau_{s.i.}$ consist of $\mathbb{R}$, $\emptyset$, and all the semi-infinite open intervals (a, ∞), $a \in \mathbb{R}$. Then $\tau_{s.i.}$ is a topology on $\mathbb{R}$.

Solution.

1. Let X be a set. Then a **topology** on X is a family τ of subsets of X such that

   [T1] τ includes X and $\emptyset$;

   [T2] τ is closed under arbitrary unions:

$$U_\lambda \in \tau \text{ for all } \lambda \in \Lambda \Longrightarrow \bigcup_{\lambda \in \Lambda} U_\lambda \in \tau;$$

[**T3**] τ is closed under finite intersections:

$$U_1, U_2, \ldots, U_n \in \tau \Longrightarrow U_1 \cap U_2 \cap \ldots \cap U_n \in \tau.$$

2. One can check that

   [**T1**]: $\tau_{s.i.}$ includes $\mathbb{R}$ and $\emptyset$.

   [**T2**]: Let (a_λ, ∞), $\lambda \in \Lambda$, be a family of members of $\tau_{s.i.}$. Then
   (i) if the a_λ, $\lambda \in \Lambda$ are bounded below, their union is $(\inf a_\lambda, \infty) \in \tau_{s.i.}$;

   (ii) if they are not bounded below, their union is $\mathbb{R} \in \tau_{s.i.}$.

   [**T3**]
$$\bigcap_{i=1}^{n} (a_i, \infty) = (\max(a_1, a_2, \ldots, a_n), \infty) \in \tau_{s.i.}.$$

   Hence $\tau_{s.i.}$ is a topology in $\mathbb{R}$.

Section 4 Hausdorff Spaces and Limits

Definitions: Hausdorff Space (4.1), convergent sequence in a metric space (4.7), convergent sequence in a topological space (4.11).
Theorems 4.5, 4.13 (statements and proofs).
Examples 4.2 -4.4, 4.9, 4.10 4.12.

Examples Sheet 2: Qu.11
Examples Sheet 3: Qu.1, Qu.2.

Example. Consider the metric space $(\mathbf{C}^4, d_1)$, where $\mathbf{C}^4 = \{\vec{v} = (v_1, v_2, v_3, v_4);\ v_i \in \mathbf{C}, i = 1, 2, 3, 4\}$ and

$$d_1(\vec{v}, \vec{w}) \ = \ \sum_{j=1}^{4} |v_j - w_j|,$$

$$\text{for } \vec{v} = (v_1, v_2, v_3, v_4), \vec{w} = (w_1, w_2, w_3, w_4).$$

Show that the sequence $(\vec{v}_n)_{n=1}^{\infty}$, where
$\vec{v}_n = (2019i + \frac{1}{n^2}, 2, \frac{i}{2^n}, -i) \in \mathbf{C}^4$,
$i^2 = -1$, converges to $\vec{u} = (2019i, 2, 0, -i)$ as $n \to \infty$ with respect to the metric d_1.

Proof. In the metric space $(\mathbf{C}^4, d_1)$, a *sequence* $(\vec{x}_n)_{n=1}^{\infty}$, $\vec{x}_n \in \mathbf{C}^4$, *converges to* $\vec{x} \in \mathbf{C}^4$, as $n \to \infty$ if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that, for all $n \geq N$, $d_1(\vec{x}_n, \vec{x}) < \varepsilon$.
Note that

$$d_1(\vec{v}_n, \vec{u}) =$$

$$d_1((2019i + \frac{1}{n^2}, 2, \frac{i}{2^n}, -i), (2019i, 2, 0, -i)) =$$

$$|2019i + \frac{1}{n^2} - 2019i| + |2 - 2| +$$

$$|\frac{i}{2^n} - 0| + |-i - (-i)| =$$

$$\{\frac{1}{n^2} + \frac{1}{2^n}\} \to 0$$

as $n \to \infty$. Thus $\lim_{n\to\infty} \vec{v}_n = \vec{u} = (2019i, 2, 0, -i)$ with respect to the metric d_1.

Recall that a topological space X is **Hausdorff** if for each pair of points x_1 and x_2 there are disjoint open sets U_1 and U_2 such that $x_1 \in U_1$ and $x_2 \in U_2$.

Example. Show that every metric space (X, d) is Hausdorff.

Proof. Suppose that X is a metric space and x_1 and x_2 are distinct points of X. Then $\delta = d(x_1, x_2) > 0$ and let $U_1 = B(x_1, \delta/2)$, $U_2 = B(x_2, \delta/2)$. Then U_1 and U_2 are open, $x_1 \in U_1, x_2 \in U_2$. Finally, U_1 and U_2 are disjoint, because if they have a common point z, it follows that

$$\delta = d(x_1, x_2) \leq$$

$$d(x_1, z) + d(z, x_2) <$$

$$\delta/2 + \delta/2 = \delta,$$

a contradiction.

Example. The topological space X with the indiscrete topology τ is not Hausdorff provided X contains more than one point.

Proof. Let x_1 and x_2 be distinct points in X. Recall that the indiscrete topology $\tau = \{X, \emptyset\}$. Every open set $U_1 \in \tau$ such that $x_1 \in U_1$ has to be X and therefore $x_2 \in U_1$. Thus there are no disjoint open sets U_1 and U_2 in the indiscrete topology such that $x_1 \in U_1$ and $x_2 \in U_2$.

Section 5 Closed Sets

Definitions: closed set, closed ball.
Propositions 5.4, 5.6 (statements and proofs).
Examples 5.2, 5.3, 5.7.

Examples Sheet 3: Qu.10, Qu.12

Example Consider the topological space $(C[0,1], \tau_{d_\infty})$ with the metric topology induced by d_∞, where $C[0,1] = \{f : [0,1] \to \mathbf{C} : f \text{ is continuous on } [0,1]\}$ and

$$d_\infty(f,g) = \sup_{0 \le t \le 1} |f(t) - g(t)|.$$

Is the following subset of $(C[0,1], \tau_{d_\infty})$ closed?

$$\bigcup_{p=1}^{3} \{f \in C[0,1] : \sup_{0 \le t \le 1} |f(t) - (\cos t)^p| \le 2p\}.$$

Justify your answer.

Proof. A closed ball $\overline{B_\infty(g,r)}$ with centre at g and radius r is closed, since

$$\overline{B_\infty(g,r)}^c = \{f \in C[0,1] : d_\infty(f,g) > r\}$$

is open in $(C[0,1], \tau_{d_\infty})$. The set

$$\bigcup_{p=1}^{3} \{f \in C[0,1] : \sup_{0 \le t \le 1} |f(t) - (\cos t)^p| \le 2p\}$$

is a finite union of closed balls $\overline{B_\infty(g_p, r_p)}$, where $g_p(t) = (\cos t)^p$ and $r_p = 2p$, $p = 1, 2, 3$. Therefore it is closed in $(C[0,1], \tau_{d_\infty})$.

Section 6 Separation axioms
Read and try to understand the material.

Section 7 Basic Topological Concepts

Definitions: interior of a set (7.1, 7.3), closure of a set (7.8, 7.9), boundary of a set (7.16, 7.17), topology on a subset.
Examples 7.4, 7.5, 7.10, 7.11, 7.18, 7.19.

Definition 0.1. The subset A° of A consisting of all the interior points of A is called **interior** of A.

Example 0.2. Consider $\mathbb{R}$ with the usual topology and the closed interval $A = [a, b] \subseteq \mathbb{R}$. Then every point r of the open interval (a, b) is an interior point of $[a, b]$. One can see that $r \in (a, b)$ and the open subset $(a, b) \subseteq A$, but a and b are not interior points. Therefore $[a, b]^\circ = (a, b)$.

Example 0.3. Consider $\mathbb{R}$ with the usual topology and the subset $\mathbb{Z} \subseteq \mathbb{R}$. Every point of $\mathbb{Z}$ is NOT an interior point of $\mathbb{Z}$, since every nonempty open interval contains points which are not in $\mathbb{Z}$. Therefore $\mathbb{Z}^\circ = \emptyset$.

Section 8 Compactness

Definitions: open cover of X (8.1), open subcover of X (8.3), compact (8.5).
Propositions/Theorems 8.7, 8.8, 8.11, 8.12, 8.15, 8.17 (statements and a number of proofs).
Heine-Borel theorem 8.14 (statements and a proof)
Examples 8.2, 8.4, 8.6, 8.9, 8.16.

Examples Sheet 4: Qu.1, Qu.2

Section 9 Continuity

Definition of a continuous function (9.1, 9.2).
Theorems 9.3, 9.8 and 9.9 (statements and some proofs).
Examples 9.5, 9.6.

Section 10 Metric Spaces Again

Theorems 10.1 and 10.2 (statements and some proofs).

Section 11 Completeness in Metric Spaces

Definitions: Cauchy sequence (11.1), completeness of a metric space (11.5), contraction (11.11), fixed point of a mapping (11.14).
Theorems/Propositions 11.2, 11.3, 11.8 (statements and some proofs)
Completeness of $C(K)$ (11.9, statement and a proof).
Examples 11.4, 11.6, 11.16.

Examples Sheet 4: Qu.17, Qu.18, Qu.19, Qu.21, Qu.22

Section 12 Connectedness
Read and try to understand the material.

Section 13 Picard's Theorem

Picard's Theorem (try to understand the proof).

Examples Sheet 5: Qu.27