Please do the following problems:
Qu.58, Qu.59, Qu.64, Qu.66 and Qu.69.

hand in your solutions on Monday the 11th of December by 16.00. The tutorial is on Monday the 4th December at 11.00 in ARMB.3.38.

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MAS3702 Linear Analysis (2017)
Examples Sheet 5

Qu.58  State the Riesz–Fréchet theorem. Define a mapping $T: C[0,1] \to \mathbb{C}$ by the formula

$$T(f) = -5i \int_0^1 f(t) t^3 dt, \quad i^2 = -1.$$ 

Prove that $T$ is a bounded linear functional with respect to the norm $\|f\|_2$, where $\|f\|_2 = \left\{ \int_0^1 |f(t)|^2 dt \right\}^{1/2}$. Find $\|T\|$.

15 marks

Qu.59  Let $W = \{f \in C[0,1]: f(1) = 0\}$ and let $R: W \to \mathbb{C}$ be defined by

$$Rf = \int_0^1 xf(x) \, dx.$$ 

(i) Show that $R$ is a linear functional.
(ii) Let $W$ have the supremum norm $\|\cdot\|_\infty$. Prove that $R$ is continuous on $W$ and find $\|R\|$.

[Hint: Find a sequence of functions, $f_n$ such that $\|f_n\|_\infty = 1$ and $|Rf_n| \to \|R\|$.

20 marks

Qu.60  Let $F: C^1[0,1] \to \mathbb{C}$ be defined by

$$F(f) = f'(1), \quad f \in C^1[0,1].$$ 

Show that $F$ is a linear functional. Prove that $F$ is discontinuous with respect to the norm $\|\cdot\|_2$, where $\|f\|_2 = \left\{ \int_0^1 |f(t)|^2 dt \right\}^{1/2}$.

Qu.61  Consider $C[-\pi, \pi]$ with the usual inner product norm $\|\cdot\|_2$, where

$$\|f\|_2 = \left\{ \int_{-\pi}^{\pi} |f(t)|^2 dt \right\}^{1/2}.$$ 

Let $F: C[-\pi, \pi] \to \mathbb{C}$ be defined by

$$Ff = \int_{-\pi}^0 f(x) \, dx.$$ 

Prove that $F$ is a continuous linear functional on $C[-\pi, \pi]$ such that $F$ is not equal to $<\cdot, g>$ for any $g \in C[-\pi, \pi]$.

Qu.62  Let $(c_0, \|\cdot\|_\infty)$ and $(\ell^1, \|\cdot\|_1)$ be as in Qu. 39 and Qu. 42 respectively. Show that the dual space $(c_0)^*$ of $c_0$ can be identified with $\ell^1$: that is, there is a mapping $T: \ell^1 \to (c_0)^*$ which is an isomorphism of vector spaces and which preserves norms.

Qu.63  Let $T: \mathbb{C}^2 \to \mathbb{C}^2$ be the linear operator defined by the formula

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2i \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$ 

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2i \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$
where $i^2 = -1$. Find the adjoint operator $T^*$ of $T$.

**Qu.64** Let $T: \ell^2 \to \ell^2$ be the left shift operator:
$$T(x_1, x_2, \ldots, x_n, \ldots) = (x_2, x_3, \ldots, x_n, \ldots).$$

(i) Find the following operators $T^*$, $T^*T$ and $TT^*$, where $T^*$ is the adjoint operator of $T$.

(ii) Is $T^*$ invertible? Is $TT^*$ invertible?

20 marks

**Qu.65** Show that if $S$ and $T$ are invertible operators in $B(\ell^2)$ then so is $ST$. Give an example of operators $S$, $T$ such that $ST$ is invertible, but neither $S$ nor $T$ is invertible.

**Qu.66** Consider $C[0, 1]$ with the norm $\| \cdot \|_\infty$, where $\| f \|_\infty = \sup_{t \in [0,1]} |f(t)|$.
Let $T: C[0,1] \to C[0,1]$ be the linear operator defined by the formula
$$(Tx)(t) = 3t^2x(t), \ x \in C[0,1], \ t \in [0,1].$$
Find the spectrum $Sp T$ of $T$.

25 marks

**Qu.67** Let $S, T \in M_n(\mathbb{C})$.

(i) Show that every non-zero eigenvalue of $ST$ with eigenvector $v$ is an eigenvalue of $TS$ with eigenvector $Tv$.

(ii) Let $S$ and $T$ be matrices which satisfy
$$ST - TS = I.$$ Show that if $\lambda$ is an eigenvalue of $ST$ then $\lambda - 1$ is an eigenvalue of $TS$, with the same eigenvector. Hence show there are no pairs of matrices $S$, $T$ such that $ST - TS = I$.

**Qu.68** Let $K$ be the linear operator $K: (C([0, 1]), \| \cdot \|_\infty) \to (C([0, 1]), \| \cdot \|_\infty)$ defined by
$$(Kf)(x) = \int_0^1 (x-t)f(t) \, dt.$$ Show that any eigenvector of $K$, with non-zero eigenvalue, is of the form $f(x) = Ax + B$, for some constants $A$ and $B$. Find the non-zero eigenvalues and corresponding eigenvectors of this operator.

**Qu.69** Let $T$ be the left shift operator on $(\ell^\infty, \| \cdot \|_\infty)$ defined by
$$T(x_1, x_2, \ldots, x_n, \ldots) = (x_2, x_3, \ldots, x_n, \ldots).$$

(i) Find the operator norm $\| T \|$ of $T$.

(ii) Show that vectors $(1, \lambda, \lambda^2, \lambda^3, \ldots) \in \ell^\infty$, where $\lambda \in \mathbb{C}$ and $|\lambda| \leq 1$, are eigenvectors of $T$.

(iii) Find the spectrum $Sp T$ of $T$.

20 marks

**Qu.70** (i) Find all the eigenvalues of the left shift operator on $\ell^2$:
$$T(x_1, x_2, \ldots, x_n, \ldots) = (x_2, x_3, \ldots, x_n, \ldots).$$

(ii) Find the spectrum $Sp T$ of $T$ (use the fact that $Sp T$ is a closed bounded subset of $\mathbb{C}$ and is contained in the closed disc of centre 0, radius $\| T \|$).