MAS 3091, Applied Mathematics

September 2009

PROJECT 1: EPIDEMICS

Instructions: Do questions 1 to 9, then do either question 10 or 11.

1. **Epidemics.** To model the spread of an epidemic (e.g. flu, smallpox) we divide the population in three classes: susceptible individuals, $S(t)$, who do not have the diseases at time t but risk catching it; infected individuals, $I(t)$, who have the disease at time t and may pass it to susceptible individuals if they get in contact with them; removed individuals, $R(t)$, who, having had the disease, are now immune to it, and cannot spread any longer. The total number of individuals is $N = S + I + R$. This model is called SIR, as individuals move from the S–class to the I–class and finally to the R–class. [1]

   The governing equations of the model are:

   $$\dot{S} = -\alpha I S, \tag{1}$$

   $$\dot{I} = \alpha I S - \beta I, \tag{2}$$

   $$\dot{R} = \beta I, \tag{3}$$

   where $\alpha > 0$ is the rate of infection, and $1/\beta > 0$ is the typical time to be recover from the disease. The quantities S, I and R, being populations, cannot be negative. We assume that $N >> 1$, so S, I and R are real numbers.

   The initial condition of the SIR model is that at the beginning of the epidemic ($t = 0$) a small number $I(0)$ of infected individuals is introduced in the population; since $R(0) = 0$, then initially $S(0) \approx N$.

   Show that the total population $N = S + I + R$ is constant, that is to say $dN/dt = 0$.

2. Show that $S(t)$ is a decreasing function, that is to say $S(0) > S(t_1) > S(t_2) > S(t_3) \cdots$ where $0 < t_1 < t_2 < t_3 \cdots$. (Hint: consider the signs of the terms in Eq. 1).

[1] Mathematically, "removed" means that members of the R–class have been removed from the other classes. Clinically, depending on the disease which we model, "removed" means that either R-class individuals are healthy again, or that they are dead !

3. Show that the epidemic will start spreading only if

$$S(0) > \frac{\beta}{\alpha}. \tag{4}$$

that is, if, for the same number of susceptible individuals, the infection rate is large enough, or the recovery time is long enough. (Hint: consider Eq. 2 at $t = 0$)

4. Derive an equation for dI/dS, solve it and show that the solution I in terms of S is

$$I(t) = N - S(t) + \frac{\beta}{\alpha} \ln\left(\frac{S(t)}{S(0)}\right), \tag{5}$$

(Hint: write $dI/dS = (dI/dt)/(dS/dt)$ and then use Eq. 2 and 1).

5. Show that, during the evolution of the epidemic, the maximum number of infected people is

$$I_{max} = N - \frac{\beta}{\alpha} + \frac{\beta}{\alpha} \ln\left(\frac{\beta}{\alpha S(0)}\right), \tag{6}$$

(Hint: maximise I as a function of S given by Eq. 5)

6. Derive and equation for dS/dR, solve it and show that the solution for S in terms of R is

$$S(t) = S(0)e^{-\alpha R(t)/\beta}, \tag{7}$$

(Hint: write $dS/dR = (dS/dt)/(dR/dt)$ and then use Eq. 1 and 3)

7. Use the previous result and the fact that $R(t) \leq N$ to show that, at the end of the epidemic, some susceptible individuals have escaped the infection.

8. Evaluate Eq. 5 at $t = \infty$ and show that

$$\frac{\beta}{\alpha} = \frac{N - S(\infty)}{\ln\left(S(0)/S(\infty)\right)} \tag{8}$$

9. Estimate α and β for an outbreak of bubonic plague in an English village which took place in 1666, according to records found in the parish church. The records(see the following table) list the number of susceptibles and infected at different times. The records also suggest that the recovery time for the disease is 11 days.

Date	Susceptibles	Infectives
1 July	254	7
15 July	200	20
1 August	150	30
15 August	120	20
1 September	100	10
15 September	90	5
1 October	85	0

(Hint: The ratio β/α can be deduced from the size N of the population before the epidemic and the number of people who survived the epidemic without being infected.)

10. **Flu outbreak.** Write a Fortran 90 program or a Maple program to integrate in time Eq. 1, 2 and 3 for given α and β with initial condition $S(0)$, $I(0)$ and $R(0)$ using the Euler method with time step Δt.

    Run the program to apply the SIR model to a town of $N = 10000$ people suffering a flu outbreak. The parameters should be $\alpha = 0.001$ and $\beta = 3$. Your initial condition should be $I(0) = 1$, $R(0) = 0$ and $S(0) = N - I(0)$. The unit of time is one month. Find the solution $S(t)$, $I(t)$ and $R(t)$ for $0 \leq t \leq 12$ months. Use time step $\Delta t = 0.001$. Plot S, I and R vs time t for $0 \leq t \leq 12$ months.

    Plot S, I and R vs t. Then make a plot of I vs S. At the peak of the infection, how many people are infected ? At the end of the infection, how many people have escaped becoming ill ?

11. **Plague outbreak.** Write a Fortran 90 program or a Maple program to integrate in time Eq. 1, 2 and 3 for given α and β with initial condition $S(0)$, $I(0)$ and $R(0)$ using the Euler method with time step Δt.

    Run the program to model the 1666 plague outbreak described in question 9 for $0 \leq t \leq 3$ months. Plot $S(t)$, $I(t)$ and $R(t)$. For comparison, plot in the same figure the data listed in the table and compare the data and the results of the SIR model. Discuss the limitations of the SIR model in this case.

TEAM PROJECT 2: CLOUDS AND DROPS

Instructions: Do all questions.

1. **Evaporating cloud.** A spherical droplet of water in a cloud has radius r, volume $V = (4\pi/3)r^3$ and surface area $A = 4\pi r^2$. The droplet evaporates at rate given by

$$\frac{dV}{dt} = -kA, \tag{9}$$

where t is time and k is a constant.

   (a) Show that the governing equation for $r(t)$ is

$$\frac{dr}{dt} = -k, \tag{10}$$

   (b) Show that the solution of the above equation is is

$$r(t) = r_0 - kt, \tag{11}$$

   where $r(0) = r_0$ is the initial radius.

   (c) Show that the time for the droplet to evaporate is $t_{evap} = r_0/k$ (that is to say, at $t = t_{evap}$ the radius becomes zero and the droplet disappears).

   (d) Show that the volume of the droplet changes with time according to

$$V(t) = V(0)\left(1 - \frac{t}{t_{evap}}\right)^3. \tag{12}$$

   where $V(0)$ is the initial volume.

2. **Falling drop.** Consider a rain droplet of constant radius r. The vertical motion of the droplet is described by the equations

$$\frac{dz}{dt} = v, \tag{13}$$

$$\frac{dv}{dt} = -\frac{v}{\tau} - g, \tag{14}$$

where z is the position of the droplet, v is the velocity, g is the acceleration due to gravity, and the parameter τ is

$$\tau = \frac{2\rho r^2}{9\mu}, \tag{15}$$

where ρ is the density of the droplet and μ is the viscosity of the surrounding air. The initial conditions at $t = 0$ are that $z(0) = z_0$ and $v(0) = v_0$.

(a) Show that

$$v(t) = (v_0 + \tau g)e^{-t/\tau} - g\tau, \tag{16}$$

and that

$$z(t) = z_0 + g\tau + \tau(v_0 + g\tau)(1 - e^{-t/\tau}) \tag{17}$$

(b) Show that, for $t >> \tau$, the droplet falls with constant velocity

$$v_\infty = -g\tau, \tag{18}$$

(the minus sign means that the direction is down, opposite to gravity).

3. **Fall and evaporation of a raindrop.** Write a Fortran 90 program or a Maple program to predict the behaviour of a falling rain droplet which evaporates (that is, its radius r changes with time). Your program should use the Euler method to solve the following governing equations:

$$\frac{dr}{dt} = -k, \tag{19}$$

$$\frac{dz}{dt} = v, \tag{20}$$

$$\frac{dv}{dt} = -\frac{v}{\tau} - g, \tag{21}$$

where

$$\tau = \frac{2\rho r^2}{9\mu}, \tag{22}$$

Note that, since here we assume $r = r(t)$, then $\tau = \tau(t)$.

The initial conditions are that $r(0) = 0.01$ cm, $z(0) = 0$ and $v(0) = 0$. The values of the parameters are: $\mu = 1.78 \times 10^{-4}$ g cm^{-1} s^{-1}, $g = 980.4$ cm s^{-2}, and $\rho = 1$ g cm^{-3}. Use time step $\Delta t = \tau/100$.

Plot $r(t)$, $z(t)$, $v(t)$ vs t in the range $0 \leq t \leq 2$ s for the following values of k: $k = 0$ (droplet which does not evaporate), $k = 0.001$, 0.005, 0.01 and 0.05 cm s^{-1}. Discuss the effect of the evaporation of the falling motion of the droplet.

4. **Visibility in fog and mist.** Consider a given volume of water V which consists of N droplets of radius r. Assuming no overlap of the line of sight, each droplet blocks off the area πr^2.

   Find the area blocked off by N droplets and explain why we see further in the rain ($r \approx 0.1$ cm) than in a drizzle ($r \approx 0.02$ cm) than in a fog ($r \approx 0.001$ cm).

5. **Should one walk or run in the rain ?** If it rains, we do not have an umbrella and we do not like to get wet, it is not entirely obvious whether we should walk or run [2]. To model the problem, assume that rain is falling with vertical velocity u, depositing water on the ground at the rate of ϕ litres per square metre per second. Assume that the wind adds a horizontal component w to the rain's velocity. The apparent angle of the rain to a stationary person is thus $tan^{-1}(w/u)$. Model a person as a rectangular block of height h, shoulder width s and body thickness b. Assume that the person wants to travel a distance d at speed v, choosing v so that the total amount of water falling on the top (head and shoulders, area sb) and the front or back (area hs) is minimised.

   (a) Assume that the person walks against the rain. Show that the amount of water collected by the top is $W_1 = sb\phi d/v$. Explain why the front, moving at effective velocity $v + w$ through water of density ϕ/u, collects it at the rate $hs(v+w)\phi/u$, hence show that the amount of water collected by the front is $W_2 = hs(\phi/u)(v + w)(d/v)$. Show that the total amount of water collected is

   $$W = W_1 + W_2 = s\phi d \left(\frac{h}{u} + \frac{1}{v} \left(b + \frac{hw}{u} \right) \right), \tag{23}$$

   hence state whether the person should run or not. Discuss what happens in various limits (e.g. $v \to 0$, $w \to 0$).

   (b) Do the same analysis in the case that the wind is from behind. Distinguish the two cases $v > w$ and $v < w$ (hint: check what happens for $b > hw/u$ or $b < hw/u$). Again, state whether the person should run or not. If the person's height and thickness are respectively $h = 180$ cm and $b = 30$ cm, and the rain falls at an angle of 10 degrees, what should the person do ? Walk or run ?

[2] The problem was solved by David E. Bell, Mathematical Gazette, **60**, 206–208 (1976)

PROJECT 3: THE WAVE EQUATION

1. Derive the wave equation, $u_{tt} = c^2 u_{xx}$, as a model for small transverse vibrations of an elastic string.

2. Solve the wave equation using the method of d'Alembert and discuss how the solution can be interpreted.

3. Use the method of separation of variables to solve the wave equation for a string with fixed endpoints $u(0, t) = u(L, t) = 0$ and with a triangular initial deflection,

$$u(x, 0) = \begin{cases} 2x/L, & 0 \leq x \leq L/2, \\ 2(L - x)/L, & L/2 \leq x \leq L, \end{cases}$$

   and $u_t(x, 0) = 0$.

There are many useful textbooks in the library. Use graphs, plotted with Maple or another package, where appropriate. Be creative and remember that historical information can provide an interesting and informative background.

PROJECT 4: THE DIVERGENCE THEOREM OF GAUSS

1. Prove the divergence theorem of Gauss.
2. Derive the diffusion equation for heat flow in a body, $u_t = ku_{xx}$, using Gauss' divergence theorem.
3. Use the method of separation of variables to solve the diffusion equation for the temperature $u(x,t)$ in a 1D bar with insulated ends $u_x(0,t) = u_x(L,t) = 0$ and an initial temperature

$$f(x) = \begin{cases} x, & 0 \leq x \leq L/2, \\ L - x, & L/2 \leq x \leq L. \end{cases}$$

There are many useful textbooks in the library. Use graphs, plotted with Maple or another package, where appropriate. Be creative and remember that historical information can provide an interesting and informative background.