Combing nilpotent and polycyclic groups

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Abstract

The notable exclusions from the family of automatic groups are those nilpotent groups which are not virtually abelian, and the fundamental groups of compact 3-manifolds based on the *Nil* or *Sol* geometries. Of these, the 3-manifold groups have been shown by Bridson and Gilman to lie in a family of groups defined by conditions slightly more general than those of automatic groups, that is, to have combings which lie in the formal language class of indexed languages. In fact, the combings constructed by Bridson and Gilman for these groups can also be seen to be real-time languages (that is, recognised by real-time Turing machines).

This article investigates the situation for nilpotent and polycyclic groups. It is shown that a finitely generated class 2 nilpotent group with cyclic commutator subgroup is real-time combable, as are also all 2 or 3-generated class 2 nilpotent groups, and groups in specific families of nilpotent groups (the finitely generated Heisenberg groups, groups of unipotent matrices over $\mathbf{Z}$ and the free class 2 nilpotent groups). Further it is shown that any polycyclic-by-finite group embeds in a real-time combable group. All the combings constructed in the article are boundedly asynchronous, and those for nilpotent-by-finite groups have polynomially bounded length functions.

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1 Introduction

The aim of this article is to investigate which finitely generated nilpotent and polycyclic groups have real-time combings, or rather asynchronous combings which are real-time languages.

The concept of a combing for a finitely generated group has grown out of the definition of an automatic group (as introduced in [9]). A formal definition is given in Section 2; informally, a combing is an orderly set of strands through the Cayley graph of the group, alternatively a set of words mapping onto the group for which words which represent closely related group elements are also closely related as words; automatic groups possess combings which are regular languages.

Our interest in combings in other formal language classes arises out of work in [7], which shows that the fundamental group of any compact geometrisable 3-manifold has an indexed combing, that is, a combing in the formal language class of indexed languages. (The indexed languages lie above the context-free languages in the formal language hierarchy, and are defined by automata with an attached system of nested stacks.) It seems natural to ask whether or not the family of indexed combable groups is large enough also to contain all finitely generated nilpotent groups. This paper arose out of an attempt to answer that question. However, our attention was diverted to the class of real-time languages by our realisation that all the combings constructed in [7] lay in that class. Further, while the combing for $\mathbf{Z}^2$ which was fundamental to the construction of the others in [7] is an indexed language, it seems unlikely that the analogous combing for $\mathbf{Z}^n$ is indexed (we have not yet proved this), but it is a real-time language. The real-time languages, recognised by real-time Turing machines, are, in particular, recognisable in linear time. Hence in this paper we address the following question: 'Does every finitely generated nilpotent group have a real-time combing?'

We certainly need to go some way up the language hierarchy to find combings for nilpotent groups. By [9], nilpotent groups, unless virtually abelian, cannot have regular combings, and by [7], if such a group has a context-free combing, it cannot be bijective (that is, it cannot contain unique representatives of each group element). Further, Burillo has proved in [8] that neither the groups $U_n(\mathbf{Z})$ of n-dimensional unipotent upper-triangular matrices over $\mathbf{Z}$, nor the $2n+1$-dimensional Heisenberg groups, defined by the presentations

$$\begin{aligned} H_{2n+1} &= \langle x_1, \ldots x_n, y_1, \ldots y_n, z \mid [x_i, y_i] = z, \forall i, \\ &\qquad [x_i, x_j] = [y_i, y_j] = [x_i, y_j] = 1, \forall i, j, i \neq j \rangle \end{aligned}$$

can have synchronous, quasigeodesic combings. (Thurston had already proved in [9] that the 3-dimensional Heisenberg group does not satisfy a quadratic isoperimetric inequality, and hence cannot be synchronously combable by quasi-geodesics; Burillo's result was proved by consideration of higher dimensional isoperimetric inequalities.) In fact both $U_n(\mathbf{Z})$ and H_{2n+1} have asynchronous combings; see the end of Section 4, or [20].

We do not have a complete answer to our question, but have some interesting partial answers. We prove that a finitely generated class 2 nilpotent group has a real-time combing if it has cyclic commutator group, or can be generated by at most 3 of its elements; we do so by relating such a group to a semidirect product, and applying a result of Bridson ([5]). We give an example of a 4-generated class 2 nilpotent group which cannot be related to a semidirect product in this way (but so far we have not proved that this group does not have such a combing for some other reason). Finally we show that any polycyclic by finite group (and hence, of course, any finitely generated nilpotent group) embeds in a real-time combable group. We verify that all these combings have fairly good geometrical properties.

An entirely different approach towards the construction of combings is described by Pittet in [18], and attributed to Gromov ([16]). Asynchronous combings for certain finitely generated nilpotent groups are constructed out of *homotopic combings* of Lie groups in which they embed. (Essentially a homotopic combing of a Lie group is a set of continuous rectifiable paths in the underlying manifold satisfying a continuous form of the fellow traveller condition of this paper.) It is observed that the free class c nilpotent groups (the quotients of free groups by the n-th terms of their lower central series) and the groups

$$G_c = \langle a_1, \dots a_c, t \mid [a_i, a_j] = 1, a_i^t = a_i a_{i+1}, i \neq c,\ a_c^t = a_c \rangle$$

(defined in [3], and also class c nilpotent) are combable for this reason, and that these groups consequently have polynomial Dehn functions of degree $c+1$. By the results of this article, the groups G_c (which are split extensions of the form $\mathbf{Z}^c \rtimes \mathbf{Z}$) and the class 2 free nilpotent groups are real-time combable, but the higher class free nilpotent groups do not obviously yield to our methods. Pittet's construction is geometric, and depends on particular properties of the Lie group; the language theoretic complexity of the combings is not examined.

The work of this paper relates also to work of Baumslag, Shapiro and Short in [4]. There the class of parallel poly-pushdown groups is defined, and proved to contain the fundamental groups of all compact geometrisable 3-manifolds and every class 2 nilpotent group. Further, every finitely generated torsion-free nilpotent group is proved to embed in a parallel poly-pushdown group. Hence the results are analogous to ours. However the class of parallel poly-pushdown groups generalises the concept of an automatic group in a rather different way; the language associated with a parallel poly-pushdown group satifies a weaker condition (which can be checked by pushdown automata) than the fellow traveller condition of this paper, and so is not necessarily a combing.

2 Definitions and notations

Let G be a group, with identity element 1, and finite generating set X. Without loss of generality, we may assume that X is inverse closed, that is, contains the inverse of each of its elements; we shall make this assumption throughout this paper. We call a product of elements in X a word over X, and denote by X^* the set of all such words. Let $\Gamma = \Gamma_{G,X}$ be the Cayley graph for G over X, with vertices corresponding to the elements of G, and, for each $x \in X$, a directed edge from the vertex g to the vertex gx, labelled by x. Let $d_{G,X}$ measure (graph theoretical) distance between vertices of $\Gamma_{G,X}$. For words $w, v \in X^*$, we write $w = v$ if w and v are identical as words, $w =_G v$ if w and v represent the same element of G. We define $l(w)$ to be the length of w as a string, and $l_G(w)$ to be the length of the shortest word in X^* representing the same element of G as w, that is, the *geodesic length* of w. It is straightforward to extend $d_{G,X}$ to a metric on the 1-skeleton of Γ. Then each word w can be associated with a path from 1 labelled by w, and parameterised by $t \in [0, \infty)$, such that, for $t < l(w)$, the path from 1 to $w(t)$ has length t, and for $t \geq l(w)$, $w(t) = w(l(w))$.

Suppose that v, w are words in X^*, and that $K \in \mathbf{N}$. We say that v and w *asynchronously K-fellow-travel* if there is a differentiable function $h : \mathbf{R} \to \mathbf{R}$, mapping $[0, l(v) + 1]$ onto $[0, l(w) + 1]$ and strictly increasing on $[0, l(v) + 1]$, with the property that, for all $t > 0$, $d_{G,X}(v(t), w(h(t))) \leq K$. (The only point of adding 1 to $l(v)$ and $l(w)$ in this definition is to deal with the cases where one of the two words is trivial.) If h is the identity function, we say that v and w *synchronously K-fellow-travel.* If for some M, and all t, h satisfies $1/M \leq h'(t) \leq M$, we say that v and w *boundedly asynchronously fellow-travel* with bound M.

We define a *language* for G over X to be a set L of words in X which contains at least one representative for each element of G; L is said to be *bijective* if it contains exactly one representative of each group element. We call a language L for G an *asynchronous combing* (or just *combing*) if for some K the asynchronous K-fellow-traveller condition is satisfied by all pairs of words $v, w \in L$ for which $w =_G vx$ for some $x \in X \cup \{1\}$; if relevant pairs of words synchronously fellow-travel, then L is a *synchronous combing*, while if relevant pairs of words boundedly asynchronously fellow-travel, for some global bound M, then L is a *boundedly asynchronous combing.*

If L is a combing for G over X then, following [6], we define the *length function* $f : \mathbf{N} \to \mathbf{N}$ for L by the rule that $f(n)$ is the maximum length of a word in L of geodesic length at most n. The language L is a *geodesic combing* if $f(n) = n$ for all n.

A group is *automatic* if it has a synchronous combing which is a regular language (that is, recognised by a finite state automaton, see [14]), and *asynchronously automatic* if it has a regular asynchronous combing. (In fact it is proved in [9], Theorem 7.2.4 that any group with a regular, asynchronous combing must have a regular boundedly asnchronous combing.) A systematic analysis of combings of various types (that is, associated with a range of fellow-travel properties) and lying in various formal language classes is given in [20]; in this article we restrict attention to real-time combable groups, that is to groups with asynchronous combings which lie in the formal language class of real-time languages. Further, the combings we construct will all be seen to be boundedly asynchronous, and will often have polynomially bounded length functions.

In this paper we assume familiarity with the basics of formal language theory, such as the definitions of Turing machines and finite state automata. An introduction to the subject, directed towards geometric group theorists, can be found in [10]; [14] is an excellent standard reference. Below we give a brief description of real-time languages. A definition of indexed languages can be found in [1], and of the nested stack automata which define them in [2], while the indexed grammars which define them are described in [7].

Real-time languages, defined by Rabin in [19], are the languages accepted by deterministic real-time Turing machines. These have a fixed number of tapes, one of which is designated as the input tape, and contains just the input string. The other tapes can be taken to be infinite in both directions. With each transition, the machine must read one input symbol and move along the input tape to the next input symbol. For each of the other tapes, the machine may write a symbol or a blank, and then either stay still or move one place to the left or right. The computation halts when it reaches the end of input. Real-time languages do not have as many nice closure properties as the others considered in this paper (for example, they are not necessarily closed under concatenation with regular languages or under homomorphism). Furthermore, it can be difficult to determine whether a given language is real-time or not. However, they do represent a very natural model of linear-time computation, and they seem to be the most appropriate language for many of the combings that arise in this paper. All of $\{a^{n^2} : n \in \mathbf{Z}^+\}, \{a^{2^n} : n \in \mathbf{Z}^+\}$, $\{a^n b^{n^2} : n \in \mathbf{Z}^+\}$, $\{a^{n!} : n \in \mathbf{Z}^+\}$ and $\{(ab^n)^n :\in \mathbf{Z}^+\}$ are real time languages. The first three are also indexed ([14]), but the latter two are not ([11, 13]). It is proved in [21] that the language

$$\{a^{n_1}ba^{n_2}b\dots a^{n_{r-1}}ba^{n_r}c^s a^{n_{r-s+1}} \,|r, s, n_1, \dots, n_r \in \mathbf{Z}^+\}$$

is deterministic context-free but not real-time

In the formal language hierarchy, the real-time languages form a subfamily of the context-sensitive languages which contains all the regular languages, but (as we see from the example above) not all context-free languages. The indexed languages lie between the context-sensitive and the context-free languages.

The following, rather surprising, result shows that real-time combings are only really interesting when they satisfy additional restrictions.

Proposition 2.1 *If G has an asynchronous combing which is recursively enumerable, then G has a real-time asynchronous combing.*

PROOF: Let L be a recursively enumerable combing for G accepted by a Turing machine M. We construct a real-time combing L' for G by replacing each $w \in L$ by a word of the form we^m, where e is a symbol representing the identity and m is the sum of the length of w and the number of moves which M needs to accept w.

L' is accepted by a real-time Turing machine which, given an input word w', first copies each symbol up to the first occurrence of e onto its work tape, and then operates as M on the contents of that tape, while continuing to read from the input tape. It accepts w' provided that it reads only e's while operating as M, and that M halts in an accept state just as the end of the input is reached. $\square$

However, a real-time combing defined as above is in general not obviously constructible; its length function $f(n)$ (defined earlier in this section) might not be recursive. By contrast, the combings constructed in this paper are much better behaved. For instance, they will all be boundedly asynchronous; this, as we shall now show, implies that the length function is at worst exponential. In fact, where the groups involved are nilpotent-by-finite, we shall see later that the length function is polynomially bounded.

Proposition 2.2 *Let L be a boundedly asynchronous combing for G. Then the length function for L is at worst exponential.*

PROOF: The boundedness of the combing implies that there is an integer M with the property that, if $h : \mathbf{R} \to \mathbf{R}$ is the relative-speed function of two words v and w that asynchronously fellow-travel, then $(l(w)+1)/(l(v)+1) \leq M$. Let c be the length of a shortest word $w_0 \in L$ satisfying $w_0 =_G 1$; the above shows that any other representative of the identity has length less than $(c+1)M$. Now let $w \in L$ be a word having geodesic length $n > 0$. We shall show by induction on n that $l(w) < (c+1)M^n$, which will prove the result. If $n = 1$, then w fellow-travels with w_0, so $|l(w)| < (c+1)M$. If $n > 1$, then $w =_G ux$, where u has geodesic length $n-1$ and x is a generator of G. By induction, any word in the combing for u has length at most $(c+1)M^{n-1}$, but such a word fellow-travels with a combing word for w, which therefore has length at most $(c+1)M^n$. $\square$

3 A particularly symmetric combing for $\mathbf{Z}^n$

The combings in this paper are all constructed using a particularly well behaved combing of the free abelian group $\mathbf{Z}^n$, due to Martin Bridson; it generalises a combing from [5]. The combing is synchronous and geodesic, and behaves particularly well under automorphisms of the group, due to the fact that it is *almost-linear*, that is, there is a constant K depending only on n such that for any $x \in \mathbf{Z}^n$ the combing path in $\mathbf{Z}^n$ from the origin to x and the corresponding straight line in $\mathbf{R}^n$ are synchronous K-fellow-travellers. Here we are considering $\mathbf{Z}^n$ as embedded in the usual way in $\mathbf{R}^n$ and thinking of a path in $\mathbf{Z}^n$ as a sequence of elements of $\mathbf{Z}^n$ with successive elements distance one apart.

Let $e_1, \ldots, e_n$ be a basis for $\mathbf{Z}^n$. Take $\Sigma_n = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$, and define Σ_n^* to be the free monoid over Σ_n. The correspondence $a_i \to e_i, a_i^{-1} \to -e_i$ induces a monoid homomorphism from Σ_n^* to $\mathbf{Z}^n$. Denote by $\overline{w}$ the image of $w \in \Sigma_n^*$ under this homomorphism. For example $\overline{\epsilon} = 0$ where ϵ is the empty word and 0 is the unit element of $\mathbf{Z}^n$. We do not distinguish between $w \in \Sigma_n^*$ and the corresponding path in $\mathbf{Z}^n$ from the origin to $\overline{w}$.

For any point $p = \sum p_i e_i \in \mathbf{Z}^n$ parametrize the straight line from 0 to p by $x_i = p_i t e_i$, $0 \le t \le 1$. Start at $t = 0$, and each time x_i assumes a positive integer value write down a_i, each time x_i assumes a negative integer value write down a_i^{-1}. If two or more x_i's assume integer values at the same time, write down the corresponding a_i's in order of decreasing i. Figure 1 shows the combing path for $(4, 3)$.

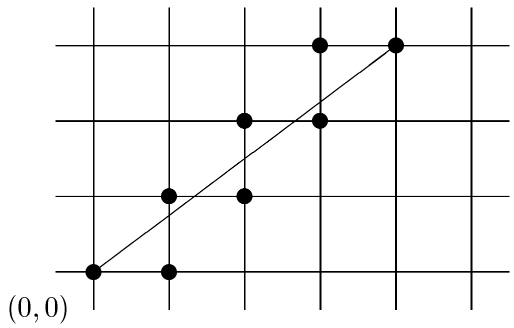

Figure 1: The combing path for $(4, 3)$ is $a_1 a_2 a_1 a_2 a_1 a_2 a_1$.

Let $L_n \subset \Sigma_n^*$ be the language defined by the recipe above. Clearly L_n is a combing of $\mathbf{Z}^n$, and it is straightforward to show that L_n has the desired fellow-traveller property.

It is shown in [7] that L_2 is an indexed language. We devote the remainder of Section 3 to the proof of the following result:

Theorem 3.1 *The language L_n is a real-time language for all $n \geq 2$.*

PROOF: The proof divides into two parts, a reduction to the consideration of a subset L_2^+ of Σ_2^*, followed by a demonstration that L_2^+ is real-time.

3.1 Reduction to a 2-dimensional problem

First we shall reduce to the positive orthant. For any $S \subset \{1, \ldots, n\}$ define $f : \Sigma_n^* \to \Sigma_n^*$ to be the monoid homomorphism which interchanges a_i and a_i^{-1} for $i \in S$ and fixes all other a_i's. A moment's consideration convinces us that f maps L_n to itself. Let L_n^+ be the sublanguage of all paths in L_n which lie in the positive orthant, i.e., the set of words $w \in L_n$ with $\overline{w} = \sum p_j e_j$, $p_j \geq 0$. Then L_n is the union of the images of L_n^+ under the monoid isomorphisms f corresponding to all $S \subset \{1, \ldots, n\}$. As real-time languages are closed under union, it follows that L_n is real-time if L_n^+ is.

Next for each $1 \leq i < j \leq n$ let $f_{i,j} : \Sigma_n^* \to \Sigma_2^*$ be the monoid homomorphism which sends a_i to a_1, a_j to a_2, and all other generators to ϵ. We claim that L_n^+ is the intersection of the inverse images of L_2^+ under all the $f_{i,j}$'s. Since real-time languages are closed under inverse homomorphism and intersection, it will follow that L_n^+ is real-time if L_2^+ is.

It follows from the definition of L_n that $f_{i,j}$ maps L_n^+ to L_2^+. Hence $L_n^+ \subset \cap f_{i,j}^{-1}(L_2^+)$. For the converse suppose w is in the intersection, and $\overline{w} = \sum p_k e_k$. For each i, j, $f_{i,j}(w)$ is the combing path for $p_i e_1 + p_j e_2$. Consequently $f_{i,j}(w) = f_{i,j}(v)$ where v is the combing word in L_n for $\sum p_k e_k$. But it is straightforward to check that if $f_{i,j}(w) = f_{i,j}(v)$ for all i, j, then $w = v$. We conclude that $\cap f_{i,j}^{-1}(L_2^+) \subset L_n^+$.

3.2 The 2-dimensional problem

It now suffices to show that L_2^+, is real-time. From now on, for ease of notation, we shall rewite a_1 as a and a_2 as b, and so consider L_2^+ as a subset of $\{a, b\}^*$. Since b^* is a regular language, it is enough to show that $L_2^\# = L_2^+ \setminus b^*$ is real-time. We work with $L_2^\#$ because it has the following recursive definition.

$$\begin{aligned} w &= w_k^n \text{ for some } k \geq 0 \text{ and } n \geq 1 && (1) \\ w_j &= w_{j-1}^{i_j} w_{j-2} \text{ for } 2 \leq j \leq k \text{ and } i_j \geq 1 && (2) \end{aligned}$$

and

$$\begin{aligned} &\text{If } k \text{ is odd, then } w_1 = b^{i_1} a \text{ and } w_0 = b && (3) \\ &\text{If } k \text{ is even, then } w_1 = a^{i_1} b \text{ and } w_0 = a. && (4) \end{aligned}$$

That is, $L_2^\#$ is the collection of all words $w \in \Sigma^*$ satisfying some instance of (1-4). For example the sequence $w_0 = a$, $w_1 = ab$, $w_2 = (ab)^3 a$ yields $abababa$, the combing path for $(4, 3)$.

We shall not verify in this paper that $L_2^\#$ is so defined. This fact follows almost immediately from parts of the the proof of Theorem (3.10) of [7], which verifies that the same language (known in that paper as L_1) is an indexed language.

We construct a real-time Turing machine $\mathcal{T}$ which accepts the language $L_2^{\#}$, working with the recursive definition of $L_2^{\#}$ above. In our proof, we shall consider only inputs beginning with a, but our arguments will always extend to the other case.

The valid inputs of $\mathcal{T}$ in a^*b^* are a and $a^{i_1}b$, and by (2) any valid input not in a^*b^* (and beginning with a) must have the form $a^{i_1}ba\ldots$. It is straightforward to cope with inputs in a^*b^* and to arrange things so that after reading $a^{i_1}ba$, $\mathcal{T}$ has one work tape containing $w_0 = a$, a second containing $w_1 = a^{i_1}b$, and so that the tape heads for these tapes are positioned at the ends of their tape contents. We shall call configurations like this distinguished.

Definition 3.2 *$\mathcal{T}$ is in a distinguished configuration if its input is a solution to (1-4), and for some $j \geq 1$*

1. *$\mathcal{T}$ has read $w_j w_{j-1}$ from its input;*
2. *$\mathcal{T}$ has one work tape containing w_{j-1}, and a second containing w_j;*
3. *The tape heads for these tapes are each positioned either at the end of their tape contents or at the beginning.*
4. *On the work tape containing w_j, the squares a distance $|w_{j-1}|$ from each end of the tape contents are marked.*

Lemma 3.5 below tells us that if $\mathcal{T}$ is in a distinguished configuration, then either there is a longer input prefix $w_{j+1}w_j$, or the total input is w_{j+1} or w_j^n (for some $n > 1$). $\mathcal{T}$ will verify that the input has one of these forms, and in the first case it will simultaneously update the work tape containing w_{j-1} so that after coming to the end of the input prefix $w_{j+1}w_j$ the work tape contains w_{j+1} and $\mathcal{T}$ is in another distinguished configuration. In the second case the input will be accepted before $\mathcal{T}$ has time to update the work tape. Thus $\mathcal{T}$ will accept all words in $L_2^{\#}$. Of course for input not in $L_2^{\#}$, $\mathcal{T}$ will reach a point where it is not able to do a required verification, and it will reject that input.

During its computation $\mathcal{T}$ will need to compare w_j to segments of input. It will do this by traversing the work tape containing w_j in either direction. For this method to be feasible we need to know that w_j is essentially a palindrome. We shall use the following notation. For any word v with length $|v| \geq 2$, $\Phi(v)$ is v with its last two letters reversed. Notice that if $u = \Phi(v)$, then $v = \Phi(u)$; and $u\Phi(v) = \Phi(uv)$. Also note that $|w_j| \geq 2$ if $j \geq 1$. Lemma 3.3, which refers to any solution to equations (1-4), is easy to prove by induction, and we omit the proof.

Lemma 3.3 *For all $j \geq 1$, $w_j w_{j-1} = w_{j-1}\Phi(w_j)$ and $w_{j-1}w_j = \Phi(w_j w_{j-1})$.*

To complete our proof of Theorem 3.1, we need Lemmas 3.4 and 3.5 below, which we also state without proof. (They follow fairly easily from Lemma 3.3.) They also refer to any solution to equations (1-4).

Lemma 3.4 *For all $j \geq 1$, w_j consists of a palindrome followed by ab or ba.*

Lemma 3.5 *Suppose $j > 0$, and $w_j w_{j-1}$ is a prefix of w. One of the following holds.*

1. *$j < k$ and w has a prefix $w_{j+1} w_j = w_j w_{j-1} \Phi(w_j)^{i_{j+1}-1} w_j$;*
2. *$j = k-1, n = 1$, and $w = w_{j+1} = w_j w_{j-1} \Phi(w_j)^{i_{j+1}-1}$;*
3. *$j = k, n > 1$, and $w = w_j^n$ consists of $w_j w_{j-1} \Phi(w_j)^{n-1}$ with the suffix w_{j-1} deleted.*

Now suppose that $\mathcal{T}$ is in a distinguished configuration as in Definition 3.2. $\mathcal{T}$ need only check that its input continues in one of the three ways indicated in Lemma 3.5, namely a power of $\Phi(w_j)$ followed by w_j, or a power of $\Phi(w_j)$, or a power of $\Phi(w_j)$ with the suffix w_{j-1} deleted. (Notice that the value of i_{j+1} is determined by the occurrence of a substring w_j of the input.) In the first case $\mathcal{T}$ must update the worktape containing w_{j-1} so that it contains w_{j+1} with the prefix and suffix of length $|w_j|$ marked, and then position each of its tape heads at one end of the contents of its tape; afterwards $\mathcal{T}$ is again in a distinguished configuration. In each of the other two cases $\mathcal{T}$ comes to the end of its input and accepts the input. Actually $\mathcal{T}$ must also verify the parity condition of (3-4), but this task is accomplished by checking that the input ends in a.

Using Lemma 3.4 it is straightforward to design $\mathcal{T}$ so that by traversing the worktape containing w_j in either direction it can check that the next segment of length $|w_j|$ of the input is either $\Phi(w_j)$ or w_j. If the input ends with $\Phi(w_j)$, then $\mathcal{T}$ accepts. If w_j is encountered, then once we show how the appropriate worktape is updated by the time w_j is read from the input, $\mathcal{T}$ will be in another distinguished configuration, namely the one corresponding to Definition 3.2 with j replaced by $j+1$. Finally using the markings on the work tape containing w_j, $\mathcal{T}$ can also tell when its input ends after a power of $\Phi(w_j)$ with the suffix w_{j-1} deleted.

It remains to see how $\mathcal{T}$ updates the worktape containing w_{j-1} while it is traversing the worktape containing w_j i_{j+1} times. Call the first worktape tape 1 and the second tape 2. Suppose that $\mathcal{T}$ is at the end of w_{j-1} on tape 1. Since $w_{j+1} = w_j^{i_{j+1}} w_{j-1} = w_{j-1} \Phi(w_j)^{i_{j+1}}$, $\mathcal{T}$ need only move to the right writing $\Phi(w_j)$ on the tape 1 each time it traverses the contents of tape 2. The square on tape 2 which marks the beginning of the suffix w_{j-1} of w_j will mark the end of the prefix of length $|w_j|$ of w_{j+1} the first time $\Phi(w_j)$ is copied onto tape 1. In order to mark the appropriate suffix of w_{j+1}, $\mathcal{T}$ makes a temporary mark at the beginning of each $\Phi(w_j)$ which it writes. The first time it scans w_{j+1} in processing the next distinguished configuration all except the first of these marks are deleted, and the first mark is made permanent.

If $\mathcal{T}$ starts at the left of w_{j-1} on tape 1, then the procedure is similar. As $w_{j+1} = w_j^{i_{j+1}} w_{j-1}$, $\mathcal{T}$ need only move to the left writing w_j from right to left each time it traverses tape 2. As w_j is essentially a palindrome, this can be done. The markings are done the same way as before with obvious modifications.

This completes the proof of Theorem 3.1. □

4 Properties of the families of real-time combable groups

We shall construct real-time combings for nilpotent groups by showing that the groups can be constructed out of free abelian pieces with combings of type L_n, as already described. In this section we prove the necessary closure properties for the families of real-time combable groups which ensure that this strategy is valid. In fact, in the following lemma we prove more than we need (asynchronous results would be enough).

Proposition 4.1 *Let G be a finitely generated group.*

(a) *If G is synchronously, asynchronously or boundedly asynchronously real-time combable, then it is so with respect to any generating set.*

(b) *Let H be a subgroup of finite index in G. Then G is synchronously, asynchronously or boundedly asynchronously real-time combable if and only if H is.*

In each of the above cases, wherever the original combing has a polynomially bounded length function, the new combing has a length function which is bounded by a polynomial of the same degree.

Similar results for asynchronous and synchronous combings in other languages families are proved in [7] and [20]. Although real-time languages do not possess the necessary properties for all of those results to apply directly, many of the ideas can be used.

Proof: To prove (a), let L be a real-time combing for G, over a finite (inverse closed) generating set $X = \{x_1, \ldots x_m\}$, and suppose that $Y = \{y_1, \ldots y_n\}$ is a second (inverse closed) generating set for G. The natural way to define a combing L' over Y is to find a set of words $w_1, \ldots w_m$ over Y, with w_i equal in G to x_i, and then define L' to be the set of words over Y formed by substituting w_i

for each occurrence of x_i in each word of L. The language L' naturally inherits asynchronous fellow-traveller properties from L.

We can ensure that L' is recognisable by a real-time Turing machine by requiring that each of the substituting words w_i is distinct and terminated by a string $y_1y_1^{-1}$, which appears nowhere else, If all the w_i have the same length k, then L' satisfies synchronous fellow traveller properties; although in general we cannot organise this, we can at least arrange that all have length either k or $k-1$, and pad out the shorter w_i's by strings $y_1^{-1}y_1$ on alternate substitutions.

To prove (b), suppose that H is a subgroup of finite index in G, and let T be a finite transversal for H in G, containing the identity element.

If H has a real-time synchronous or (boundedly) asynchronous combing, then the language for G formed by concatenating that combing with the elements of T is certainly a real-time combing of the same type for G.

Now suppose that L is a real-time synchronous or (possibly boundedly) asynchronous combing for G over an (inverse closed) generating set X. Let T be a finite transversal for H in G. An appropriate language for H over the Schreier generators for H with respect to X and T can be constructed using the Reidemeister-Schreier rewriting process; the construction is described in [20]. That the language is real-time is not hard to verify (the arguments of [20] do not in fact apply to real-time languages, but can be modified).

Finally we need to verify that the various combings constructed have polynomially bounded length functions, given that the same is true of the original combings.

We observe first that the substitutions corresponding to change of generators have the effect of changing the length of a word by at most a constant factor (the maximum length of the old generators written as words in the new ones). Similarly, the geodesic length of the substituted word is smaller than the geodesic length of the original word by at most another constant factor (the maximum length of the new generators written as words in the old). Hence changing the generating set changes the length function by at most a constant factor, which can be expressed in terms of the above constants and the polynomial degree of the original length function.

For each of the remaining cases, the proofs that the length function continues to be polynomial degree are very similar. Hence we shall give details of the proof only in one case. Suppose that H has finite index in G, and associated transversal T. Given a combing L_H, with polynomial length function f, we want to show that the combing L_HT for G has polynomial length function.

We suppose that L_H is defined over a generating set X for H, and that Y is the set of Schreier generators for H associated with $X \cup T$ and T. A geodesic representative w' over $X \cup T$ of a word w in L_H can be rewritten as a word of the same length over Y (using the Reidemeister-Schreier rewriting process). Its geodesic length over X is at most a constant factor longer than this (since both X and Y are generating sets for H). So, for words representing elements of H, geodesic lengths over $X \cup T$ are at most a constant factor less than those over X, and hence, for some c, $cf(n)$ bounds the length of any word in L_G which represents an element of H and has geodesic length n over $X \cup T$. It remains, for each t, to consider elements of L_G which represent elements of Ht. For such a word vt, of geodesic length n over $X \cup T$, consideration of v, which must have geodesic length at most $n+1$, shows that vt has length at most $cf(n+1)+1$. The result follows. □

Proposition 4.2 *A direct product of real-time combable groups is real-time combable. If the factors have boundedly asynchronous combings, then so does the direct product. If the combing of each of the factors has polynomial length function, then so does the combing constructed for the direct product.*

PROOF: The concatenation of combings for the direct factors is clearly an asynchronous combing. Since disjoint generating sets can be chosen for the two factors, the concatenation of the two languages is easily seen to be in the same family of languages as the original languages (The concatenation of two real-time languages over non-disjoint alphabets need not be a real-time language).

The final statement follows from the fact that geodesic words in the direct product can always be found which are concatenations of geodesic words in each of its factors. □

Proposition 4.3 *Let $G = \mathbf{Z}^n \rtimes H$ be a split extension of an n-generated free abelian group and a combable group H. Let L_H be the given combing for H, and let L_n be the combing for $\mathbf{Z}^n$ described in Section 3.*

Then $L_G = L_H L_n$ is a combing for G. If L_H is boundedly asynchronous then so is L_G. If L_H is a real-time language, then so is L_G.

PROOF: The proof that L_G is a combing is given in [5] (Theorem B). Since the concatenation of real-time languages over disjoint alphabet sets is easily seen to be a real-time language, the fact that L_G is real-time follows immediately from Theorem 3.1. That the asynchronicity of the combing is bounded is not proved in [5], but is clear from examination of the proof. □

In order to get a polynomial bound on the length functon for a combing of a nilpotent group of the form $\mathbf{Z}^n \rtimes H$, we need to look more closely at the action associated with the extension. We say that a group H acting on a group N *acts nilpotently* if a series $N = N_0 \supseteq N_1 \supseteq \ldots N_k = 1$ of subgroups of N can be found with $[N_i, H] \subseteq N_{i+1}$ for $0 \leq i < k$. We call such a series an *H-central series* for N. We define the *relative class* of the action (or of the associated split extension) to be the minimum such k. If N is $\mathbf{Z}^n$ and H acts nilpotently on N, with relative class k, then it can be shown that an H-central series for N of length k can be found in which the factors N_i/N_{i+1} are all torsion-free.

This will be proved in Section 5, Lemma 5.4. Note that if $N \rtimes H$ is nilpotent of class c, then H acts nilpotently on N with relative class at most c.

Proposition 4.4 *Let $G = \mathbf{Z}^n \rtimes H$ be as in the previous lemma, and assume in addition that H acts nilpotently on $\mathbf{Z}^n$, with relative class k, and that the combing L_H of H has polynomial length function of degree m.*

Then a generating set for $\mathbf{Z}^n$ can be found such that the associated combing $L_G = L_H L_n$ of G has length function bounded by a polynomial of degree at most the maximum of m and k.

PROOF: Lemma 5.4 ensures the existence of an H-central series $\mathbf{Z}^n = N_0 \supseteq N_1 \supseteq \ldots N_k = 1$ of $\mathbf{Z}^n$ with torsion-free factors. Select the generating set $X = \{x_1, \ldots, x_n\}$ for $\mathbf{Z}^n$ so that $x_{i_j+1}, \ldots, x_n$ generate N_j minimally, for each $j < k$, $i_0 = 0$ and $i_k = n$. Let c be the maximum of the lengths in X of any of the elements x_j^y, where y is an element of the generating set Y of H over which L_H is defined. For each $y \in Y$, if $i_j < i \leq i_{j+1}$, then $x_i^h = x_i w$, for some word w of length at most $c-1$ in $x_{i_{j+1}+1}, \ldots x_n$. We now establish a bound on the length of x_i^h in terms of $l(h)$, c and k. Let $h = y_{t_1} y_{t_2} \ldots y_{t_l}$ with $y_{t_j} \in Y$. Define elements $g_0, g_1, \ldots g_l \in \mathbf{Z}^n$ by $g_0 = x_i$, $g_j = g_{j-1}^{y_{t_j}}$ for $1 \leq j \leq l$, and hence $g_l = x_i^h$. It is convenient to assign *dates* to the occurrences of the generators x_r in the words $g_0, \ldots g_l$. The single occurrence of x_i has date 0, and in general, if an occurrence of x_r in g_{j-1} has date m and $x_r^{y_{t_j}} = x_r w$, then the corresponding occurrences of x_r and the generators of w in g_j have dates m and $m+1$ respectively. Let $t(j,m)$ be the total number of occurrences of generators of date m in the word g_j. Then the generators of date m in g_{j+1} include those same $t(j,m)$ generators together with at most $(c-1)t(j,m-1)$ new ones arising from the conjugations by $y_{t_{j+1}}$. Hence $t(j+1,m) \leq t(j,m) + (c-1)t(j,m-1)$. From this inequality and the conditions $t(j,0) = 1$ for all j and $t(0,m) = 0$ for all $m > 1$, it follows by induction that $t(j,m) \leq \binom{j}{m}(c-1)^m$ for all j, m. However, since $t(j,m) = 0$ for all $m \geq k$, we have $l(x_i^h) \leq a \times l(h)^{k-1}$ for some constant a which depends on c and k.

Now let v be a geodesic word over $X \cup Y$, containing r generators from X and s from Y. Let u be the concatenation of the s generators in Y in the order in which they appear in v, and let $u' \in L_H$ with $u' =_H u$. Then we have $v =_G u'w'$, where w' is a product of r elements of X each conjugated by some suffix of u. From the preceding paragraph, there exists $v' \in L_G$ with $v' =_G v$ and with v' equal to u' times a word of length at most $r \times a \times s^{k-1}$. Since, by assumption, $l(u')$ is bounded by a polynomial of degree at most m in $l(u) = s$ and $ras^{k-1} \leq a(r+s)^k$, the result follows. □

The following corollary follows immediately from repeated application of this lemma.

Corollary 4.5 *Suppose that the nilpotent group G is isomorphic to a tower of split extensions of the form $\mathbf{Z}^{n_1} \rtimes (\mathbf{Z}^{n_2} \rtimes (\mathbf{Z}^{n_3} \rtimes \ldots \mathbf{Z}^{n_r}) \ldots)$. Then G has a real-time combing L_G, whose length function is bounded by a polynomial with degree the maximum, k, of the relative classes of the extensions.*

Note that k is no larger than the nilpotency class of G.

Corollary 4.6 *For any n, the unipotent groups $U_n(\mathbf{Z})$ and the Heisenberg groups H_{2n+1}, as defined in Section 1, as well as the free nilpotent groups of class 2, $Fr_n/\gamma_2(Fr_n)$, are real-time combable. The combings are boundedly asynchronous, with length functions bounded by polynomials of degrees $n-1$, 2 and 2, respectively.*

Proof: For the Heisenberg and free nilpotent groups we apply Propositions 4.3 and 4.4; for the unipotent groups we apply Corollary 4.5. □

Details of the above constructions, and further non-nilpotent examples are given in [20]. These examples are the building blocks of many others. The combability of class 2 nilpotent groups with cyclic commutator subgroup, proved in Section 6, is basically a consequence of the combability of the Heisenberg groups; similarly the embedding theorem for polycyclic-by-finite groups proved in Section 7 follows essentially from the combability of the groups $U_n(\mathbf{Z})$.

5 Useful properties of nilpotent groups

In this section, we list some facts about nilpotent groups, which we shall use in the following two sections. They are of a standard nature, and only outlines of proofs will be included. Let $G = \gamma_1(G) \supset \gamma_2(G) \supset \ldots \supset \gamma_{c+1}(G) = 1$ be the lower central series of G, where G is nilpotent of class c. The first result is well-known; see, for example, Lemma 2.6, Corollary 1 of [12], and the second is of a similar nature.

Lemma 5.1 *If H is a subgroup of the nilpotent group G with $HG' = G$, then $H = G$.*

Lemma 5.2 *Let H be a subgroup of the finitely generated nilpotent group G for which $|G : HG'|$ is finite. Then $|G : H|$ is finite.*

Proof: Using induction on the class c, we may assume that $|G : H\gamma_c(G)|$ is finite, and so we must show that $|H\gamma_c(G) : H| = |\gamma_c(G) : H \cap \gamma_c(G)| < \infty$.

An element of $\gamma_c(G)$ is a product of commutators $[g, k]$ with $g \in G$ and $k \in \gamma_{c-1}(G)$. Let $|G : HG'| = m$ and $|\gamma_{c-1}(G) : (H \cap \gamma_{c-1}(G))\gamma_c(G)| = l$. Then

$g^m \in HG'$ and $k^l \in (H \cap \gamma_{c-1}(G))\gamma_c(G)$, and so $[g,k]^{ml} = [g^m, k^l] \in H \cap \gamma_c(G)$. Thus $\gamma_c(G)/(H \cap \gamma_c(G))$ is finitely generated of finite exponent, and the result follows. □

The next result allows us to reduce consideration of an arbitrary nilpotent group G to one where both G and G/G' are torsion free.

Lemma 5.3 *Let $G = \langle g_1, \dots g_n \rangle$ be a finitely generated nilpotent group. Then G has an n-generator subgroup G_0 of finite index such that both G_0 and G_0/G_0' are torsion-free.*

PROOF: By Theorem 7.8 of [12], there is an integer r such that G^r is torsion-free. Let $H = \langle g_1^r, \dots, g_n^r \rangle$. Then $|G : HG'|$ is finite, so by Lemma 5.2, $|G : H|$ is finite. Since $H \subseteq G^r$, H is torsion-free.

Now choose $h_1, \dots, h_m \in H$ such that $h_1 H', \dots h_m H'$ freely generate a maximal free abelian subgroup of H/H', and let $G_0 = \langle h_1, \dots, h_m \rangle$. Clearly $m \leq n$. By Lemma 5.2, $|H : G_0|$ and hence $|G : G_0|$ is finite. Finally, since $G_0' \subseteq H'$, G_0/G_0' is itself free abelian of rank m and hence torsion-free, so the result follows. □

Lemma 5.4 *If H acts nilpotently on $N = \mathbf{Z}^n$ with relative class k, then N has an H-central series of length k with torsion-free factors.*

PROOF: Let the given H-central series be $M_0 = 1 \subseteq M_1 \subseteq M_2 \subseteq \dots M_k = N$, and define the series $Z_0 = 1 \subseteq Z_1 \subseteq Z_2 \subseteq \dots$ by letting Z_{i+1} be the inverse image in N of $C_{N/Z_i}(H)$ for all $i \geq 0$. Induction shows that $M_i \subseteq Z_i$ for all i. Hence $Z_k = N$.

We show that Z_{i+1}/Z_i is torsion-free by induction on i. This is clear for $i = 1$. For $i > 1$, let $g \in Z_{i+1} \setminus Z_i$ and suppose that $g^m \in Z_i$. Then there exists $h \in H$ with $[h,g] \notin Z_{i-1}$; but $[h,g]^m Z_{i-1} = [h, g^m] Z_{i-1} = Z_{i-1}$, contradicting the induction hypothesis. □

6 Class 2 nilpotent groups

In the following we shall denote by $F_{k,c}$ (or sometimes, where there is no ambiguity, simply by F) the *free nilpotent group* on k generators of class c, that is the quotient of the free group Fr_k on k generators by its subgroup $\gamma_{c+1}(Fr_k)$.

Suppose that G is nilpotent of class 2. In this section we prove that if G has cyclic commutator subgroup, or can be generated by at most 3 of its elements, then it is real-time combable. The method is basically to decompose finite index subgroups of G as split extensions, and then to apply the results of Section 4. However, we have an example of a 4 generator class 2 nilpotent group which has no finite index subgroup which is a split extension.

Lemma 6.1 *Suppose that G is finitely generated nilpotent of class 2, and that G/G' is torsion-free. Then, for some k, G can be expressed as a quotient $F_{k,2}/K$ with $K \subseteq F'_{k,2}$.*

PROOF: By Lemma 5.1, if G/G' has rank k, then G can be generated by k elements. The result follows. □

Theorem 6.2 *Any class 2 finitely generated nilpotent group G with G' cyclic has a real-time combing. The combing is boundedly asynchronous and has a length function which is at most quadratic.*

PROOF: By Lemma 5.3 and Proposition 4.1 (b), we may assume that both G and G/G' are torsion free. Let $G' = \langle c \rangle$. We claim that some finite index subgroup of G decomposes as a direct product of a central abelian subgroup Z and a subgroup $H = \langle a_1, \ldots a_n, b_1, \ldots b_n \rangle$ such that for any i, $[a_i, b_i]$ is a power of c, but for $i \neq j$, $[a_i, a_j] = [a_i, b_j] = 1$. Then H decomposes as a semidirect product of the subgroups $\langle a_1, \ldots a_n, c \rangle$ and $\langle b_1, \ldots b_n \rangle$, and the results of Section 4 imply that G is real-time combable. We establish the claim by induction on the size of a minimal generating set X for G. By Lemma 5.1, $|X|$ is equal to the rank of G/G'.

First we select any two non-commuting elements $a_1, b_1 \in X$ and suppose that $[a_1, b_1] = c^i$ For each other $x \in X$, where $[a_1, x] = c^j$ and $[b_1, x] = c^k$, the element $y = a_1^k b_1^{-j} x^i$ commutes with a_1 and b_1. Let G_1 be the group generated by all such elements y. Then $|G : \langle a_1, b_1, G_1 \rangle|$ is finite, and $G_1 \cap \langle a_1, b_1 \rangle \subseteq \langle c^i \rangle$. If G_1 is abelian, then $G_1 \cap \langle a_1, b_1 \rangle = 1$, and we have the required decomposition with $Z = G_1$. Otherwise we apply induction to G_1.

Bounded asynchronicity and the quadratic bound on the length function follow from Propositions 4.1 and 4.3, and Corollary 4.4. □

Theorem 6.3 *Any two or three generator class 2 nilpotent group G has a real-time combing. The combing is boundedly asynchronous and has a length function which is at most quadratic.*

PROOF: The two generator groups are covered by Theorem 6.2, so we assume that G is three generated. Lemma 5.3 and the results of Section 4 allow us to assume that G and G/G' are torsion-free.

By Lemma 6.1, G is a quotient of $F_{3,2}$ by a normal subgroup K in the commutator subgroup. Since we may assume that $G' \cong F'_{3,2}/K$ is non-cyclic and torsion-free, and since $F'_{3,2}$ has rank 3, we need only consider the case where K is 1-generated, that is, where G is defined by one relator which is a product of commutators. In this case, the single relator can be put into the form $[a, b]^k [a^i b^j, c]^{-1}$. We consider the cases $i = 0$ and $i \neq 0$ separately.

First let $i \neq 0$, and let $N = \langle a^i b^j, [a,b] \rangle$, $H = \langle b, c \rangle$. Then $|G : NH|$ is finite, and N is abelian and normal in G. Working mod G', we see that, since $a \notin H$, $N \cap H \subseteq \langle [a,b] \rangle$. Hence if $N \cap H \neq 1$ we have a relation between $[b,c]$ and $[a,b]$; but such a relation cannot be a consequence of the relator $[a,b]^k [a^i b^j, c]^{-1}$. So NH is a split extension, and hence, by the results of Section 4, G has a real-time combing.

When $i = 0$, the one relation can be written as $[b, a^k c^j] = 1$. Then the group $\langle a^k, b, c^j \rangle$, which has finite index in G, can be written as a semidirect product of $N = \langle a^k c^j, [a,c] \rangle$ and $H = \langle b, a^k \rangle$. Hence again G has a real-time combing.

Bounded asynchronicity and the quadratic bound on the length function follow again from Propositions 4.1, 4.3, and 4.4. □

Once we move to four generators, the situation is much less clear. Let F be the free nilpotent group of class two and rank four, with generating set $\{a,b,c,d\}$. Then F/F' is free abelian of rank 4, and $F' = Z(F)$ is free abelian of rank 6, and is generated by the six commutators $[a,b]$, $[a,c]$, $[a,d]$, $[b,c]$, $[b,d]$, $[c,d]$. Let K be the subgroup $\langle [a,b][c,d] \rangle$ of F, and let $G = F/K$. Then $G' = F'/K$ is free abelian of rank 5, and $G/G' \cong F/F'$.

Proposition 6.4 *No subgroup G_0 of finite index in G can be decomposed as a semidirect product $N \rtimes H$, where N is a nontrivial abelian normal subgroup of G_0.*

For the proof, assume that there is such a subgroup with a decomposition of this form. Note that, since G is torsion-free, N must be a nontrivial free abelian group. The idea is to show that the free abelian groups $[N, G_0] \subseteq N$ and $H' \subseteq H$ both have rank 3. Since both lie within G', which has rank 5, they must then intersect non-trivially. Hence N and H intersect non-trivially, which is a contradiction. This argument follows from a series of lemmas.

Lemma 6.5 *Let E be any subgroup of F with 3 (or fewer) generators. Then no nontrivial element of K lies in E'.*

Proof: Let $E = \langle e, f, g \rangle$ and suppose that $1 \neq ([a,b][c,d])^t \in E'$. Choose an odd prime p that does not divide t, and let $P = F/F^p$. Then P is a special p-group of order p^{10}, with $|P'| = p^6$, and P' is generated by the six commutators of the pairs of the generators. Let $Q = EF^p/F^p$.

To simplify notation, we shall use a,b,c,d,e,f,g to denote the images of these elements in P. Note that $1 \neq ([a,b][c,d])^t \in Q'$. We can regard e,f,g as elements of the vector space P/P', and assume that they are in reduced echelon form with respect to the basis a,b,c,d, and we may as well assume that none of e,f,g equals zero in P/P'. This leaves the following four essentially different possibilities for e,f,g:

(*i*) b, c, d; (*ii*) ab^i, c, d $(0 \leq i < p)$;
(*iii*) ac^i, bc^j, d $(0 \leq i, j < p)$; (*iv*) ad^i, bd^j, cd^k $(0 \leq i, j, k < p)$.

In all cases, $Q' = \langle [e,f], [e,g], [f,g] \rangle$. Cases (i) and (ii) are impossible, since none of $[e,f], [e,g], [f,g]$ involves $[a,b]$. In Case (iv), we have $[e,f] = [a,b][a,d]^j[b,d]^{-i}, [e,g] = [a,c][a,d]^k[c,d]^{-i}$ and $[f,g] = [b,c][b,d]^k[c,d]^{-j}$. Of these, only $[e,g]$ involves $[a,c]$ and only $[f,g]$ involves $[b,c]$, so $([a,b][c,d])^t$ would have to be a power of $[e,f]$, which it clearly is not. A similar argument rules out Case (iii). □

Lemma 6.6 *If* $[g,h] = 1$ *with* $g, h \in F$, *then* $\langle g, h \rangle F'/F'$ *is cyclic.*

Proof: Since F' is central in F, we may assume that $g = a^i b^j c^k d^l$ and $h = a^{i'} b^{j'} c^{k'} d^{l'}$ for some $i, j, k, l, i', j', k', l' \in \mathbf{Z}$ and then

$$[g,h] = [a,b]^{ij'-ji'}[a,c]^{ik'-ki'}[a,d]^{il'-li'}[b,c]^{jk'-kj'}[b,d]^{jl'-lj'}[c,d]^{kl'-lk'},$$

and so we have $ij' = ji'$, etc. It can be checked that, for any solution of these six equations, g and h are powers of a common element, and the result follows. □

Lemma 6.7 NG'/G' *is an infinite cyclic group.*

Proof: NG'/G' is free abelian, because G/G' is. We cannot have $N \subseteq G'$, because this would imply $HG' = G_0$, and then $H' = G_0'$ would have finite index in G', so $N \cap H$ could not be trivial. So we may assume that N contains two elements $\bar{g}$ and $\bar{h}$ such that $\langle \bar{g}, \bar{h} \rangle G'/G'$ is not cyclic. But then, if g and h are inverse images of $\bar{g}$ and $\bar{h}$ in F, $\langle g, h \rangle F'/F'$ is also not cyclic, so $[g,h] \neq 1$ by Lemma 6.6. But N is abelian, so $[\bar{g}, \bar{h}] = 1$, which means that $[g,h] \in K$, contradicting Lemma 6.5 (applied to $\langle g, h \rangle$). □

Lemma 6.8 HG'/G' *is free abelian of rank 3.*

Proof: Again HG'/G' is free abelian, and Lemma 6.7 implies that its rank is at least 3. But if it had rank 4, $|G : HG'|$ would be finite. so (by Lemma 5.2) $|G : H|$ and $|G' : H'|$ would be finite, and $N \cap H$ could not be trivial. □

Lemma 6.9 $[N, G_0]$ *is free abelian of rank 3.*

Proof: Let $\hat{N}$ and F_0 be the complete inverse images of N and G_0 in F. Then $\hat{N}F'/F'$ is infinite cyclic by Lemma 6.7; let gF'/F' be a generator. Considering the homomorphism $\phi : F/F' \to [F, \hat{N}]$ defined by $h \mapsto [g,h]$, we see that $F/C_F(\hat{N})F' \cong [F, \hat{N}]$, and so these groups are free abelian of the same rank. By Lemma 6.6, the group $C_F(\hat{N})F'/F'$ has rank exactly one. Thus $F/C_F(\hat{N})F'$ and $[F, \hat{N}]$ have rank $4-1 = 3$; since $|F : F_0|$ is finite, the same is true of $[F_0, \hat{N}]$.

Now the cyclicity of $\hat{N}F'/F'$ implies that all elements of $[\hat{N}, F_0]$ are commutators of the form $[g^i, h]$. Then Lemma 6.5 applied to $E = \langle g, h \rangle$ implies that $[\hat{N}, F_0] \cap K = 1$, and the result follows. □

Lemma 6.10 *H' is free abelian of rank 3.*

PROOF: Let $\hat{H}$ be the complete inverse image of H in F. Then, by Lemma 6.8, $\hat{H}F'/F'$ is free abelian of rank 3; let $\hat{H}F'/F' = \langle eF', fF', gF' \rangle$. Then $\hat{H}' = \langle [e,f], [e,g], [f,g] \rangle$, so its rank is at most 3. If it were less than 3, then we would have $[e,f]^x[e,g]^y[f,g]^z = 1$ for integers x, y, z not all zero. But it can be shown that this product of commutators is equal to a single commutator $[h,k]$ for some $h, k \in F$ with $\langle h,k \rangle F'/F'$ non-cyclic, which contradicts Lemma 6.6.

To finish, we need only show that $\hat{H}' \cap K = \{1\}$, which follows from Lemma 6.5. □

This completes the proof of theorem 6.4

7 Embedding in real-time combable groups

In this section we prove the following result.

Theorem 7.1 *Any polycyclic-by-finite group embeds as a subgroup of a real-time combable group. Furthermore, any nilpotent-by-finite group embeds as a subgroup of a group with a real-time combing with polynomially bounded length function.*

This result seems interesting in the context of the result of Gersten and Short ([15]) that no polycyclic group can embed in a biautomatic group, unless abelian by finite.

PROOF: We start by applying results of Malčev and Hall ([17, 12]), which imply that any polycyclic group (and hence also any polycyclic-by-finite group) G has a subgroup G_0 of finite index which embeds in a group $T_n(\mathcal{F})$ of upper triangular matrices over an algebraic number field $\mathcal{F}$. In fact, by using a result in an Exercise on page 36 of [22], we can even embed G_0 into $T_n(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers of an algebraic number field. Note that $T_n(\mathcal{O})$ is a finitely generated group. By inducing this representation from G_0 to G, we see that G itself embeds in $T_n(\mathcal{O}) wr S_m$, where $m = |G : G_0|$.

To finish the general, polycyclic-by-finite case, we need only to observe that $T_n(\mathcal{O}) wr S_m$ is real-time combable. $T_n(\mathcal{O})$ can be decomposed as a semidirect product of the form $E_n(\mathcal{O}) \rtimes (T_{n-1}(\mathcal{O}) \times A)$, where $E_n(\mathcal{O})$ is the group of $n \times n$ matrices over $\mathcal{O}$ with 1's on the diagonal, and the only other non-zero entries being in the right hand column, and A is the group of of $n \times n$ diagonal matrices with a unit of $\mathcal{O}$ in the bottom right hand corner, and all other diagonal entries equal to 1. Both $E_n(\mathcal{O})$ and A are clearly finitely generated and abelian; an obvious induction argument (on n), using Propositions 4.3 and 4.2, then proves that $T_n(\mathcal{O})$ is real-time combable. Then since $T_n(\mathcal{O}) wr S_m$ contains a direct product of copies of $T_n(\mathcal{O})$ as a subgroup of finite index, Propositions 4.1 and 4.2 now imply that $T_n(\mathcal{O}) wr S_m$ is real-time combable.

When G is in fact nilpotent-by-finite, G has a subgroup G_1, of finite index m', which is torsion-free nilpotent, and hence embeds in some $U_{n'}(\mathbf{Z})$, by [12], Theorem 7.5. Then G embeds in $U_{n'}(\mathbf{Z}) wr S_{m'}$, which is real-time combable with polynomially bounded length function, by Corollary 4.6 and Propositions 4.1 and 4.2. $\square$

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