Generalising some results about right-angled Artin groups to graph products of groups

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Abstract

We prove three results about the graph product $G = \mathcal{G}(\Gamma; G_v, v \in V(\Gamma))$ of groups G_v over a graph Γ. The first result generalises a result of Servatius, Droms and Servatius, proved by them for right-angled Artin groups; we prove a necessary and sufficient condition on a finite graph Γ for the kernel of the map from G to the associated direct product to be free (one part of this result already follows from a result in S. Kim's Ph.D. thesis). The second result generalises a result of Hermiller and Šunić, again from right-angled Artin groups; we prove that, for a graph Γ with finite chromatic number, G has a series in which every factor is a free product of vertex groups. The third result provides an alternative proof of a theorem due to Meier, which provides necessary and sufficient conditions on a finite graph Γ for G to be hyperbolic.

Keywords: groups, graph product, hyperbolic groups

1. Introduction

Given a simplicial graph Γ, and vertex groups G_v for each $v \in V(\Gamma)$, we use the notation $\mathcal{G}(\Gamma; G_v, v \in V(\Gamma))$ to denote the associated graph product of groups. Where there is no ambiguity in the choice of the vertex groups G_v, we may abbreviate this to $\mathcal{G}(\Gamma)$.

The group $\mathcal{G}(\Gamma)$ is the 'largest' group generated by the vertex groups such that G_u and G_v commute whenever u and v are joined by an edge in Γ. More precisely, it is the quotient of the free product of the groups G_v by the normal

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closure of the set of all commutators $[g_u, g_v]$ with $g_u \in G_u$, $g_v \in G_v$ and $u \sim v$. The right-angled Artin groups, also known as graph groups, are precisely the graph products in which all vertex groups are infinite cyclic groups, while the right-angled Coxeter groups arise from vertex groups of order 2.

In this article we prove three results about the graph product $G = \mathcal{G}(\Gamma; G_v, v \in V(\Gamma))$ of groups G_v over a graph Γ. The first result, found in Theorem 3.1 and Theorem 3.2, generalises a result of Servatius, Droms and Servatius, proved for right-angled Artin groups in [10]; we prove a necessary and sufficient condition on a finite graph Γ for the kernel of the map from G to the associated direct product to be free. Part of this result was already known, proved in [15]. The second result, Theorem 4.1, generalises a result of Hermiller and Šunić [12] again from right-angled Artin groups; we prove that, for a graph Γ with finite chromatic number, G has a series in which every factor is a free product of vertex groups. The third result, Theorem 5.1, provides an alternative proof of a theorem due to Meier [18], which provides necessary and sufficient conditions on a finite graph Γ for G to be hyperbolic.

While the graph product construction might seem merely to be a generalisation of direct and free products, it has produced some groups with rather interesting properties, including a group (a subgroup of a right-angled Artin group) that has FP but is not finitely presented [1] and a group $(F_2 \times F_2)$ with insoluble subgroup membership problem [19]. Right-angled Artin groups [2] and Coxeter groups have been particularly studied; their actions on CAT(0) cubical complexes give them interesting geometrical properties.

Graph products were introduced by Green in her PhD thesis [9], where in particular a normal form was developed and the graph product construction was shown to preserve residual finiteness; this work was extended by Hsu and Wise in [13], where right-angled Artin groups were shown to embed in right-angled Coxeter groups and hence to be linear. The preservation of semihyperbolicity, automaticity (as well as asynchronous automaticity and biautomaticity) and the possession of a complete rewrite system under graph products is proved in [11], necessary and sufficient conditions for the preservation of hyperbolicity in [18], the question of when the group is virtually free in [16], of preservation of orderability in [5]. The representation of the group as a graph product of directly indecomposable groups was shown to be unique in [21]. Automorphisms and the structure of centralisers for graph products of groups have been the subject of recent study, and in particular graph products defined over random graphs have provoked some interest for their applications [6, 4, 3].

2. Notation and basic properties

We use standard notation from graph theory. So for $v, w \in V(\Gamma)$ we say that v, w are adjacent if $\{v, w\}$ is an edge, and write $v \sim w$. If v, w are non-adjacent we write $v \not\sim w$.

A normal form for elements of graph products is developed in [9]. Each element $g \in \mathcal{G}(\Gamma)$ can be written as a product $g_1 \cdots g_k$ with each g_i in a vertex group G_{v_i}. We shall call such a product an *expression* for g, and the elements g_i the *syllables* of the expression. The expression is *reduced* if the syllable length k is minimal among all such expressions for g. Note that the identity element of G has syllable length 0.

The operation of replacing $g_i g_{i+1}$ by $g_{i+1} g_i$ within an expression, when $v_i \sim v_{i+1}$, is called a *shuffle.* We need the following technical result.

Proposition 2.1. *[9, Theorem 3.9] For $g \in \mathcal{G}(\Gamma)$, any expression for g can be transformed to any reduced expression for g by a succession of shuffles and group multiplications within the vertex groups.*

Now suppose that, for each vertex v of Γ, X_v is a generating set for G_v, and that $X = \cup_{v \in V(\Gamma)} X_v$. The next result is a consequence of Proposition 2.1.

Proposition 2.2. *For $g \in \mathcal{G}(\Gamma)$ a non-geodesic word for g over X can be reduced to a shorter representative by a succession of shuffles followed by a replacement of a subword over X_v by a shorter such word, for some vertex v of Γ.*

Proof: Suppose that w is a non-geodesic word over X representing g. We can write w as a concatenation $w_1 \cdots w_k$ of maximal subwords over the generating sets X_v and hence find an expression $g_1 \cdots g_k$ for g for which g_i is an element of a vertex group G_{v_i}, represented by the subword w_i.

We prove the result by induction on k.

If the expression is reduced, then one of the subwords w_i must be non-geodesic, and hence can be replaced by a shorter subword. This includes the case where $k = 1$.

Otherwise, the expression can be transformed to a reduced expression by a succession of shuffles and multiplications within vertex groups. The first such multiplication must be applied to an expression represented by a word that can be derived from w by a series of shuffles only. The expression resulting from that multiplication, which is represented by the same word, has shorter syllable length, and the result now follows by induction. $\square$

3. Generalising Servatius, Droms and Servatius

In this section we generalise two theorems of Servatius, Droms and Servatius.

In [10, Theorem 1] the following is proved:

> Let G be the right-angled Artin group defined by the n-cycle C_n with $n \geq 3$. Then G' has a subgroup isomorphic to the fundamental group of the orientable surface of genus $1 + (n-4)2^{n-3}$.

We generalise this to prove the same statement for a graph product of groups, namely:

Theorem 3.1. *Let $G = \mathcal{G}(\Gamma; G_v, v \in V(\Gamma))$ be a graph of non-trivial groups, for which Γ is a cycle of length n with $n \geq 3$. Then the kernel of the natural map from G to $G_{v_1} \times \cdots \times G_{v_n}$ contains the fundamental group of the orientable surface of genus $1 + (n-4)2^{n-3}$.*

In fact this result was already known for $n \geq 5$ (and it is straightforward to prove for n=3,4); however our proof is short and elementary. For $n \geq 5$ the result is proved in Sang-hyun Kim's 2007 thesis [15, Theorem 3.6] using Van Kampen diagrams, and for $n \geq 6$ another quite different proof is provided in [14, Corollary 1.3].

In [10, Theorem 2] the following is proved:

> Let G be the right-angled Artin group defined by a finite graph Γ. Then G' is free if and only if Γ does not contain C_n as an induced subgraph for any $n \geq 4$.

We generalise this in the following result.

Theorem 3.2. *Let $G = \mathcal{G}(\Gamma; G_v, v \in V(\Gamma))$ be a graph product of non-trivial groups with $V(\Gamma) = \{v_1, \ldots, v_n\}$ finite. Then the kernel of the natural map from G to $G_{v_1} \times \cdots \times G_{v_n}$ is free if and only if Γ does not contains a cycle of length 4 or more as an induced subgraph.*

From now on we shall use the notation $\mathcal{K}(G)$ for the kernel of that natural map that appears in both theorems.

Proof of Theorem 3.1: This proof closely follows and generalises the proof of [10, Theorem 1]. We have modified the notation and terminology in places.

We begin with a technical lemma.

Lemma 3.3. *Suppose that Γ is a graph, Δ an induced subgraph of Γ, and that $G_v, v \in V(\Gamma)$, $H_v, v \in V(\Delta)$ are groups with $H_v \subseteq G_v$ for all $v \in V(\Delta)$. Suppose that $G = \mathcal{G}(\Gamma; G_v(v \in V(\Gamma)))$ and $H = \mathcal{G}(\Delta; G_v(v \in V(\Delta)))$. Then*

(1) H embeds naturally in G as the subgroup $\langle H_v : v \in V(\Delta)\rangle$, and
(2) $\mathcal{K}(H) = \mathcal{K}(G) \cap H$.

PROOF: The natural embedding of H in G as a subgroup follows from [9, Proposition 3.31], and the rest is then immediate. $\square$

Applying Lemma 3.3 in the case where $\Delta = \Gamma$ and the H_v's are cyclic subgroups of the G_v's, we see that it is sufficient to prove the result of the theorem in the case where the vertex groups are non-trivial cyclic. In that case, the kernel is simply G'.

We choose generators x_v $(v \in V(\Gamma))$ for the cyclic groups G_v and suppose that x_v has order $k_v \in \mathbb{N} \cup \{\infty\}$, with $k_v > 1$.

The group G has the presentation

$$\langle x_v,\ (v \in V(\Gamma)) \mid x_v^{k_v}, [x_v, x_u]\ (v \in V(\Gamma),\ u \sim v)\rangle.$$

We define X_G to be the associated presentation complex of G, which has a single 0-cell, a 1-cell for each x_i and a 2-cell for each relator, attached along the appropriate sequence of 1-cells. Each relator is either a commutator of generators or a power of a generator, and we call the two kinds of associated 2-cells *commutator* cells and *power* cells.

We write K for $\mathcal{K}(G) = G'$ and define $X_{G/K}$ to be the presentation complex of G/K, using the presentation

$$\langle x_v\ (v \in V(\Gamma)) \mid x_v^{k_v}, [x_v, x_u]\ (v, u \in V(\Gamma))\rangle.$$

Clearly X_G embeds in $X_{G/K}$.

We define $\tilde{X}_{G/K}$ to be the universal cover of $X_{G/K}$, and then Z to be the subcomplex of $\tilde{X}_{G/K}$ obtained from it by deleting the lifts of all 2-cells of $X_{G/K}$ that correspond to commutators not present in the presentation of G. Then Z covers X_G and has K as its fundamental group.

Now let w be a word in the generators x_v representing the identity in G. Then it is a consequence of Proposition 2.1 (using an argument similar to the proof of Proposition 2.2), that w can be transformed to the empty word ϵ by a combination of the following moves:

M1 delete a subword $x_v x_v^{-1}$ or $x_v^{-1} x_v$,
M2 delete a subword of the form $x_v^{k_v}$ or $x_v^{-k_v}$,
M3 replace a subword $x_v^{\pm 1} x_u^{\pm 1}$ by $x_u^{\pm 1} x_v^{\pm 1}$, where $u \sim v$.

Since the 1-skeleton of $\tilde{X}_{G/K}$ can be identified with the Cayley graph of G/K, and is also equal to the 1-skeleton of Z, the edges in the 1-skeleton of Z can be oriented and labelled by the generators x_v of G. Suppose that p is a path in Z

that is homotopic to the trivial loop, and let w be the label of p. Then w represents the identity of G, and there is a sequence of words $w = w_1, w_2, \dots, w_k = \epsilon$, such that each w_i transformed into w_{i+1} by a move of type M1, M2 or M3, and a corresponding sequence of loops

$$p = p_1 \to p_2 \to \cdots \to p_k,$$

with p_k the trivial loop, where p_i is labelled by w_i. Then p_{i+1} is contained within p_i if the move has type M1 or M2, or within $p_i \cup F$, where F is a face of Z intersecting p_i in 2 consecutive edges, if the move has type M3.

So now if Y is a subcomplex of Z such that

(1) Y contains any power cell of Z of which it contains the 1-skeleton and

(2) Y contains any commutator cell of Z (and its incident vertices and edges) of which it contains two adjacent edges,

then the natural embedding of Y in Z induces an embedding of $\pi_1(Y)$ as a subgroup of $\pi_1(Z)$.

Both conditions are satisfied by the following subcomplex Y. Fix a vertex v of Z. Then Y consists of all the vertices and edges of Z on paths from v labelled by words $x_{v_1} \cdots x_{v_k}$ with all v_j distinct, together with all 2-cells of Z whose 1-skeletons are in Y. Intuitively Y is the intersection of the unit cube with Z; note that Y contains no power cells, and so this is the same subcomplex as in [10]; hence Condition (1) (which does not arise in [10]) holds vacuously.

Just as in [10] we observe that Y is an orientable surface of Euler characteristic $(4-n)2^{n-2}$ and hence genus $1 - \frac{1}{2}\chi(Y) = 1 + (n-4)2^{n-3}$. □

The remainder of this section is devoted to the proof of Theorem 3.2. The 'only if' part of the theorem follows from Theorem 3.1 and Lemma 3.3. Then the 'if' part is an immediate consequence of the following result.

Proposition 3.4. *Let Γ be a finite graph that does not contain a cycle of length 4 or more as an induced subgraph, let G_v, for $v \in V(\Gamma)$, be arbitrary groups, and let $G = \mathcal{G}(\Gamma; G_v, v \in V(\Gamma))$. Then $\mathcal{K}(G)$ is free.*

The proof of Proposition 3.4 follows closely and generalises the proof of [10, Theorem 2].

Proof of Proposition 3.4: Write $K = \mathcal{K}(G)$. If Γ is the complete graph, then G is the direct product of the vertex groups and K is the identity subgroup, so the result holds.

The proof is by induction on the number n of vertices, the case $n = 1$ being covered by the above.

For $n > 1$ and Γ not complete, it follows from [17, Solution to Problem 9.29b] (as explained in the proof of the lemma in [8]) that Γ can be written as the union of proper subgraphs Γ_1 and Γ_2, whose intersection Γ_{12} is either complete, or empty (if Γ is disconnected). Then $G \cong G_1 *_{G_{12}} G_2$, where $G_1 = \mathcal{G}(\Gamma_1; G_v, v \in V(\Gamma_1))$, $G_2 = \mathcal{G}(\Gamma_2; G_v, v \in V(\Gamma_2))$, and $G_{12} = \mathcal{G}(\Gamma_{12}; G_v, v \in V(\Gamma_{12}))$. Now by Lemma 3.3, $\mathcal{K}(G_1) = K \cap G_1$, $\mathcal{K}(G_2) = K \cap G_2$ and $\mathcal{K}(G_{12}) = K \cap G_{12}$. By the induction hypothesis, $\mathcal{K}(G_1)$ and $\mathcal{K}(G_2)$ are free and, since Γ_{12} is complete, $\mathcal{K}(G_{12})$ is the identity subgroup, and hence so is $K \cap G_{12}$. Now, as an amalgamated free product, G is the fundamental group of a graph of groups (as defined in [7]), for the graph consisting of a single edge, with vertex groups G_1, G_2 and edge group G_{12}. Since $K \cap G_{12}$ and hence also (by the normality of K) the intersections of K with all conjugates of G_{12} are trivial, we can apply [7, Theorem I.7.7] to deduce that K is a free product of a free group F with subgroups of conjugates of the vertex groups $\mathcal{K}(G_1)$ and $\mathcal{K}(G_2)$. So K is free as claimed. $\square$

4. Generalising Hermiller and Šunić

Theorem 4.1. *Suppose that* Γ *is a graph with finite chromatic number* $\mathrm{chr}(\Gamma)$*, and* $G = \mathcal{G}(\Gamma; G_v, v \in V(\Gamma))$ *a graph of groups. Then there is a series*

$$G = G_0 \rhd G_1 \rhd \cdots \rhd G_{\mathrm{chr}(\Gamma)} = 1,$$

such that each factor G_{i-1}/G_i *is the free product of copies of vertex groups* G_v*.*

PROOF: The proof is based on the proof of [12, Theorem A], which deals with the special case where G is a right-angled Artin group. We use induction on $\mathrm{chr}(\Gamma)$. If $\mathrm{chr}(\Gamma) = 1$ then there are no edges, and so G is a free product of the vertex groups G_v.

Now suppose that $\mathrm{chr}(\Gamma) > 1$. Then given a colouring of the graph with $\mathrm{chr}(\Gamma)$ colours, we define the subset D of the vertices to be all those of one particular colour, and put $L = V(\Gamma) \setminus D$. (This follows the notation of [12], where D and L are referred to as the *dead* and *living* vertices.) Let Γ_L be the induced subgraph with vertex set L, and define $G_L := \mathcal{G}(\Gamma_L; G_v, v \in L)$. For each vertex v, we let $\langle X_v \mid S_v \rangle$ be a presentation for G_v; and we let $\langle X_L \mid S_L \rangle$ be a presentation for G_L, where $X_L = \cup_{l \in L} X_l$ and S_L is the union of S_l $(l \in L)$ and the commutator relations derived from Γ_L.

The result holds for G_L by the induction hypothesis. Hence we can prove the result by showing that G is isomorphic to a split extension of the form $H \rtimes G_L$, where H is a free product of vertex groups.

We define $H := *_{d \in D} H_d$ to be the free product of groups H_d, for $d \in D$, where the H_d are defined as follows. For each $d \in D$, we define T_d to be the set of all those elements $t \in G_L$ for which there is no reduced expression starting with an element of G_v with $v \sim d$; note that $1 \in T_d$. Then we define $H_d := *_{t \in T_d} G_{d,t}$ to be a free product of copies $G_{d,t}$ of G_d, which are indexed by the elements of T_d.

For each $d \in D$, we shall now define an action of G_L on T_d. This induces an action of G_L on H_d by permuting the free factors, and this in turn induces an action of G_L on H, which will be used to define the semidirect product $H \rtimes G_L$.

We start by defining an action of each G_l on T_d. For each element a of some G_l ($l \in L$), we define a map $\alpha_{d,a} : T_d \to T_d$, by:

$$t\alpha_{d,a} := \begin{cases} ta, & ta \in T_d \\ t, & ta \notin T_d \end{cases}$$

Using (without modification) the argument in the proof of [12, Theorem A], we find that

> $(*)$ if $t \in T_d$ and $1 \neq a \in G_l$, then $ta \notin T_d$ if and only if $d \sim l$ and all the syllables in any reduced expression for t are from vertex groups G_v with $v \sim l$.

Hence if a, b are non-identity elements of the same vertex group G_l, and $t \in T_d$, then either t is fixed by both $\alpha_{d,a}$ and $\alpha_{d,b}$ or t is fixed by neither of $\alpha_{d,a}$ and $\alpha_{d,b}$. It follows from this observation that for any $a, b \in G_l$ (including the identity) $\alpha_{d,a}\alpha_{d,b} = \alpha_{d,ab}$. So for each vertex group G_l with $l \in L$, $a \mapsto \alpha_{d,a}$ defines an action of G_l on T_d.

Exactly as in the proof of [12, Theorem A] we that, if $a \in G_l$ and $b \in G_m$ with $l \sim m$, then $\alpha_{d,a}\alpha_{d,b} = \alpha_{d,b}\alpha_{d,a}$.

Hence we can extend our actions of the vertex groups of L on T_d to an action of G_L on T_d by defining $\alpha_{d,a_1 \ldots a_k}$ to be the composite $\alpha_{d,a_1} \cdots \alpha_{d,a_k}$, where each a_i is in some vertex group.

We also observe that

> $(**)$ if t is in the subset $T_d \subseteq G_L$ and τ is a reduced expression for t, then every prefix of τ is also a reduced expression for an element of T_d, and hence $1\alpha_{d,t} = t$.

Now for $t \neq 1$ we let $\theta_{d,t}$ be an isomorphism from $G_{d,1}$ to $G_{d,t}$, and we identify $G_{d,1}$ with G_d. Then, for each $x \in G_L$, we define an isomorphism β_x of H that permutes the free factors $G_{d,t}$ of each of the free factors H_d of H by defining, for $g \in G_{d,t}$:

$$g\beta_x := g\theta_{d,t}^{-1}\theta_{d,t\alpha_{d,x}}.$$

Hence $x \mapsto \beta_x$ defines an action of G_L on H, and we define the associated semidirect product $H \rtimes G_L$.

Our proof is complete once we have shown that $H \rtimes G_L$ is isomorphic to G.

For each $d \in D$, we define $X_{d,1} := X_d$ to be our given generating set of $G_{d,1} = G_d$ and, for each $1 \neq t \in T_d$, let $X_{d,t} = \{x\theta_{d,t} \mid x \in X_d\}$ be the associated generating set of $G_{d,t}$. Then

$$\begin{aligned} H \rtimes G_L &= \langle X_L \cup \bigcup_{d\in D} \bigcup_{t\in T_d} X_{d,t} \mid S_L \cup \bigcup_{d\in D} (S_d \cup R_d)\rangle \\ \text{where} \quad R_d &= \bigcup_{t\in T_d} \{g^x = g\beta_x : g \in X_{d,t}, x \in X_L\}. \end{aligned}$$

For each t in $\bigcup_{d\in D} T_d$, choose a word $\rho(t)$ over X_L that represents t. We deduce from our observation $(**)$ above that, for $g \in G_{d,1}$ and $t \in T_d$, we have $g\beta_t = g\theta_{d,t}$ and hence (by the definition of the semidirect product using the action defined by β_t) $g^{\rho(t)} = g\theta_{d,t}$. We use these expressions to eliminate from the presentation all the generators in the sets $X_{d,t}$, for each $d \in D$ and $1 \neq t \in T_d$. Let $g^x = g\beta_x$ be a relator from R_d with $g \in X_{d,t}$. Then, if $t \neq 1$, g rewrites as $g_1^{\rho(t)}$, where $g_1 = g\theta_{d,t}^{-1} \in X_d$, and we put $g_1 = g$ if $t = 1$. So g^x rewrites (or remains when $t = 1$) as $g_1^{\rho(t)x}$, while $g\beta_x$ becomes $g_1^{\rho(tx)}$ when $tx \in T_d$ or $g_1^{\rho(t)}$ when $tx \notin T_d$. In the first case, $\rho(tx) =_{G_L} \rho(t)x$, and the relation is a consequence of relations of G_L, and so can be omitted.

So our presentation simplifies to

$$\begin{aligned} H \rtimes G_L &= \langle X_L \cup \bigcup_{d\in D} X_d \mid S_L \cup \bigcup_{d\in D} (S_d \cup R'_d)\rangle \\ \text{where} \quad R'_d &= \{g^{\rho(t)x} = g^{\rho(t)} : g \in X_d, t \in T_d, x \in X_L, tx \notin T_d\}. \end{aligned}$$

Now we know from $(*)$ that whenever g, x, t satisfy the conditions in the definition of R'_d, then x and t must commute, and $x \in X_l$ with $l \sim d$. Since $x, t \in G_L$, $xt = tx$ must be a consequence of the relations in S_L, and hence the corresponding relation $g^{\rho(t)x} = g^{\rho(t)}$ can be replaced by $g^x = g$. So we can replace R'_d by

$$R''_d = \{g^x = g : g \in X_d, x \in X_l, l \sim d\}.$$

that is by the set of commutator relations implied by those edges of Γ between vertices in L and vertices in D, and we see that

$$H \rtimes G_L = \langle X_L \cup \bigcup_{d\in D} X_d \mid S_L \cup \bigcup_{d\in D} (S_d \cup R''_d)\rangle,$$

which we recognise as a presentation for $\mathcal{G}(\Gamma)$. $\square$

The following immediate consequence is already known [5].

Corollary 4.2. *If $G_v, v \in V(\Gamma)$ are right orderable groups, then for any graph Γ with finite chromatic number, the graph group $\mathcal{G}(\Gamma; G_v, v \in V(\Gamma))$ is right orderable.*

PROOF: Let $G = \mathcal{G}(\Gamma)$. We find a series

$$G = G_0 \rhd G_1 \rhd \cdots \rhd G_{\mathrm{chr}(\Gamma)} = 1$$

for G, where each factor G_{i-1}/G_i is a free product of groups G_v. The result now follows from the fact that the set of right orderable groups is closed under free products and extensions. $\square$

5. When is a graph product of groups hyperbolic?

In this section, we provide a alternative proof of the following theorem, which is the main result of [18]. Our proof makes use of the result proved in [20] that a group is (word-)hyperbolic if and only if geodesic bigons in its Cayley graph are uniformly thin, and is otherwise elementary.

Theorem 5.1. *Let $G = \mathcal{G}(\Gamma; G_v, v \in V(\Gamma))$ be a graph product of non-trivial groups with $V(\Gamma) = \{v_1, \ldots, v_n\}$ finite. Then G is hyperbolic if and only if all the following conditions hold.*

- *(i) Each vertex group G_v is hyperbolic.*
- *(ii) No two vertices of Γ with infinite vertex groups are adjacent.*
- *(iii) If G_v is infinite for some vertex v, then any two vertices that are adjacent to v are adjacent to each other.*
- *(iv) Γ has no cycle of length 4 as an induced subgraph.*

PROOF: For each vertex v of Γ we choose a finite symmetric generating set X_v of G_v, equal to $G_v \setminus \{1\}$ if G_v is finite. Then we let $X = \cup_{v \in V(\Gamma)} X_v$. We denote by $v(x)$ the vertex of Γ such that $x \in X_{v(x)}$.

Suppose first that G is hyperbolic. Then geodesic triangles in the Cayley graph $\mathcal{C}(G, X)$ are uniformly thin. It follows from Proposition 2.2 that, for any $v \in X_v$, words labelling geodesics within $\mathcal{C}(G_v, X_v)$ remain geodesic within $\mathcal{C}(G, X)$ and so geodesic triangles within $\mathcal{C}(G_v, X_v)$ are also uniformly thin. Hence each G_v is hyperbolic, that is Condition (i) holds. Just as in the proof in [18], we see easily that if any of the Conditions (ii)–(iv) fails, then G contains a subgroup isomorphic to $\mathbb{Z}^2$, and so cannot be hyperbolic.

This completes the 'only if' part of the result. The 'if' part is a consequence of the following result together with the main result of [20]. $\square$

Proposition 5.2. *Let $G = \mathcal{G}(\Gamma, G_v, v \in V(\Gamma))$ be a graph product as in Theorem 5.1 and such that the Conditions (i)–(iv) are satisfied. Then geodesic bigons in $\mathcal{C}(G, X)$ are uniformly thin.*

Before embarking on the proof of this proposition, we prove two technical lemmas, which will be used repeatedly.

We first clarify our notation. We let $n = |V(\Gamma)|$. If α is a word over a generating set X then, for $x \in X$, $x \in \alpha$ will mean that x or x^{-1} occurs in the word α.

Lemma 5.3. *Let α be a word over X. Suppose that a sequence of shuffles of $|\alpha_2|$ letters to the right hand end transforms α to $\alpha_1\alpha_2$. Then α and $\alpha_1\alpha_2$ fellow travel at distance at most $2\min(|\alpha_1|, |\alpha_2|)$.*

PROOF: We prove by induction on $|\alpha_2|$ that α and $\alpha_1\alpha_2$ fellow travel at distance at most $2|\alpha_2|$. If $|\alpha_2| = 1$ then there are words α_{11}, α_{12} with $\alpha = \alpha_{11}\alpha_2\alpha_{12}$ and $\alpha_1 = \alpha_{11}\alpha_{12}$ such that α_2 commutes with all letters in α_{12}. It is then easily verified that $\alpha = \alpha_{11}\alpha_2\alpha_{12}$ and $\alpha_1\alpha_2 = \alpha_{11}\alpha_{12}\alpha_2$ 2-fellow travel. For $|\alpha_2| > 1$, first shuffle the last letter of α_2 to the end of the word. Then the result follows from the case $|\alpha_2| = 1$ and the inductive hypothesis. By a similar argument, α and $\alpha_1\alpha_2$ fellow travel at distance at most $2|\alpha_1|$, and the result follows. $\Box$

Lemma 5.4. *For $x \in X_v$, suppose that α is a geodesic word over X, but αx is non-geodesic. Then α has a suffix β, with first letter in X_v, and all other letters either in X_v or in X_w with $w \sim v$, such that βx is non-geodesic. If G_v is finite, then we can choose β so that only its first letter is in X_v.*

PROOF: We choose β to be the shortest suffix of α such that βx is not geodesic, and let $\beta = z\gamma$ with $z \in X$. Now we apply Proposition 2.2 to see that we can replace βx by a G-equivalent shorter subword by first shuffling generators from commuting vertex groups, and then replacing a word from the generators in one of the vertex groups G_u by a G_u-equivalent shorter word. If $u \neq v$, then we could carry out those shuffles that do not involve x in β and shorten β, contradicting the geodesity of β. So we must have $u = v$, and similarly $u = v(z)$, so $v(z) = v$ as claimed. Furthermore, the word over X_u that is shortened must start with z and end with x.

Let $y \in \beta$ with $v(y) \neq v$. If $v(y) \not\sim v$, then y would remain between z and x after the shuffles, and so it would not be possible to create a subword over X_v starting with z and ending with x; hence we must have $v(y) \sim v$.

The final statement now follows immediately from the facts that the choice of generating set for a finite vertex group ensures that geodesic words in that group have length at most 1. $\Box$

The remainder of the section is devoted to the proof of Proposition 5.2.

We need to prove the uniform thinness of geodesic bigons. From the Condition (i) that vertex groups are hyperbolic we deduce the existence of a constant k such that for each vertex v bigons over X_v are uniformly k-thin.

The endpoints of geodesic bigons in $\mathcal{C}(G, X)$ may be at vertices or on edges of $\mathcal{C}(G, X)$ but it is sufficient to prove their uniform thinness in two situations: when both endpoints of the bigon lie on vertices of $\mathcal{C}(G, X)$, and when one endpoint lies at a vertex and the other lies at the midpoint of an edge. In other words, we have to prove that there exist constants K, K' such that:

(a) If α_1, α_2 are geodesic words over X with $\alpha_1 =_G \alpha_2$ than α_1 and α_2 K-fellow travel;

(b) If α_1, α_2 are geodesic words over X with $|\alpha_1| = |\alpha_2|$ and $\alpha_1 x =_G \alpha_2$ for some $x \in X$, then α_1 and α_2 K'-fellow travel.

In fact we can reduce consideration of Case (b) easily to Case (a), as follows.

Suppose that α_1, α_2, x are as in (b), and let $\alpha_1 = \eta\beta$, where β is the suffix of α_1 arising from Lemma 5.4. Let $v = v(x)$. Now suppose that shuffling all the letters of X_v to the right hand end transforms β to $\beta'_v\beta_v$, where β is a maximal suffix over X_v. It follows from Lemma 5.4 that $\beta_v x$ is non-geodesic, and hence equal in G_v to some word γ over X_v with $|\gamma| = |\beta_v|$.

If $|G_v|$ is finite, then $|\beta_v| = |\gamma| = 1$, and this fact together with Lemma 5.3 ensures that βx and $\beta'_v\gamma$ 2-fellow travel.

If $|G_v|$ is infinite then the Conditions (ii) and (iii) imply that the vertex groups G_u containing the letters of β'_v are finite and commute with each other. Then geodesity of β'_v implies that for any vertex u at most one letter of β'_v comes from X_u, and so $|\beta'_v| \leq n$, and we can apply Lemma 5.3 to see that βx $2n$-fellow travels with $\beta'_v\beta_v x$. Since $\beta_v x$ and γ k-fellow travel, it follows that βx and $\beta'_v\gamma$ $(2n + k)$-fellow travel.

In either case, we see that $\alpha_1 x = \eta\beta x$ and $\eta\beta'_v\gamma$ $(2n + k)$-fellow travel and so, since $\eta\beta'_v\gamma$ is a geodesic equal in G to α_2, (b) would follow from (a).

So now assume that α_1 and α_2 are as in (a). We shall prove by induction on $|\alpha_1|$ that they fellow travel at distance $K = 4n + k$. This is clear when $|\alpha_1| = 0$, so assume that $|\alpha_1| > 0$. Let $\alpha_1 = \eta_1\gamma_1$ and $\alpha_2 = \eta_2\gamma_2$, where, for $i = 1, 2$, γ_i is the maximal suffix of α_i all of whose letters come from commuting vertex groups. Geodesity of α_i implies that γ_i can contain at most one letter from each finite vertex group.

Suppose first that at least one of the words γ_i, say γ_1, contains a letter from an infinite vertex group, G_v.

Let y be the last such letter in γ_1. Then y can be shuffled to the end of γ_1 and so $\alpha_2 y^{-1}$ is not geodesic. We apply Lemma 5.4 to α_2 and y^{-1}, and let β be the resulting suffix of α_2. By Conditions (ii) and (iii), for each $z \in \beta$, either $v(z) = v$ or $G_{v(z)}$ is finite, and any two such vertices are adjacent in Γ. So β is a suffix of γ_2, and hence γ_2 also contains a letter from X_v. From Condition (ii), we see that G_v is the unique infinite vertex group with letters in γ_1 or γ_2.

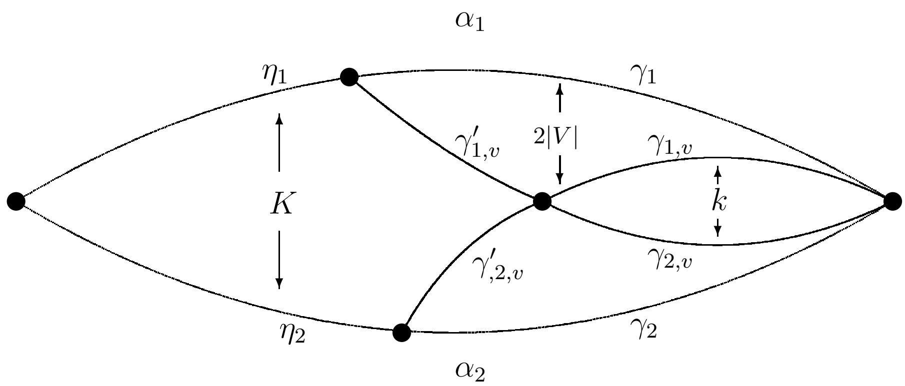

Figure 1: Infinite vertex group in tails

For $i = 1, 2$, let $\gamma_{i,v}$ and $\gamma'_{i,v}$ be the words obtained from γ_i by deleting all letters not in X_v, or in X_v, respectively. Conditions (ii) and (iii) ensure that $|\gamma'_{i,v}| \leq n$, and hence by Lemma 5.3 $\gamma_i =_G \gamma'_{i,v}\gamma_{i,v}$, and γ_i $2n$-fellow travels with $\gamma'_{i,v}\gamma_{i,v}$.

We claim that $\gamma_{1,v} =_G \gamma_{2,v}$. For if not, assume without loss that $|\gamma_{2,v}| \leq |\gamma_{1,v}|$, and let α be a geodesic word over X_v for $\gamma_{2,v}\gamma_{1,v}^{-1}$. Then $\eta_2\gamma'_{2,v}\alpha =_G \eta_1\gamma'_{1,v}$ with $|\eta_2\gamma'_{2,v}\alpha| > |\eta_1\gamma'_{1,v}|$ and we can apply Lemma 5.4 again to the shortest non-geodesic prefix of $\eta_2\gamma'_{2,v}\alpha$ and conclude that $\gamma'_{2,v}$ contains a letter from X_v, a contradiction.

Hence $\gamma_{1,v} =_G \gamma_{2,v}$, and so (since $\alpha_1 =_G \alpha_2$), we also have $\eta_1\gamma'_{1,v} =_G \eta_2\gamma'_{2,v}$. Since $\gamma_{1,v}$ and $\gamma_{2,v}$ are geodesic they must have the same length. We know them to be non-trivial, and hence the words $\eta_1\gamma'_{1,v}$ and $\eta_2\gamma'_{2,v}$ both have length less that $|\alpha_1|$, and we can apply the induction hypothesis to see that they K-fellow travel.

The situation is displayed in Fig. 1. We deduce that α_1 and α_2 K-fellow travel, as claimed.

So now we may suppose that γ_1 and γ_2 only involve letters from finite vertex groups, and hence that $|\gamma_i| \leq n$ for $i = 1, 2$. We consider two cases.

First suppose that some vertex group G_v is involved in both γ_1 and γ_2; that is, for some $x, y \in G_v$, $x \in \gamma_1, y \in \gamma_2$. Then by a similar argument to that in the previous paragraph, we can use Lemma 5.4 to prove that $x = y$ and, for $i = 1, 2$, we have $\alpha_i =_G \eta_i\gamma'_{i,v}x$, where $\gamma'_{i,v}$ is the result of deleting the single occurrence of x from γ_i. By the inductive hypothesis, $\eta_1\gamma'_{1,v}$ and $\eta_2\gamma'_{2,v}$ K-fellow travel, and since $|\gamma_i| \leq n$, we see that α_1 and α_2 must do too. See Fig. 2.

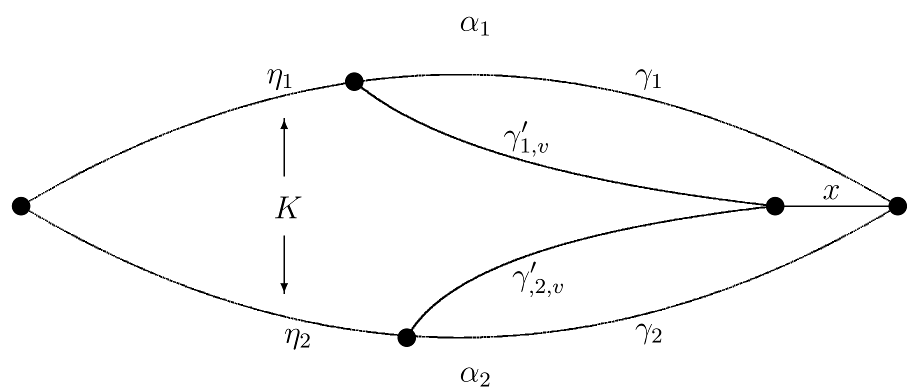

Figure 2: Common finite vertex group in tails

It remains to consider, and eliminate the case where no vertex group contains letters in both γ_1 and γ_2. If η_1 or η_2 is empty, then $|\alpha_i| \leq n$ and the result follows immediately, so let z_1, z_2 be the final letters of η_1 and η_2, respectively.

We shall now show that $v(z_2) \sim v(x)$ for every $x \in \gamma_1$, and $v(z_1) \sim v(y)$ for every $y \in \gamma_2$.

For suppose that $x \in \gamma_1$. By definition of γ_1,$z_1 \notin G_{v(x)}$, so $v(z_1) \neq v(x)$. Since x can be shuffled to the end of γ_1, $\alpha_2 x^{-1}$ is not geodesic. We apply Lemma 5.4 to α_2 and x^{-1} and let β be the resulting suffix of α_2; then β starts with a letter from $X_{v(x)}$. Since $v(x) \neq v(y)$ for all $y \in \gamma_2$, γ_2 must be a proper suffix of β, and hence, by Lemma 5.4, for each letter $y \in \gamma_2$, $v(x) \sim v(y)$. Hence $v(x) \neq v(z_2)$, and so $z_2\gamma_2$ is also a proper suffix of β and thus $v(x) \sim v(z_2)$. Similarly, if $y \in \gamma_2$ then $v(y) \sim v(z_1)$.

Note that it follows from the above that $v(z_1) \neq v(z_2)$, since the maximality of γ_1 as a suffix of α_1 of letters from commuting vertex groups ensures that we cannot have $v(z_1) \sim v(x)$ for all $x \in \gamma_1$.

Next we show that $v(z_1) \sim v(z_2)$. For if we multiply α_1 by each of the letters of γ_2^{-1} in turn, then Lemma 5.4 ensures that each such letter shuffles past the suffix $z_1\gamma_1$ of α_1 and merges with an earlier letter in η_1. We end up with a geodesic word α'_1 with $\alpha'_1 =_G \eta_2$, where α'_1 still has $z_1\gamma_1$ as a suffix. Then $\alpha'_1 z_2^{-1}$ is not geodesic, and Lemma 5.4 now implies that $v(z_1) \sim v(z_2)$.

So now, we can choose $x \in \gamma_1$ with $v(z_1) \not\sim v(x)$ and $y \in \gamma_2$ with $v(z_2) \not\sim v(y)$, and then the subgraph of Γ induced on $v(z_1), v(y), v(x), v(z_2)$ is a cycle of length 4, and Condition (iv) is violated.

Hence this final case cannot occur, and the proofs of Proposition 5.2 and Theorem 5.1 are complete.

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