Rapid Decay is Preserved by Graph Products

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Abstract

We prove that the rapid decay property (RD) of groups is preserved by graph products defined on finite simplicial graphs.

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1 Introduction

The main result proved in this paper is that the graph product of finitely many groups with the rapid decay property (RD) has the same property. This is stated formally in Section 4 as Theorem 4.1.

We call a function $\ell : G \to \mathbb{R}$ a *length function* for a group G if it satisfies

$$\ell(1_G) = 0, \quad \ell(g^{-1}) = \ell(g), \quad \ell(gh) \leq \ell(g) + \ell(h), \quad \forall g, h \in G.$$

Following Jolissaint [17], a group G is said to have RD if the operator norm $||.||_*$ for the group algebra $\mathbb{C}G$ is bounded by a constant multiple of the Sobolev norm $||.||_{2,r,\ell}$, a norm that is a variant of the l^2 norm weighted by a length function ℓ for G.

More precisely, RD holds for G if there are constants C, r and a length function ℓ on G such that for any $\phi, \psi \in \mathbb{C}G$,

$$||\phi||_* := \sup_{\psi \in \mathbb{C}G} \frac{||\phi * \psi||_2}{||\psi||_2} \leq C||\phi||_{2,r,\ell}.$$

Here, $\phi * \psi$ denotes the convolution of ϕ and ψ, $||.||_2$ the standard l^2 norm, and $||.||_{2,r,\ell}$ the Sobolev norm of order r with respect to ℓ. So we have:

$$\begin{aligned} \phi * \psi(g) &= \sum_{h \in G} \phi(h)\psi(h^{-1}g), \\ ||\psi||_2 &= \sqrt{\sum_{g \in G} |\psi(g)|^2}, \\ ||\phi||_{2,r,\ell} &= \sqrt{\sum_{g \in G} |\phi(g)|^2 (1 + \ell(g))^{2r}}. \end{aligned}$$

A brief introduction to RD is given in [2]; it is relevant to the Baum-Connes and Novikov conjectures (see an account in [22]). RD was originally studied by Jolissaint [17], after it emerged from work of Haagerup [11], who proved it for free groups. Jolissaint extended Haagerup's methods to prove it for classical hyperbolic groups (i.e. discrete cocompact groups of isometries of hyperbolic space), and to prove that free and direct products of groups with RD inherit the property, as do subgroups; de la Harpe [14] extended Jolissaint's arguments for classical hyperbolic groups to derive RD for word hyperbolic groups. More recently, Drutu and Sapir have shown that a group that is hyberbolic relative to a family of parabolic subgroups with RD must itself have RD [9]. In all this work the proofs focus on the factorisations of geodesic words. Other authors prove RD through examination of actions of the group [21, 2].

The graph product construction is a natural generalisation of both direct and free products. Given a finite simplicial graph Γ with a group attached to each vertex, the associated *graph product* is the group generated by the vertex groups with the added relations that elements of groups attached to adjacent vertices commute; the representation of such a group as a graph product of directly indecomposable groups is proved to be unique [20].

Right-angled Artin groups [3] (also known as graph groups) and right-angled Coxeter groups arise in this way, as the graph products of infinite cyclic groups and cyclic groups of order 2 respectively, and have been widely studied; some groups with rather interesting properties arise via graph products, including a group (a subgroup of a right-angled Artin group) that has FP but is not finitely presented [1] and a group ($F_2 \times F_2$) with insoluble subgroup membership problem [19]. Both right-angled Artin groups and right-angled Coxeter groups are already known to possess RD, through their actions on CAT(0) cube complexes [2] (indeed all finite rank Coxeter groups possess RD for this reason [6]).

Graph products were introduced by Green in her PhD thesis [10] where, in particular, a normal form was developed and the graph product construction was shown to preserve residual finiteness; this work was extended by Hsu and Wise in [13] where, in particular, right-angled Artin groups were shown to embed in right-angled Coxeter groups and hence to be linear. The preservation of semihyperbolicity, automaticity (as well as asynchronous automaticity and biautomaticity) and the possession of a complete rewrite system under graph products is proved in [12], necessary and sufficient conditions for the preservation of hyperbolicity in [18], the question of when the group is virtually free in [16], of orderability in [7]. Automorphisms and the structure of centralisers for graph products of groups have been the subject of recent study, and in particular graph products defined over random graphs have provoked some interest for their applications [8, 5, 4].

Our proof that the graph product construction preserves rapid decay will build on the methods used by Jolissaint for direct and free products [17]. We use a reformulation of RD due to Jolissaint, explained in Section 2 below, which compares norms on elements of $\mathbb{C}G$ with restricted support. We examine in Section 3 the geodesic decompositions of elements in a graph product, and show that they satisfy particularly useful properties, which will be applied in our proof. Section 4 states our main result, Theorem 4.1, and reduces its proof to the proof of a further technical condition, which is similar to that of Jolissaint's reformulation, but easier to verify in the context of graph and free products; the same condition is used in [17] for free products. The remaining two sections are devoted to the proof of Proposition 4.3, thereby completing the proof of Theorem 4.1

2 A reformulation of the rapid decay property

By [17, Lemma 2.1.3], any length function ℓ on G is equivalent to one with $\ell(G) \subseteq \mathbb{N}$ and $\ell(g) > 0\ \forall g \neq 1$. Since RD is invariant under length equivalence, by [17, Remark 1.1.7], we shall assume from now on that all length functions have this property.

Given a length function ℓ on G, and $k \in \mathbb{N}$, we define $C_k(\ell)$ to be the set $\{g \in G \mid \ell(g) = k\}$. We write χ_k for the characteristic function on C_k, and for $\phi \in \mathbb{C}G$, we write ϕ_k for the pointwise product $\phi.\chi_k$.

It is proved by Jolissaint [17, Proposition 1.2.6] that RD for G is equivalent to the following condition:

$(*)$ There exist $c, r > 0$ such that $\forall \phi, \psi \in \mathbb{C}G,\ k, l, m \in \mathbb{N}:$

$$\begin{array}{lcll} ||(\phi_k * \psi_l)_m||_2 & \leq & c||\phi_k||_{2,r,\ell}||\psi_l||_2 & \text{if } |k-l| \leq m \leq k+l, \\ ||(\phi_k * \psi_l)_m||_2 & = & 0 & \text{otherwise.} \end{array}$$

It follows from the properties of a length function that $||(\phi_k * \psi_l)_m||_2 = 0$ for m outside the range $[|k-l|, k+l]$. Hence we shall establish RD by verifying the following condition:

$(**)$ There exists a polynomial $P(x)$ such that $\forall \phi, \psi \in \mathbb{C}G,\ k, l, m \in \mathbb{N}:$

$$|k-l| \leq m \leq k+l \quad \Rightarrow \quad ||(\phi_k * \psi_l)_m||_2 \leq P(k)||\phi_k||_2||\psi_l||_2.$$

3 Graph products

Graph products of groups are studied in detail in [10], and we shall make use of the results from that thesis. Let $\Gamma = (V, E)$ be a finite simplicial graph, together with vertex groups G_v for each $v \in V$. The associated graph product G is defined to be the quotient of the free product of the groups G_v by the normal closure of all the commutators $[g, g']$ for which $g \in G_v$, $g' \in G_w$ and $\{v, w\}$ is an edge of the graph. We write $G = G(\Gamma; G_v, v \in V)$.

Given such a group G, we define $\mathcal{K}$ to be the set of cliques of the associated graph Γ, which we can identify with a set of subsets of the vertex set V of Γ. We define $\mathcal{K}_m$ to be the subset of $\mathcal{K}$ of cliques of size m. Given any subset J of V we define G_J to be the subgroup of G generated by the elements of its subgroups G_v with $v \in J$. It follows from [10, Proposition 3.31] that G_J is naturally isomorphic to the graph product defined by the induced subgraph of Γ with vertex set J. If J is a clique, then G_J is the direct product of its vertex groups.

Every element of the group G can be written as a product $y_1 \cdots y_k$ for $k \geq 0$ and with each y_i in a vertex group G_{v_i}; that is, each element of the group has a representation as a word over the set $S = \cup_{v \in V}(G_v \setminus \{1\})$. We call such a representation an *expression*, the elements y_i the *syllables* of the expression, and say that the expression has *syllable length* k; we define the syllable length of g, $\lambda(g)$, to be the minimum of the syllable lengths of expressions for g, and say that an expression for g is *reduced* if it has syllable length $\lambda(g)$. The function λ is easily seen to be a length function, but it is not the one that we shall use to prove RD.

We define $\Lambda_k = \{g \in G : \lambda(g) = k\}$.

Note that when G is a free product, every expression for which consecutive y_is come from distinct vertex groups is reduced, and each reduced expression corresponds to a distinct element of G. That is not true in general. But it is proved in [10, Theorem 3.9] that any expression for g can be transformed to any reduced expression by a sequence of replacements of the form $y'y \to yy'$ where y, y' belong to commuting vertex groups, or $y'y'' \to y$, where $y'y'' = y$ is a relation holding between three elements of one of the vertex groups, or deletion of $y'y''$ where the y', y'' are mutually inverse elements of a vertex group. Hence any reduced expression for an element g involves the same syllables, but the order of the syllables in the expression is not determined.

We shall need to estimate the number of factorisations of elements in graph products. For $g \in \Lambda_{k+l}$, we use the notation

$$\begin{aligned} \mathcal{F}_k^l(g) &:= \{(g_1, g_2) \mid g =_G g_1 g_2,\ \lambda(g_1) = k,\ \lambda(g_2) = l\}, \\ \mathrm{F}_k^l &:= \sup_{g \in \Lambda_{k+l}} |\mathcal{F}_k^l(g)|. \end{aligned}$$

Similarly, given a clique $J \in \mathcal{K}$ and $g \in \Lambda_{k+l+|J|}$, we use the notation

$$\begin{aligned} \mathcal{F}_k^l(J, g) &:= \{(g_1, s, g_2) \mid g =_G g_1 s g_2,\ \lambda(g_1) = k,\ \lambda(g_2) = l,\ s \in G_J,\ \lambda(s) = |J|\}, \\ \mathrm{F}_k^l(J) &:= \sup_{g \in \Lambda_{k+l+|J|}} |\mathcal{F}_k^l(J, g)|. \end{aligned}$$

By considering factorisations of g^{-1}, we see that $\mathrm{F}_k^l = \mathrm{F}_l^k$ and $\mathrm{F}_k^l(J) = \mathrm{F}_l^k(J)$.

Let $g, h \in G$. We say that h is a left divisor of g, if $\lambda(g) = \lambda(h) + \lambda(h^{-1}g)$, or, equivalently, if h has a minimal expression v that is a prefix of a minimal expression w for g. We define right divisors similarly.

We shall need

Lemma 3.1. *For each $J \in \mathcal{K}$, $\mathrm{F}_k^l(J)$ is bounded by a polynomial in* $\min(k, l)$.

Proof. Since $\mathrm{F}_k^l(J) = \mathrm{F}_l^k(J)$, it is sufficient to prove that $\mathrm{F}_k^l(J)$ is bounded by a polynomial in k.

Let $g \in \Lambda_m$ with $m \geq k+l$, and consider factorisations $g = g_1 s g_2$ with $g_1 \in \Lambda_k$, $g_2 \in \Lambda_l$, and $s \in G_J$ with $\lambda(s) = |J|$.

We start by bounding the number of left divisors g_1 of g with $g_1 \in \Lambda_k$. Suppose that $y_1 y_2 \cdots y_m$ is a reduced expression for g, with each y_i an element of a vertex group. Then $g_1 = y_{\sigma(1)} \cdots y_{\sigma(k)}$, where σ is a permutation of $\{1, \ldots, m\}$. When we transform the original expression $y_1 y_2 \cdots y_m$ to the new expression, using shuffles, there is no need to swap any of the syllables in g_1 among themselves, so we can assume that $\sigma(1) \leq \sigma(2) \leq \cdots \leq \sigma(k)$.

Let us call the syllable $y_{\sigma(i)}$ with $1 \leq i \leq k$ *unconstrained* if its vertex group commutes with the vertex groups of each of $y_{\sigma(i+1)}, \ldots, y_{\sigma(k)}$. (In particular, $y_{\sigma(k)}$ is unconstrained.)

We claim that g_1 is uniquely determined by its unconstrained syllables $y_{\sigma(i)}$ with $1 \leq i \leq k$. To show this, suppose that $g_1, g_1' \in \Lambda_k$ are left divisors of g corresponding to permutations σ, σ' of $\{1, \ldots, m\}$, where g_1, g_1' have the same unconstrained syllables. Suppose that $g_1 \neq g_1'$, and let i be maximal such that $y_{\sigma(i)}$ is a syllable of g_1 but not of g_1'. Then, by assumption, $y_{\sigma(i)}$ is not unconstrained, so there exists j with $i < j \leq k$ such that $y_{\sigma(i)}$ does not commute with $y_{\sigma(j)}$. By maximality of i, $y_{\sigma(j)}$ is a syllable of g_1'. Since $y_{\sigma(i)}$ and $y_{\sigma(j)}$ do not commute, $y_{\sigma(i)}$ must remain to the left of $y_{\sigma(j)}$ after any shuffles of g. So $y_{\sigma(i)}$ must be a syllable of g_1', a contradiction.

If there were two unconstrained syllables in g_1 from the same vertex group G_v, they could be moved together, thereby shortening the expression; hence there is at most one. Since g_1 has syllable length k, an unconstrained syllable $y_{\sigma(i)}$ in g_1 from G_v must be among the first k syllables in $y_1 y_2 \cdots y_m$ that come from G_v. Hence there are $k+1$ possible ways of selecting (0 or 1) syllables of $y_1 y_2 \cdots y_m$ from G_v that will be unconstrained in g_1. This gives $(k+1)^{|V|}$ choices in total for the unconstrained syllables of g_1, and hence $(k+1)^{|V|}$ choices for the left divisor g_1.

Similarly, we have an upper bound of $(|J|+1)^{|V|}$ for the number of left divisors of length $|J|$ of $g_1^{-1} g$ for a given g_1. In fact, the bound for the number of choices for s is $(|J|+1)^{|J|}$, as $s \in G_J$. The right divisor g_2 is then completely determined by g, g_1 and s. This results in the inequality $\mathrm{F}_k^l(J) \leq (k+1)^{|V|}(|J|+1)^{|J|}$, giving the required polynomial bound in k. ☐

Lemma 3.2. *Suppose that $g, h_1, h_2 \in L$ with $g =_G h_1 h_2$, $h_1 \in \Lambda_k, h_2 \in \Lambda_l$, and $g \in \Lambda_{k+l-q}$ with $q > 0$. Then $h_1 =_G g_1 s_1 w$ and $h_2 =_G w^{-1} s_2 g_2$, where:*

(1) $s_1, s_2 \in J$ for some $J \in \mathcal{K}$, and $\lambda(s_1) = \lambda(s_2) = \lambda(s_1 s_2) = |J|$,

(2) $q = |J| + 2\lambda(w)$.

Proof. Suppose first that there is no cancellation between h_1 and h_2, that is, when we reduce the expression $h_1 h_2$, no syllable becomes equal to the identity. Then pick a right divisor s_1 of h_1, and a left divisor s_2 of h_2, both minimal with respect to syllable length, such that $\lambda(s_1 s_2) = \lambda(s_1) + \lambda(s_2) - q$. Minimality ensures that every syllable of s_1 (or s_2) must merge without cancelling with a symbol of s_2 (or s_1). So $\lambda(s_1) = \lambda(s_2) = \lambda(s_1 s_2)$, the syllables within each of s_1 and s_2 must commute, and there is a clique $J \subset \Gamma$ of size q such that $s_1, s_2, s_1 s_2 \in G_J$.

If there is cancellation between h_1 and h_2, then we can find a right divisor w of h_1 such that w^{-1} is a left divisor of h_2. We choose w of maximal syllable length, so that $h_1 = h_1' w$ and $h_2 = w^{-1} h_2'$ for some $h_1', h_2' \in G$, and no cancellation occurs between h_1' and h_2'. Applying the above argument to h_1' and h_2', we find a clique J of size $q - 2\lambda(w)$, a right divisor s_1 of h_1', and a left divisor s_2 of h_2', with $\lambda(s_1) = \lambda(s_2) = \lambda(s_1 s_2) = |J|$. □

4 Main result

We prove

Theorem 4.1. *Suppose that $G = G(\Gamma; G_v, v \in V)$ is a graph product of groups defined with respect to a finite simplicial graph $\Gamma = (V, E)$, and suppose that each vertex group G_v satisfies RD. Then G satisfies RD.*

The proof generalises Jolissaint's proof in [17] of RD for free products, which is itself a generalisation of Haagerup's proof for free groups. We use the result of [17, Lemma 2.1.2], which implies (although it is not stated explicitly) that RD is preserved by taking direct products, and rely on the existence of polynomial bounds relating to factorisation in graph products proved in Lemma 3.1 and Lemma 3.2.

The following easy consequence of the Cauchy-Schwarz inequality will be used frequently.

Lemma 4.2. *For any positive integer M and real numbers $a_1, \ldots, a_M$,*

$$\left(\sum_{i=1}^{M} a_i\right)^2 \leq M \left(\sum_{i=1}^{M} a_i^2\right).$$

The proof of Theorem 4.1 that follows depends on Proposition 4.3, which is stated within the proof; we defer the proof of that to Section 6.

Proof of Theorem 4.1: For each $v \in V$ we choose a length function ℓ_v on G_v with respect to which G_v has rapid decay. Then, given $g \in G$ and a reduced expression $g = y_1 \cdots y_k$ for G, with $y_j \in G_{v_j}$, we define

$$\ell(g) = \sum_{j=1}^{k} \ell_{v_j}(y_j).$$

That ℓ is both well defined and a length function follows easily from [10, Theorem 3.9]. On subgroups G_J with $J \in \mathcal{K}$, we see that ℓ restricts to ℓ_J, defined in the same way as a sum of functions ℓ_{v_j} with $v_j \in J$; it follows from [17, Lemma 2.1.2] that G_J satisfies RD with respect to ℓ_J.

We prove rapid decay with respect to the length function ℓ by verifying the condition $(**)$ of Section 2. But rather than prove that directly we shall deduce it from a condition on the length function λ, that is stated as Proposition 4.3.

Recall that Λ_k is defined as the set $\{g \in G : \lambda(g) = k\}$. We now define $\chi_{(k)}$ to be the characteristic function on Λ_k and $\phi_{(k)}$ to be the pointwise product $\phi.\chi_{(k)}$. In general a function labelled with the subscript (k) is understood to have support on Λ_k.

We shall prove the following in Section 6.

Proposition 4.3. *$\exists\ c, r > 0$ such that for all $\phi, \psi \in \mathbb{C}G, k, l, m \in \mathbb{N}$, and $|k-l| \leq m \leq k+l$,*

$$||(\phi_{(k)} * \psi_{(l)})_{(m)}||_2 \leq c||\phi_{(k)}||_{2,r,\ell}||\psi_{(l)}||_2.$$

To derive our main result from this proposition, we shall use it to deduce the condition $(**)$ of Section 2, that is, we shall deduce the similar condition in which restrictions to $\Lambda_k, \Lambda_l, \Lambda_m$ are replaced by restrictions to C_k, C_l, C_m. (But note that both of the length functions λ and ℓ are involved in the proposition statement.) Jolissaint's proof of RD for free products follows exactly the same strategy, and the argument below is basically his, with some slight modification of notation to match our own. We suppose that k, l, m are fixed in the appropriate range. We write ϕ' rather than ϕ_k and ψ' rather than ψ_l, to make our notation less cumbersome.

Since $\ell(g) \geq 1$ for all $g \in G \setminus \{1\}$, we have $C_k \subseteq \cup_{j=0}^k \Lambda_j$ and $C_l \subseteq \cup_{i=0}^l \Lambda_i$, and hence $\phi' = \sum_{j=0}^k \phi'_{(j)}$, $\psi' = \sum_{i=0}^l \psi'_{(i)}$. Similarly, for fixed j, we have $|(\phi'_{(j)} * \psi')_m(g)| \leq |\sum_{p=0}^m (\phi'_j * \psi')_{(p)}(g)|$, for all $g \in G$. (Note that we dropped the restriction to C_m on the right hand side of this inequality.) Hence

$$||(\phi'_{(j)} * \psi')_m||_2^2 \leq \sum_{p=0}^m ||(\phi'_{(j)} * \psi')_{(p)}||_2^2.$$

Now, in any product of group elements $g_1 g_2 = g$ with $g_1 \in \Lambda_i$, $g_2 \in \Lambda_j$, $g \in \Lambda_p$, we must have $p \leq i+j$, $i \leq p+j$, $j \leq p+i$. Hence in each of the terms $(\phi'_{(j)} * \psi')_{(p)}$ in the sum on the right hand side of the above inequality, for fixed j and p, the support of ψ' lies in the union of the Λ_i with $|j-p| \leq i \leq j+p$, and so

$$\sum_{p=0}^m ||(\phi'_{(j)} * \psi')_{(p)}||_2^2 \leq \sum_{p=0}^m || \sum_{i=|j-p|}^{j+p} (\phi'_{(j)} * \psi'_{(i)})_{(p)}||_2^2$$

Since there are at most $2j+1$ values of i in each of the ranges $[|j-p|, j+p]$, we can use Lemma 4.2 to bound the right hand side above by

$$(2j+1) \sum_{p=0}^m \sum_{i=|j-p|}^{j+p} ||(\phi'_{(j)} * \psi'_{(i)})_{(p)}||_2^2.$$

It follows from Proposition 4.3 that, for $|j-i| \leq p \leq j+i$,

$$||(\phi'_{(j)} * \psi'_{(i)})_{(p)}||_2 \leq c||\phi'_{(j)}||_{2,r,\ell}||\psi'_{(i)}||_2.$$

For other values of p, the left hand side is zero. Hence

$$||(\phi'_{(j)} * \psi')_m||_2^2 \leq c^2(2j+1)||\phi'_{(j)}||_{2,r,\ell}^2 \sum_{p=0}^{m} \sum_{i=|j-p|}^{j+p} ||\psi'_{(i)}||_2^2.$$

Since $|i-j| \leq p \leq i+j$, for a given value of i, there are at most $2j+1$ values of p in the above summation, and so we have

$$||(\phi'_{(j)} * \psi')_m||_2^2 \leq c^2(2j+1)^2||\phi'_{(j)}||_{2,r,\ell}^2 \sum_{i=0}^{j+m} ||\psi'_{(i)}||_2^2 \leq c^2(2j+1)^2||\phi'_{(j)}||_{2,r,\ell}^2||\psi'||_2^2.$$

Now, using the triangle inequality together with Lemma 4.2 again, we have

$$\begin{aligned}
||(\phi_k * \psi_l)_m||_2^2 = ||(\phi' * \psi')_m||_2^2 &\leq (\sum_{j=0}^{k} ||\phi'_{(j)} * \psi')_m||)^2 \\
&\leq (k+1)\sum_{j=0}^{k} ||(\phi'_{(j)} * \psi')_m||^2 \\
&\leq c^2(k+1)\sum_{j=0}^{k}(2j+1)^2||\phi'_{(j)}||_{2,r,\ell}^2||\psi'||_2^2 \\
&\leq c^2(k+1)(2k+1)^2||\phi_k||_{2,r,\ell}^2||\psi_l||_2^2 \\
&= P(k)||\phi_k||_2^2||\psi_l||_2^2,
\end{aligned}$$

where the polyonomial P has degree $3+2r$. So we have deduced $(**)$.

The proof of the theorem will be complete once Proposition 4.3 is proved. $\square$

5 Technicalities of the proof of Proposition 4.3

In an attempt to make the proof of Proposition 4.3 more readable we start with some technical results and definitions.

For a function $\phi_{(k)}$ and $p \geq 0$, we can define functions $\phi_{(k-p)}^{(p)}$ and ${}^{(p)}\phi_{(k-p)}$ by

$$\begin{aligned}
\phi_{(k-p)}^{(p)}(u) &= \begin{cases} \sqrt{\sum_{w\in\Lambda_p} |\phi_{(k)}(uw)|^2} & \text{if } u \in \Lambda_{k-p} \\ 0 & \text{otherwise} \end{cases} \\
{}^{(p)}\phi_{(k-p)}(u) &= \begin{cases} \sqrt{\sum_{w\in\Lambda_p} |\phi_{(k)}(w^{-1}u)|^2} & \text{if } u \in \Lambda_{k-p} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Note that these functions are non-negative real-valued; we shall sometimes make use of that fact as we bound sums.

Lemma 5.1.

$$||\phi_{(k-p)}^{(p)}||_2^2 \;\leq\; \mathrm{F}_{k-p}^{p}||\phi_{(k)}||_2^2 \quad \text{and} \quad ||{}^{(p)}\phi_{(k-p)}||_2^2 \;\leq\; \mathrm{F}_{k-p}^{p}||\phi_{(k)}||_2^2.$$

Proof.

$$
\begin{aligned}
||\phi^{(p)}_{(k-p)}||_2^2 &= \sum_{u\in\Lambda_{k-p}}\sum_{w\in\Lambda_p}|\phi_{(k)}(uw)|^2\\
&\le \mathrm{F}^p_{k-p}\sum_{h\in\Lambda_k}|\phi_{(k)}(h)|^2\\
&= \mathrm{F}^p_{k-p}||\phi_{(k)}||_2^2
\end{aligned}
$$

The second inequality follows similarly, since $\mathrm{F}^p_{k-p}=\mathrm{F}^{k-p}_p$. ☐

For $0\le i\le k$ and $g\in\Lambda_{k-i}$, we define functions $\phi^g_{(i)}$ and ${}^g\phi_{(i)}$ by

$$
\begin{aligned}
\phi^g_{(i)}(v) &= \begin{cases}\phi_{(k)}(vg) & \text{if } vg\in\Lambda_k\\ 0 & \text{otherwise}\end{cases}\\
{}^g\phi_{(i)}(v) &= \begin{cases}\phi_{(k)}(gv) & \text{if } gv\in\Lambda_k\\ 0 & \text{otherwise}\end{cases}
\end{aligned}
$$

6 Proof of Proposition 4.3

Proof. First we deal with the case where $m=k+l$.

By Lemma 4.2, for each $g\in\Lambda_{k+l}$ we have

$$
\begin{aligned}
|\phi_{(k)}*\psi_{(l)}(g)|^2 &= |\sum_{(h_1,h_2)\in\mathcal{F}^l_k(g)}\phi_{(k)}(h_1)\psi_{(l)}(h_2)|^2\\
&\le \mathrm{F}^l_k\sum_{(h_1,h_2)\in\mathcal{F}^l_k(g)}|\phi_{(k)}(h_1)|^2|\psi_{(l)}(h_2)|^2.
\end{aligned}
$$

Hence

$$
\begin{aligned}
||(\phi_{(k)}*\psi_{(l)})_{(k+l)}||_2^2 &\le \mathrm{F}^l_k\sum_{g\in\Lambda_{k+l}}\sum_{(h_1,h_2)\in\mathcal{F}^l_k(g)}|\phi_{(k)}(h_1)|^2|\psi_{(l)}(h_2)|^2\\
&\le \mathrm{F}^l_k\sum_{h_1\in\Lambda_k,h_2,\in\Lambda_l}|\phi_{(k)}(h_1)|^2|\psi_{(l)}(h_2)|^2\\
&= \mathrm{F}^l_k\sum_{h_1\in\Lambda_k}|\phi_{(k)}(h_1)|^2\sum_{h_2\in\Lambda_l}|\psi_{(l)}(h_2)|^2\\
&= \mathrm{F}^l_k||\phi_{(k)}||_2^2||\psi_{(l)}||_2^2,
\end{aligned}
$$

which proves $(**)$.

So now suppose that $m=k+l-q$, for $q>0$, and that $g\in\Lambda_{k+l-q}$.

By Lemma 3.2, for each factorisation of g as a product h_1h_2 with $h_1\in\Lambda_k$, $h_2\in\Lambda_l$, there is a 5-tuple (g_1,g_2,s_1,s_2,w) of elements with syllable lengths $k-q+p$, $l-q+p$, $q-2p$, $q-2p$, p, for which $h_1=g_1s_1w$, $h_2=w^{-1}s_2g_2$,

$s := s_1 s_2 \in G_J$ has syllable length $q-2p$, and s_1, s_2, s are all elements of G_J for some $J \in \mathcal{K}_{q-2p}$.

For ease of notation we now define, for $s \in G_J$,

$$\begin{aligned} \mathcal{F}(J,s) &:= \{(s_1,s_2) \in G_J \times G_J : s = s_1 s_2\}, \\ \mathcal{F}_J(J,s) &:= \{(s_1,s_2) \in G_J \times G_J : s = s_1 s_2, \lambda(s_1) = \lambda(s_2) = |J|\}. \end{aligned}$$

Now for any $g \in \Lambda_{k+l-q}$, $|\phi_{(k)} * \psi_{(l)}(g)|$ is bounded above by

$$\sum_{\substack{p=1, \\ J \in \mathcal{K}_{q-2p}}}^{p=\lfloor q/2 \rfloor} \sum_{\substack{(g_1,s,g_2) \in \\ \mathcal{F}^{l-q+p}_{k-q+p}(J,g)}} \sum_{(s_1,s_2) \in \mathcal{F}(J,s)} \sum_{w \in \Lambda_p} |\phi_{(k)}(g_1 s_1 w) \psi_{(l)}(w^{-1} s_2 g_2)|,$$

by the triangle inequality.

By Cauchy-Schwarz

$$\begin{aligned} \sum_{w \in \Lambda_p} |\phi_{(k)}(g_1 s_1 w) \psi_{(l)}(w^{-1} s_2 g_2)| &\leq \sqrt{\sum_{w \in \Lambda_p} |\phi_{(k)}(g_1 s_1 w)|^2} \sqrt{\sum_{w \in \Lambda_p} |\psi_{(l)}(w^{-1} s_2 g_2)|^2} \\ &= \phi^{(p)}_{(k-p)}(g_1 s_1) \times {}^{(p)}\psi_{(l-p)}(s_2 g_2) \\ &= {}^{g_1}\phi^{(p)}_{(q-2p)}(s_1) \times {}^{(p)}\psi^{g_2}_{(q-2p)}(s_2). \end{aligned}$$

Further,

$$\sum_{(s_1,s_2) \in \mathcal{F}_J(J,s)} {}^{g_1}\phi^{(p)}_{(q-2p)}(s_1) \times {}^{(p)}\psi^{g_2}_{(q-2p)}(s_2) \leq {}^{g_1}\phi^{(p)}_{(q-2p)} * {}^{(p)}\psi^{g_2}_{(q-2p)}(s),$$

where the convolution here is over G_J, not over G.

Then we apply the Lemma 4.2 to see that

$$\begin{aligned} |\phi_{(k)} * \psi_{(l)}(g)|^2 &\leq \left(\sum_{\substack{p=1, \\ J \in \mathcal{K}_{q-2p}}}^{p=\lfloor q/2 \rfloor} \sum_{\substack{(g_1,s,g_2) \in \\ \mathcal{F}^{l-q+p}_{k-q+p}(J,g)}} {}^{g_1}\phi^{(p)}_{(q-2p)} * {}^{(p)}\psi^{g_2}_{(q-2p)}(s) \right)^2 \\ &\leq \mathrm{MF}(k,q,l) \sum_{\substack{p=1, \\ J \in \mathcal{K}_{q-2p}}}^{p=\lfloor q/2 \rfloor} \sum_{\substack{(g_1,s,g_2) \in \\ \mathcal{F}^{l-q+p}_{k-q+p}(J,g)}} \left({}^{g_1}\phi^{(p)}_{(q-2p)} * {}^{(p)}\psi^{g_2}_{(q-2p)}(s) \right)^2. \end{aligned}$$

where $\mathrm{MF}(k,q,l) := \sum_{p=1}^{\lfloor q/2 \rfloor} \sum_{J \in \mathcal{K}_{q-2p}} \mathrm{F}^{l-q+p}_{k-q+p}(J)$ is bounded by $Q(k)$ for some polynomial Q, by Lemma 3.1, and since there are finitely many possible cliques J. Hence

$$
\begin{aligned}
||(\phi_{(k)} * \psi_{(l)})_{(m)}||_2^2 &= \sum_{g\in\Lambda_m} |\phi_{(k)} * \psi_{(l)}(g)|^2 \\
&\leq Q(k) \sum_{\substack{p=1,\\ J\in\mathcal{K}_{q-2p}}}^{p=\lfloor q/2\rfloor} \sum_{\substack{g_1\in\Lambda_{k-q+p}\\ s\in G_J,\\ g_2\in\Lambda_{l-q+p}}} ({}^{g_1}\phi^{(p)}_{(q-2p)} * {}^{(p)}\psi^{g_2}_{(q-2p)}(s))^2 \\
&= Q(k) \sum_{\substack{p=1,\\ J\in\mathcal{K}_{q-2p}}}^{p=\lfloor q/2\rfloor} \sum_{\substack{g_1\in\Lambda_{k-q+p}\\ g_2\in\Lambda_{l-q+p}}} ||{}^{g_1}\phi^{(p)}_{(q-2p)} * {}^{(p)}\psi^{g_2}_{(q-2p)})||^2_{2;G_J}.
\end{aligned}
$$

But now, since RD holds with respect to ℓ_J in each of the groups G_J, we have

$$
||{}^{g_1}\phi^{(p)}_{(q-2p)} * {}^{(p)}\psi^{g_2}_{(q-2p)}||^2_{2;G_J} \quad\leq\quad c_J^2||{}^{g_1}\phi^{(p)}_{(q-2p)}||^2_{2,r_J,\ell_J;G_J} \quad ||{}^{(p)}\psi^{g_2}_{(q-2p)}||^2_{2;G_J}.
$$

We deduce easily from this that

$$
\begin{aligned}
\sum_{\substack{g_1\in\Lambda_{k-q+p},\\ g_2\in\Lambda_{l-q+p}}} ||{}^{g_1}\phi_{(q-2p)} * \psi^{g_2}_{(q-2p)}||^2_{2;G_J} \quad\leq\quad c_J^2 \sum_{g_1\in\Lambda_{k-q+p}} ||{}^{g_1}\phi^{(p)}_{(q-2p)}||^2_{2,r_J,\ell_J;G_J} \times \\
\sum_{g_2\in\Lambda_{l-q+p}} ||{}^{(p)}\psi^{g_2}_{(q-2p)}||^2_{2;G_J}.
\end{aligned}
$$

Then, for $c = \max c_J$ and $r = \max r_J$, we have

$$
\begin{aligned}
&\sum_{\substack{p=1,\\ J\in\mathcal{K}_{q-2p}}}^{p=\lfloor q/2\rfloor} \sum_{\substack{g_1\in\Lambda_{k-q+p}\\ g_2\in\Lambda_{l-q+p}}} ||{}^{g_1}\phi^{(p)}_{(q-2p)} * {}^{(p)}\psi^{g_2}_{(q-2p)}||^2_{2;G_J} \\
\leq\; & c^2 \sum_{\substack{p=1,\\ J\in\mathcal{K}_{q-2p}}}^{p=\lfloor q/2\rfloor} \left(\sum_{g_1\in\Lambda_{k-q+p}} ||{}^{g_1}\phi^{(p)}_{(q-2p)}||^2_{2,r_J,\ell_J;G_J} \sum_{g_2\in\Lambda_{l-q+p}} ||{}^{(p)}\psi^{g_2}_{(q-2p)}||^2_{2;G_J} \right) \\
\leq\; & c^2 \left(\sum_{\substack{p=1,\\ J\in\mathcal{K}_{q-2p}}}^{p=\lfloor q/2\rfloor} \sum_{g_1\in\Lambda_{k-q+p}} ||{}^{g_1}\phi^{(p)}_{(q-2p)}||^2_{2,r,\ell_J;G_J} \right) \times \\
&\qquad \left(\sum_{\substack{p=1,\\ J\in\mathcal{K}_{q-2p}}}^{p=\lfloor q/2\rfloor} \sum_{g_2\in\Lambda_{l-q+p}} ||{}^{(p)}\psi^{g_2}_{(q-2p)}||^2_{2;G_J} \right).
\end{aligned}
$$

But

$$\sum_{\substack{p=1,\\ J\in\mathcal{K}_{q-2p}}}^{p=\lfloor q/2\rfloor} \sum_{g_1\in\Lambda_{k-q+p}} ||^{g_1}\phi^{(p)}_{(q-2p)}||^2_{2,r,\ell_J;G_J}$$

$$= \sum_{\substack{p=1,\\ J\in\mathcal{K}_{q-2p}}}^{p=\lfloor q/2\rfloor} \sum_{\substack{g_1\in\Lambda_{k-q+p},\\ s\in G_J\\ w\in\Lambda_p}} |\phi_{(k)}(g_1sw)|^2(1+\ell_J(s))^{2r}.$$

Since $\lfloor q/2\rfloor \leq k$ and the number of sets J is bounded, Lemma 3.1 implies that the number of factorisations of a fixed $g'\in\Lambda_k$ as g_1sw in the above sum is at most $P(k)$ for some polynomial P. So the sum is bounded above by

$$P(k)\sum_{g'\in\Lambda_k}|\phi_{(k)}(g')|^2(1+\ell(g'))^{2r} = P(k)||\phi_{(k)}||^2_{2,r,\ell},$$

and similarly

$$\sum_{\substack{p=1,\\ J\in\mathcal{K}_{q-2p}}}^{p=\lfloor q/2\rfloor} \sum_{g_2\in\Lambda_{l-q+p}} ||^{(p)}\psi^{g_2}_{(q-2p)}||^2_{2;G_J} \leq P(k)||\psi_{(l)}||^2_2.$$

Hence

$$\begin{aligned} ||(\phi_{(k)}*\psi_{(l)})_{(m)}||^2_2 &\leq c^2Q(k)P(k)^2||\phi_{(k)}||^2_{2,r,\ell}||\psi_{(l)}||^2_2 \\ &\leq c^2||\phi_{(k)}||^2_{2,r+\deg(Q)+2\deg(P),\ell}||\psi_{(l)}||^2_2, \end{aligned}$$

where the final inequality uses the fact that $k\leq\ell(g)$ for all $g\in\Lambda_k$. This completes the proof of Proposition 4.3. $\square$

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