Rapid decay and Baum-Connes for large type Artin groups

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Abstract

We prove that many Artin groups of large type satisfy the rapid decay property, including all those of extra-large type. For many of these, including all 3-generator groups of extra-large type, a result of Lafforgue applies to show that the groups satisfy the Baum-Connes conjecture without coefficients.

Our proof of rapid decay combines elementary analysis with combinatorial techniques, and relies on properties of geodesic words in Artin groups of large type that were observed in [17] by two of the authors of this current article.

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1 Introduction

An Artin group G is defined to be a finitely generated group with presentation

$$\langle x_1, \ldots, x_n \mid {}_{m_{ij}}(x_i, x_j) = {}_{m_{ij}}(x_j, x_i) \quad \text{for each} \quad i \neq j \quad \text{with} \quad m_{ij} < \infty \rangle,$$

where the integers m_{ij} are the entries in a Coxeter matrix (a symmetric $n \times n$ matrix over $\mathbb{N} \cup \{\infty\}$, with $m_{ii} = 1$, and $m_{ij} \geq 2$ for all $i \neq j$), and, for generators x, y and $m \in \mathbb{N}$, the word ${}_m(x, y)$ is defined to be the product of m alternating x's and y's that starts with x. The associated Coxeter group is defined by adding the relations $x_i^2 = 1$ for all i.

A Coxeter matrix can be specified using an undirected labelled graph Γ with n vertices and an edge labelled m_{ij} between distinct vertices i and j whenever $m_{ij} < \infty$, but no edge between i and j if m_{ij} is infinite. (We note that an alternative and widely used convention deletes instead edges labelled 2 and leaves those with infinite labels.) We denote the Artin group defined in this way by $G(\Gamma)$.

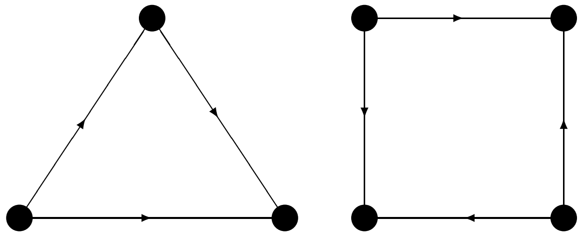
The Artin group G is said to be of *large type* if $m_{ij} \geq 3$ for all $i \neq j$, and of *extra-large type* if $m_{ij} \geq 4$ for all $i \neq j$. The concepts of large and extra-large type for Artin groups were introduced by Appel and Schupp [2, 1], who used small cancellation arguments to prove solubility of the word and conjugacy problems, the embedding of parabolic subgroups, the freeness of the subgroups $\langle x_i^2 \mid 1 \leq i \leq n \rangle$, and more, for these groups. An Artin group is said to be of *dihedral type* if 2-generated, of *spherical type* if the associated Coxeter group is finite, or of *right-angled type* if $m_{ij} \in \{2, \infty\}$ for all i, j.

The purpose of this paper is to prove the following two results.

Theorem 1.1. *Let $G = G(\Gamma)$ be an Artin group of large type for which Γ has no triangles with edge labels $3, 3, m$ with $3 \leq m < \infty$. Then G satisfies the rapid decay property. In particular, all Artin groups of extra-large type satisfy this property.*

Corollary 1.2. *Let $G = G(\Gamma)$ be an Artin group satisfying the hypotheses of Theorem 1.1. Then G satisfies the Baum-Connes conjecture without coefficients whenever one of the following holds.*

1. *G is 3-generated.*
2. *Γ is triangle-free.*
3. *The edges of Γ can be oriented in such a way that the oriented graph contains no subgraph of either of the two types shown.*



In particular, any large type Artin group $G(\Gamma)$ for which Γ is triangle-free satisfies the Baum-Connes conjecture without coefficients.

Following Jolissaint [18], we say that a finitely generated group G has the rapid decay property (RD) if the operator norm $||.||_*$ for the group algebra $\mathbb{C}G$ is bounded by a constant multiple of the Sobolev norm $||.||_{s,r,\ell}$, a norm that is a variant of the l^2 norm weighted by a length function for the group.

More precisely, rapid decay holds for G if there are constants C, r and a length function ℓ on G such that for any $\phi, \psi \in \mathbb{C}G$,

$$||\phi||_* := \sup_{\psi \in \mathbb{C}G} \frac{||\phi * \psi||_2}{||\psi||_2} \leq C||\phi||_{2,r,\ell}.$$

Here, $\phi * \psi$ denotes the convolution of ϕ and ψ, $||.||_2$ is the standard l^2 norm, and $||.||_{2,r,\ell}$ the Sobolev norm of order r, with respect to the length function ℓ, that is

$$\begin{aligned} \phi * \psi(g) &= \sum_{h \in G} \phi(h)\psi(h^{-1}g), \\ ||\psi||_2 &= \sqrt{\sum_{g \in G} |\psi(g)|^2} \\ ||\phi||_{2,r,\ell} &= \sqrt{\sum_{g \in G} |\phi(g)|^2 (1 + \ell(g))^{2r}}. \end{aligned}$$

We call a function $\ell : G \to \mathbb{R}$ a length function for G if it satisfies

$$\ell(1_G) = 0, \quad \ell(g^{-1}) = \ell(g), \quad \ell(gh) \leq \ell(g) + \ell(h), \quad \forall g, h \in G.$$

An account of the rapid decay property is given in [8].

The main interest in the rapid decay property for a group G stems from Lafforgue's result [19] that the combination of rapid decay with an appropriate action of G implies that the Baum-Connes conjecture without coefficients holds for G. That conjecture relates the K-theory of the reduced C^*-algebra $C_r^*(G)$ to the K^G-homology of the classifying space $\underline{E}G$ for proper G-actions, claiming that the assembly maps

$$\mu_i^G : RK_i^G(\underline{E}G) \to K_i(C_r^*(G)), \quad i = 0, 1,$$

are isomorphisms. This in turn implies the Novikov conjecture and (when G is torsion-free) the Kaplansky-Kadison idempotent conjecture; see [24].

The conjecture as stated above is commonly referred to as the Baum-Connes conjecture 'without coefficients' in order to distinguish it from a more general conjecture, with coefficients from a C^*-algebra admitting an action of G, to which counter-examples were claimed in [16].

Braid groups (Artin groups of type A_n) are shown to satisfy rapid decay in [5], where the result is proved for mapping class groups; braid groups are proved to satisfy Baum-Connes in [23, Corollary 14]. We note that since Artin groups of type B_n and D_n are split extensions of free groups by braid groups [11], it follows immediately from [23, Corollary 14] that those spherical type Artin groups also satisfy Baum-Connes (but we do not know if the result holds for the remaining spherical type groups). Right-angled Artin groups are proved to satisfy both rapid decay and Baum-Connes through their action on CAT(0) cube complexes [8]; alternative proofs of their rapid decay follow from their small cancellation properties [22], or the fact that they are graph products of infinite cyclic groups [10]. Dihedral Artin groups are easily proved to satisfy rapid decay since they are virtually direct products of free and cyclic groups. For the same reason they satisfy the Haagerup property (see [9]), which implies Baum-Connes [15]. We are not aware of any other classes of Artin groups known to satisfy either rapid decay or Baum-Connes prior to our results.

We shall call the hypothesis in Theorem 1.1 that there no triangles in Γ with edge labels $3, 3, m$ the $(3, 3, m)$*-hypothesis*. We shall prove rapid decay for Artin groups of large type that satisfy this hypothesis, using as our length function the word length metric over the standard generating set; that is, for each element g we shall define $\ell(g)$ to be the length of the shortest word over the standard generators that represents g.

Our proof makes critical use of the results and techniques developed in the earlier paper [17] by the second and third authors, in which the sets of geodesics and shortlex minimal geodesics of Artin groups of large type in their standard presentations are studied. It is proved there that those sets of geodesics are regular, and that the groups are shortlex automatic. Furthermore, a method was developed for rewriting arbitary words in the group generators to their shortlex normal forms. In Sections 6 and 7 of this paper, we shall recall some of these results and techniques, and develop them further, proving that certain conditions hold for factorisations of geodesics. In particular, our proof of Theorem 1.1 depends essentially on Proposition 7.5, which turns out to be false in all Artin groups of large type that do not satisfy the $(3, 3, m)$-hypothesis, so we are currently unable to dispense with this hypothesis.

We shall prove that certain conditions hold for the factorisation of geodesics, and derive rapid decay in groups for which these conditions hold. For many of those groups we can build on work of [6] to construct an action of the group that allows the application of Lafforgue's result to deduce Baum-Connes.

Our strategy to prove rapid decay by examining the factorisation of geodesics was also used by Jolissaint [18] and de la Harpe [14] to prove rapid decay for word hyperbolic groups, extending Haagerup's proof for free groups [13]; Drutu and Sapir used similar techniques to prove rapid decay for a group hyperbolic relative to a family of subgroups with the property [12]. Other authors have used more geometric techniques [7, 8].

In Section 2 we explain how Corollary 1.2 can be deduced from Theorem 1.1. Then Section 3 shows how rapid decay can be deduced from a pair of combinatorial conditions D1 and D2 relating to the factorisation of geodesics in the group. Section 5 introduces the notation we need for the remainder of the article. Section 6 examines dihedral Artin groups, first recalling from [17] some technical results on the structure of geodesics, and then using these to deduce the conditions D1 and D2 for these groups. Section 7 extends the results of Section 6 to deduce the same conditions for the large type groups considered in Theorem 1.1.

2 Deducing Baum-Connes from rapid decay

Any discrete group that acts continuously, isometrically, properly and co-compactly on a CAT(0) metric space is in the class $\mathcal{C}'$ defined by Lafforgue in [19]; hence by [19, Corollary 0.4], for such a group the Baum-Connes conjecture is a consequence of rapid decay.

We shall deduce Corollary 1.2 from Theorem 1.1 using results of [6] to construct an appropriate CAT(0) action for a large class of Artin groups of large type.

A CAT(0) space is defined to be a metric space in which distances across a triangle with geodesic sides of lengths p, q, r are always bounded above by distances between equivalent points in a Euclidean triangle with the same side lengths. A space is locally-CAT(0) or equivalently non-positively curved if any point has a neighbourhood that is CAT(0). By the Cartan-Hadamard theorem of [4], a simply connected locally-CAT(0) space is actually CAT(0).

Brady and McCammond define non-positively curved presentation complexes for non-standard presentations of 3-generator Artin groups of large type (and some others) in [6]. We shall see that the actions of the groups on the universal covers of these complexes satisfy the conditions we need.

Theorems 4, 6 and 7 of that paper consider the three cases of our theorem. In each of the three cases we define a complex K_Γ, the presentation complex for a presentation I_Γ of the Artin group $G(\Gamma)$ in which all relators have length 3, formed from the standard presentation by the addition of some generators. The presentation complex K_Γ has a single 0-cell v_0, a 1-cell for each generator of I_Γ, and a triangular 2-cell for each relator of I_Γ. We define L_Γ to be the link of v_0; this is the intersection with the surrounding complex of a small sphere centred on v_0, and can be viewed as a graph to which each edge of K contributes two vertices, and each corner of a 2-cell contributes an edge. $\tilde{K_\Gamma}$ is the universal cover of K_Γ; this is simply connected, and also has L_Γ as the link of each 0-cell.

In each of the three cases of our theorem, it is proved in [6] that some specification of edge lengths of the triangles which are the 2-cells of K_Γ extends to a piecewise Euclidean metric on K_Γ; in the first and third cases of the theorem all triangles are made equilateral with side length 1, and in the second case all triangles are made isosceles and right-angled with their shorter sides of length 1. The piecewise Euclidean metric induces a metric on the link L_Γ of v_0, where the length of each edge is equal to the angle at the corner through v_0 of the corresponding 2-cell. The metric can also be lifted to the cover $\tilde{K_\Gamma}$.

An action of G by isometries on $\tilde{K_\Gamma}$ is inherited from the left regular action of G on its Cayley graph (the 1-skeleton of $\tilde{K_\Gamma}$), in which the vertex h is mapped by g to the vertex gh. The action is free, so clearly it is both proper and continuous. The quotient, K_Γ, of $\tilde{K_\Gamma}$ by G is certainly compact. In order to apply Lafforgue's result, we need simply to verify that $\tilde{K_\Gamma}$ equipped with this metric is CAT(0).

That K_Γ is non-positively curved (locally-CAT(0)) is proved in [6]. In each of the three cases the length of a closed path in the link of a vertex is seen to be at least 2π; hence [3, Theorem 15] applies to show that K_Γ is non-positively curved. Exactly the same argument can be applied to the link of a vertex in $\tilde{K_\Gamma}$ to deduce that the simply connected cover $\tilde{K_\Gamma}$ is also non-positively curved, and hence by the Cartan-Hadamard theorem of [4], $\tilde{K_\Gamma}$ is CAT(0). Baum-Connes in each of these cases now follows by [19, Corollary 0.4].

3 Reformulation of the rapid decay condition

Let ℓ be a length function on a group G. Given $k \in \mathbb{N}$, we define C_k to be the set of elements of G with $\ell(g) = k$. We write χ_k for the characteristic function on C_k, and for $\phi \in \mathbb{C}G$, we write ϕ_k for the pointwise product $\phi.\chi_k$. (Generally, in this article we use a subscript k to indicate that a function has support on C_k.)

It is proved by Jolissaint [18, Proposition 1.2.6] that rapid decay for G is equivalent to the following condition $(*)$:

$$\begin{array}{l} \forall \phi, \psi \in \mathbb{C}G, k, l, m \in \mathbb{N}, \\ \qquad |k-l| \leq m \leq k+l, \quad \Rightarrow \quad ||(\phi_k * \psi_l)_m||_2 \leq ||\phi_k||_{2,r,\ell}||\psi_l||_2, \\ \qquad\qquad \text{otherwise} \qquad ||(\phi_k * \psi_l)_m||_2 = 0 \end{array}$$

We observe that it follows from the properties of a length function that the norm $||(\phi_k * \psi_l)_m||_2$ is zero for m outside the range $[|k-l|, k+l]$. Hence we shall verify rapid decay by verifying the following condition $(**)$:

there exists a polynomial $P(x)$ such that:

$$\begin{array}{l} \forall \phi, \psi \in \mathbb{C}G, k, l, m \in \mathbb{N}, \\ \qquad |k-l| \leq m \leq k+l, \quad \Rightarrow \quad ||(\phi_k * \psi_l)_m||_2 \leq P(k)||\phi_k||_2||\psi_l||_2. \end{array}$$

We can clearly assume that the coefficients of such a polynomial P are non-negative, and hence that P is an increasing function of x, and we shall assume throughout this paper that all polynomials that arise have this property.

4 Geodesic factorisation

For $g \in G$ we call a decomposition of g as a product $g_1 \cdots g_k$ a *geodesic factorisation* of g if $\sum \ell(g_i) = \ell(g)$, and call the elements g_i *divisors* of g. In particular

g_1 is called a *left divisor* and g_k is called a *right divisor*. Given $g \in C_{k+l}$ we define

$$\begin{aligned} \mathrm{Fact}_{k,l}(g) &:= \{(g_1, g_2) : g_1 \in C_k, g_2 \in C_l : g_1 g_2 =_G g\} \\ \mathrm{F}_{k,l} &:= \sup_{g \in C_{k+l}} |\mathrm{Fact}_{k,l}(g)| \end{aligned}$$

Let $\mathcal{P}$ be a subset of G^2 and, for $g \in G$, let $\mathcal{P}(g) = \{(g_1, g_2) \in \mathcal{P} : g_1 g_2 = g\}$. We shall refer to the factorisations of g in $\mathcal{P}(g)$ as the *permissible* factorisations of g. We define

$$\begin{aligned} \mathrm{Fact}_{\mathcal{P},k,l}(g) &:= \{(g_1, g_2) : g_1 \in C_k, g_2 \in C_l, (g_1, g_2) \in \mathcal{P}(g)\} \\ \mathrm{F}_{\mathcal{P},k,l} &:= \sup_{g \in C_{k+l}} |\mathrm{Fact}_{\mathcal{P},k,l}(g)| \end{aligned}$$

We shall verify the condition $(**)$ above for Artin groups satisfying the hypotheses of Theorem 1.1, and hence verify rapid decay, by finding a suitable set $\mathcal{P}$ of permissible factorisations for which the size of both $\mathrm{Fact}_{\mathcal{P},k,l}(g)$ and a related set are polynomially bounded.

We define two conditions D1, D2 that we shall require to hold on a suitable subset $\mathcal{P}$ of G^2.

D1 $\mathrm{F}_{\mathcal{P},k,l}$ is bounded above by $P_1(\min(k, l))$ for some polynomial $P_1(x)$.

D2 For each $g \in G$, each $k, l \geq 0$, there is a subset $S(g, k, l)$ of G^3 as follows.

For each decomposition of g as a product $g_1 g_2$ with $g_1 \in C_k$, $g_2 \in C_l$, $S(g, k, l)$ contains a triple $(f_1, \hat{g}, f_2)$, for which $g = f_1 \hat{g} f_2$, and $\hat{g} = h_1 h_2$, where $(f_1, h_1) \in \mathrm{Fact}_{\mathcal{P},k-p_1,p_1}(g_1)$ and $(h_2, f_2) \in \mathrm{Fact}_{\mathcal{P},p_2,l-p_2}(g_2)$, for some $p_1, p_2 \leq K \min(k, l)$, for some global constant K.

Furthermore, there are polynomials $P_2(x), P_3(x)$ such that

(a) for all g, k, l, $|S(g, k, l)| \leq P_2(\min(k, l))$,

(b) $|T(k, l)| \leq P_3(\min(k, l))$, where $T(k, l) := \{\hat{g} : \exists g,\ (f_1, \hat{g}, f_2) \in S(g, k, l)\}$.

Figure 1 illustrates the condition D2.

Theorem 4.1. *Let G be a group and $\mathcal{P}$ a subset of G^2 for which the conditions D1 and D2 hold. Then G satisfies rapid decay with respect to the word length metric.*

Before launching into the proof of Theorem 4.1 we prove a few technical results. Suppose first that ϕ_k is a function with support on C_k. Then for any $p \geq 0$, we can define functions $\phi^{(p)}_{\mathcal{P},k-p}$ and ${}^{(p)}\phi_{\mathcal{P},k-p}$ with support on C_{k-p} by

$$\begin{aligned} \phi^{(p)}_{\mathcal{P},k-p}(g) &= \sqrt{\sum_{h \in C_p, (g,h) \in \mathcal{P}} |\phi_k(gh)|^2}, \quad \text{if} \quad g \in C_{k-p} \\ &= 0 \quad \text{otherwise} \\ {}^{(p)}\phi_{\mathcal{P},k-p}(g) &= \sqrt{\sum_{h \in C_p, (h,g) \in \mathcal{P}} |\phi_k(hg)|^2}, \quad \text{if} \quad g \in C_{k-p} \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

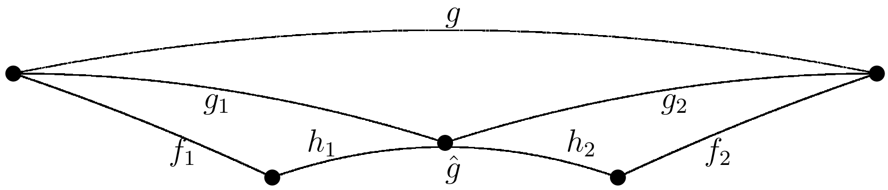


Figure 1: Condition D2

Lemma 4.2.

$$
\begin{aligned}
||\phi^{(p)}_{\mathcal{P},k-p}||_2^2 &\leq \mathrm{F}_{\mathcal{P},k-p,p}||\phi_k||_2^2 \\
||^{(p)}\phi_{\mathcal{P},k-p}||_2^2 &\leq \mathrm{F}_{\mathcal{P},p,k-p}||\phi_k||_2^2
\end{aligned}
$$

Proof.

$$
\begin{aligned}
||\phi^{(p)}_{\mathcal{P},k-p}||_2^2 &= \sum_{g\in C_{k-p}} \sum_{h\in C_p,(g,h)\in\mathcal{P}} |\phi_k(gh)|^2 \\
&\leq \mathrm{F}_{\mathcal{P},k-p,p} \sum_{g_1\in C_k} |\phi_k(g_1)|^2 \\
&= \mathrm{F}_{\mathcal{P},k-p,p}||\phi_k||_2^2
\end{aligned}
$$

The second inequality follows similarly. □

We shall make frequent use of the following, which is an easy application of the Cauchy-Schwarz inequality.

Lemma 4.3. *For $a_i \in \mathbb{C}$, $n \in \mathbb{N}$,*

$$
|\sum_{i=1}^{n} a_i|^2 \leq n \sum_{i=1}^{n} |a_i|^2
$$

We are now ready to prove the theorem.

Proof of Theorem 4.1: Suppose that D1 and D2 hold, and let $K, P_1(x), P_2(x), P_3(x)$ and the sets $\mathcal{P}$, $S(g,k,l)$ and $T(k,l)$ be as specified by those conditions.

Choose k,l,m such that $|k-l| \leq m \leq k+l$, and define $\hat{k} := \min(k,l)$. Let $g \in G$ with $\ell(g) = m$. Then

$$
|(\phi_k * \psi_l)(g)| = |\sum_{\substack{g_1 \in C_k,\, g_2 \in C_l \\ g = g_1 g_2}} \phi_k(g_1)\psi_l(g_2)| \leq \sum_{\substack{g_1 \in C_k,\, g_2 \in C_l \\ g = g_1 g_2}} |\phi_k(g_1)\psi_l(g_2)|
$$

Now the condition D2 defines an injection from $\mathrm{Fact}_{k,l}(g)$ to $G^5 \times \mathbb{N}^2$ that maps (g_1, g_2) to $(f_1, \hat{g}, f_2, h_1, h_2, p_1, p_2)$, where $h_1 \in C_{p_1}$, $h_2 \in C_{p_2}$, $(f_1, h_1), (h_2, f_2) \in \mathcal{P}$ and $h_1 h_2 = \hat{g}$. We define

$$H(f_1, \hat{g}, f_2, p_1, p_2) := \{h_1 \in C_{p_1} : h_2 := h_1^{-1}\hat{g} \in C_{p_2}, (f_1, h_1), (h_2, f_2) \in \mathcal{P}\}.$$

So the right hand side of the above inequality is bounded above by:

$$\sum_{p_1,p_2=0}^{K\hat{k}} \sum_{(f_1,\hat{g},f_2)\in S(g,k,l)} \sum_{\substack{h_1 \in H(f_1,\hat{g},f_2,p_1,p_2) \\ h_2 := h_1^{-1}\hat{g}}} |\phi_k(f_1 h_1)\psi_l(h_2 f_2)|$$

Note that this summation is over a set that is possibly larger than the image of that injection, and hence we have an upper bound rather than equality.

Now by the Cauchy-Schwartz inequality, this sum is at most

$$\sum_{p_1,p_2=0}^{K\hat{k}} \sum_{(f_1,\hat{g},f_2)\in S(g,k,l)} \sqrt{\sum_{\substack{h_1 \in C_{p_1} \\ (f_1,h_1)\in\mathcal{P}}} |\phi_k(f_1 h_1)|^2} \sqrt{\sum_{\substack{h_2 \in C_{p_2} \\ (h_2,f_2)\in\mathcal{P}}} |\psi_l(h_2 f_2)|^2}$$

$$= \sum_{p_1,p_2=0}^{K\hat{k}} \sum_{(f_1,\hat{g},f_2)\in S(g,k,l)} \phi^{(p_1)}_{\mathcal{P},k-p_1}(f_1) \quad {}^{(p_2)}\psi_{\mathcal{P},l-p_2}(f_2)$$

where $\phi^{(p)}_{\mathcal{P},k-p}$ and ${}^{(p)}\psi_{\mathcal{P},l-p}$ are as defined above.

So now, we have

$$\begin{aligned} ||(\phi_k * \psi_l)_m||_2^2 &= \sum_{g\in C_m} |\phi_k * \psi_l(g)|^2 \\ &\le \sum_{g\in C_m} |\sum_{p_1,p_2=0}^{K\hat{k}} \sum_{(f_1,\hat{g},f_2)\in S(g,k,l)} \phi^{(p_1)}_{\mathcal{P},k-p_1}(f_1) \quad {}^{(p_2)}\psi_{\mathcal{P},l-p_2}(f_2)|^2 \end{aligned}$$

Using Lemma 4.3 twice we see that

$$\begin{aligned} &\sum_{g\in C_m} |\sum_{p_1,p_2=0}^{K\hat{k}} \sum_{(f_1,\hat{g},f_2)\in S(g,k,l)} \phi^{(p_1)}_{\mathcal{P},k-p_1}(f_1) \quad {}^{(p_2)}\psi_{\mathcal{P},l-p_2}(f_2)|^2 \\ \le\ & (K\hat{k}+1)^2 \sum_{g\in C_m} \sum_{p_1,p_2=0}^{K\hat{k}} | \sum_{(f_1,\hat{g},f_2)\in S(g,k,l)} \phi^{(p_1)}_{\mathcal{P},k-p_1}(f_1) \quad {}^{(p_2)}\psi_{\mathcal{P},l-p_2}(f_2)|^2 \\ \le\ & (K\hat{k}+1)^2 \sum_{p_1,p_2=0}^{K\hat{k}} \sum_{g\in C_m} |S(g,k,l)| \sum_{(f_1,\hat{g},f_2)\in S(g,k,l)} |\phi^{(p_1)}_{\mathcal{P},k-p_1}(f_1) \quad {}^{(p_2)}\psi_{\mathcal{P},l-p_2}(f_2)|^2 \\ \le\ & (K\hat{k}+1)^2 P_2(\hat{k}) \sum_{p_1,p_2=0}^{K\hat{k}} \sum_{g\in C_m} \sum_{(f_1,\hat{g},f_2)\in S(g,k,l)} |\phi^{(p_1)}_{\mathcal{P},k-p_1}(f_1) \quad {}^{(p_2)}\psi_{\mathcal{P},l-p_2}(f_2)|^2 \end{aligned}$$

The last sum is bounded above by

$$(K\hat{k}+1)^2 P_2(\hat{k}) \sum_{p_1,p_2=0}^{K\hat{k}} \sum_{\hat{g}\in T(k,l)} \left(\sum_{f_1\in C_{k-p_1}} |\phi^{(p_1)}_{\mathcal{P},k-p_1}(f_1)|^2 \quad \sum_{f_2\in C_{l-p_2}} |{}^{(p_2)}\psi_{\mathcal{P},l-p_2}(f_2)|^2 \right)$$

which is a sum of the same terms but over a possibly larger set. We bound the last sum above by

$$\begin{aligned} & (K\hat{k}+1)^2 P_2(\hat{k})P_3(\hat{k}) \sum_{p_1,p_2=0}^{K\hat{k}} |\phi^{(p_1)}_{\mathcal{P},k-p_1}||_2^2||^{(p_2)}\psi_{\mathcal{P},l-p_2}||_2^2 \\ \leq\ & (K\hat{k}+1)^2 P_2(\hat{k})P_3(\hat{k}) \sum_{p_1,p_2=0}^{K\hat{k}} \mathrm{F}_{\mathcal{P},k-p_1,p_1}\mathrm{F}_{\mathcal{P},p_2,l-p_2}||\phi_k||_2^2||\psi_l||_2^2 \\ \leq\ & (K\hat{k}+1)^2 P_2(\hat{k})P_3(\hat{k}) \sum_{p_1,p_2=0}^{K\hat{k}} P_1(\min(k-p_1,p_1))P_1(\min(p_2,l-p_2))||\phi_k||_2^2||\psi_l||_2^2, \end{aligned}$$

using Lemma 4.2 to relate the l^2-norms. Since $\min(k-p_1,p_1)$ and $\min(p_2,l-p_2)$ above are bounded by $K\hat{k}$, condition $(**)$ now follows easily. $\square$

5 Notation for Artin groups

Our Artin groups will be defined in their standard presentations

$$\langle x_1,\ldots,x_n \mid {}_{m_{ij}}(x_i,x_j) = {}_{m_{ij}}(x_j,x_i) \quad \text{for each} \quad i\neq j\rangle.$$

We prove rapid decay for Artin groups satisfying the hypotheses of Theorem 1.1 with respect to the word length metric, by verifying the conditions D1 and D2 above. In order to do that we need to examine the structure of geodesics in these groups; we build on the results of [17].

Given an Artin group G as before we let X be the set of generators in the standard presentation and let A be the set $X\cup X^{-1}$; we call the elements of A *letters*. We shall generally use symbols like x,y,z,t for generators in X and a,b for letters in A. A letter is *positive* if it is a generator, *negative* otherwise. For each $x\in X$, we define x to be the *name* of the two letters x and x^{-1}. We say that a word $w\in A^*$ *involves* the generator x if w contains a letter with name x, and we call w a *2-generator* word if it involves exactly two of the generators. Words in A^* will be denoted by u,v,w (possibly with subscripts) or $\alpha,\beta,\gamma,\eta,\xi$. (Roughly speaking, the difference is that u,v,w will be used for interesting subwords of a specified word, and the Greek letters for subwords in which we are not interested.) A *positive word* is one in X^* and a *negative word* one in $(X^{-1})^*$; otherwise it is *unsigned*. For $u,v\in A^*$, $u=v$ denotes equality as words, whereas $u=_G v$ denotes equality within the Artin group. The length of the word w is denoted by $|w|$, while as above $|w|_G$ denotes the length of a geodesic representative and, for $g\in G$, $|g|$ denotes the length of a geodesic word representing g. A *syllable* in a word is a subword that is a power of a single generator, and is not a subword of a higher power.

For any distinct letters a and b and a positive integer r, we define alternating products ${}_r(a,b)$ and $(b,a)_r$. The product ${}_r(a,b)$, is defined, as it was earlier (over generators), to be the word of length r of alternating a's and b's starting with a, while $(b,a)_r$ is defined to be the word of length r of alternating a's and b's ending with a. For example, ${}_6(a,b) = ababab = (a,b)_6$, ${}_5(a,b) = ababa = (b,a)_5$. We define both ${}_0(a,b)$ and $(b,a)_0$ to be the empty word. For any nonempty word w, we define $\mathrm{f}[w]$ and $\mathrm{l}[w]$ to be respectively the first and last letter of

w, and pre$[w]$ and suf$[w]$ to be the maximal proper prefix and suffix of w. So $w = \text{pre}[w]\text{l}[w] = \text{f}[w]\text{suf}[w]$.

6 Dihedral Artin groups

In this section, we prove that dihedral Artin groups of large type satisfy RD by verifying the properties D1 and D2 in these groups. As mentioned in the introduction, the fact that these groups satisfy RD can be deduced immediately from results already in the literature, but the techniques that we develop here are needed in the proof of Theorem 1.1 in Section 7.

6.1 The structure of geodesics in dihedral Artin groups

We need some results on the structure of geodesics from [21, 17]. We summarise what we need, and refer to those articles for details.

Let

$$\text{DA}(m) = \langle x_1, x_2 \mid {}_m(x_1, x_2) = {}_m(x_2, x_1)\rangle$$

be a 2-generator (dihedral) Artin group with $2 \leq m < \infty$. Conjugation by the Garside element

$$\Delta := {}_m(x_1, x_2) =_{\text{DA}(m)} {}_m(x_2, x_1)$$

induces a permutation δ of order 2 or 1 (depending on the parity of m) on the letters in A, and hence an automorphism δ of order 2 or 1 of the free monoid A^*.

Let w be a freely reduced word over $A = \{x_1, x_2, x_1^{-1}, x_2^{-1}\}$. Then we define $p(w)$ to be the minimum of m and the length of the longest subword of w of alternating x_1's and x_2's (that is the length of the longest subword of w of the form ${}_r(x_1, x_2)$ or ${}_r(x_2, x_1)$). Similarly, we define $n(w)$ to be the minimum of m and the length of the longest subword of w of alternating x_1^{-1}'s and x_2^{-1}'s. It is proved in [21, Proposition 4.3] that w is geodesic in DA(m) if and only if $p(w) + n(w) \leq m$; if $p(w) + n(w) < m$, then w is the unique geodesic representative of the group element it defines, but if $p(w) + n(w) = m$ then there are other representatives. Note that a non-geodesic word is always unsigned.

For example, suppose that $m = 3$, so $\text{DA}(m) = \langle x_1, x_2 \mid x_1x_2x_1 = x_2x_1x_2\rangle$. Then $w := x_1x_2x_1$ and $w' := x_2x_1x_2$ are two geodesic representatives of the same element with $p(w) = p(w') = 3, n(w) = n(w') = 0$, and $w = x_1x_2^2x_1^{-1}$ and $w' = x_2^{-1}x_1^2x_2$ are two geodesic representatives of the same element with $p(w) = p(w') = 2, n(w) = n(w') = 1$. In fact all four words are examples of *critical words*, as defined in [17].

As in [17], we define a freely reduced word w with $p(w) + n(w) = m$ to be critical if one of the following holds, where ξ is a subword with the obvious restrictions.

(i) w is unsigned of one of the two forms ${}_p(x, y)\xi(z^{-1}, t^{-1})_n$ or ${}_n(x^{-1}, y^{-1})\xi(z, t)_p$, with $\{x, y\} = \{z, t\} = \{x_1, x_2\}$,

(ii) w is positive of one of the two forms ${}_m(x, y)\xi$ or $\xi(x, y)_m$, and only the one positive alternating subword of length m,

(iii) w is negative of one of the two forms ${}_m(x^{-1}, y^{-1})\xi$ or $\xi(x^{-1}, y^{-1})_m$, and only the one negative alternating subword of length m.

An involution τ on the set of critical words swaps critical words that represent the same element. We define τ by

$$\begin{array}{rcl}
{}_p(x,y)\,\xi\,(z^{-1},t^{-1})_n & \leftrightarrow^\tau & {}_n(y^{-1},x^{-1})\,\delta(\xi)\,(t,z)_p,\\
{}_m(x,y) & \leftrightarrow^\tau & {}_m(y,x),\\
{}_m(x^{-1},y^{-1}) & \leftrightarrow^\tau & {}_m(y^{-1},x^{-1})\\
{}_m(x,y)\,\xi & \leftrightarrow^\tau & \delta(\xi)\,(z,t)_m, \quad \text{where} \quad z = \mathrm{l}[\xi],\ \{x,y\}=\{z,t\},\\
{}_m(x^{-1},y^{-1})\,\xi & \leftrightarrow^\tau & \delta(\xi)\,(z^{-1},t^{-1})_m, \quad \text{where} \quad z = \mathrm{l}[\xi]^{-1},\ \{x,y\}=\{z,t\}.
\end{array}$$

Where a critical word w' occurs as a subword of a word w, we call the substitution of the subword w' by $\tau(w')$ a τ-move on w.

The following lemma is proved in [17, Lemma 2.3].

Lemma 6.1. *Suppose that $w \in A^*$ is geodesic and $a \in A$. If wa is non-geodesic, then either $\mathrm{l}[w] = a^{-1}$ or w has a critical suffix v such that $\mathrm{l}[\tau(v)] = a^{-1}$. Similarly, if aw is non-geodesic, then either $\mathrm{f}[w] = a^{-1}$ or w has a critical prefix v such that $\mathrm{f}[\tau(v)] = a^{-1}$.*

It is proved in [17] that, whenever w is a freely reduced word that is not minimal under the shortlex ordering, w has a factorisation as $w_1w_2w_3$, where w_2 is critical and either $w_1\tau(w_2)w_3 <_{\mathrm{lex}} w$ or $w_1\tau(w_2)w_3$ is not freely reduced. In that case, we call the τ-move on w that replaces w_2 by $\tau(w_2)$ together with any subsequent free reduction within $w_1\tau(w_2)w_3$ a critical reduction of w. It follows from [17, Theorem 2.4] that a succession of critical reductions reduces any word to its shortlex minimal representative. Further, any two geodesic representatives are related by a sequence of τ-moves. In particular this implies that two representatives of the same group element must either both be signed or both unsigned.

In order to deal with the reduction of non-geodesic words, we extend the concept of τ-moves.

A freely reduced subword u of a word w is said to be *over-critical* if it is has either of the forms

$$_p(x,y)\xi(z^{-1},t^{-1})_n \quad \text{or} \quad {}_n(x^{-1},y^{-1})\xi(z,t)_p,$$

with $p,n \le m$, $p+n > m$ and $\{x,y\} = \{z,t\} = \{x_1,x_2\}$, and the extra condition that, if $p < m$, then ${}_p(x,y)$ or $(z,t)_p$ is a maximal positive alternating subword of w and if $n < m$ then $(z^{-1},t^{-1})_n$ or ${}_n(x^{-1},y^{-1})$ is a maximal negative alternating subword of w.

Note that we do not require that $p = p(w)$ or $n = n(w)$. But since the conditions on p,n force $p(w)+n(w) > m$, over-critical words are necessarily non-geodesic.

We define τ on over-critical words by

$$\begin{array}{rcl}
\tau({}_p(x,y)\,\xi\,(z^{-1},t^{-1})_n) & := & {}_{m-p}(y^{-1},x^{-1})\,\delta(\xi)\,(t,z)_{m-n},\\
\tau({}_n(x^{-1},y^{-1})\,\xi\,(z,t)_p) & := & {}_{m-n}(y,x)\,\delta(\xi)\,(t^{-1},z^{-1})_{m-p}.
\end{array}$$

We refer to these as length reducing τ-moves. Note that any such move can also be achieved by an ordinary τ-move followed by free reduction. So it follows from Lemma 6.1 that any word can be reduced to a geodesic by a sequence of length reducing τ-moves and free reduction. We call the move *positive* if $p = m$, *negative* if $n = m$ and *unsigned* otherwise. (So if $p = n = m$ it is both positive and negative, but we won't need to use that case.)

Lemma 6.2. *Let w be freely reduced with $0 < p(w), n(w) < m$. Then w can be reduced to a geodesic using a sequence of unsigned length reducing τ-moves alone (i.e. with no free reduction).*

Proof. This follows from lemma 6.1 together with the fact that applying an unsigned length reducing τ-move to w results in a freely reduced word, and does not increase $p(w)$ or $n(w)$. $\square$

We shall also need the following lemma when we come to verify D2 for Artin groups of extra large type.

Lemma 6.3. *Let $g, h \in DA(m)$. If $gh \in \langle x_i \rangle$, then g and h have geodesic factorisations $x_i^s w$ and $w^{-1}x_i^t$, for some element w and integers s, t.*

Proof. The result is trivial if either g or h (and hence both) is in $\langle x_i \rangle$, so we suppose not.

Suppose that $g = a^k h^{-1}$ with $a = x_i^{\pm 1}$ and $k \geq 0$. Suppose first that $|ah^{-1}| = 1 + |h|$. Then $|g| = k + |h|$ and the result is clear. Otherwise, by Lemma 6.1, a^{-1} is a left divisor of h^{-1}. Let l be maximal such that a^{-l} is a left divisor of h^{-1}, and let $h^{-1} = a^{-l}w$. Then $a^{k-l}w$ is a geodesic factorisation of g, and the result follows. $\square$

6.2 Verifying D1 and D2 for dihedral Artin groups

Our aim in this section is to verify that the properties D1 and D2 hold in any dihedral Artin group G. In fact, it can be shown that the kernel of the natural homomorphism of $\mathrm{DA}(m)$ onto the dihedral group of order $2m$ in which the images of x_1 and x_2 have order 2 is a direct product of an infinite cyclic group and a free group of rank $m-1$. The fact that $\mathrm{DA}(m)$ has rapid decay then follows from [18, Propositions 2.1.5, 2.1.9], so we do not need to verify D1 and D2 in order to prove rapid decay in these groups. However we need the properties here in order to prove that corresponding properties hold for Artin groups satisfying the hypotheses of Theorem 1.1.

We shall assume throughout this section that $m \geq 3$. We assume also that $k \leq l$, and deduce D1 and D2 in that case; the case $k \geq l$ then follows immediately by symmetry.

We first need to define our set $\mathcal{P}$ of permissible geodesic factorisations of elements $g \in G$. For unsigned elements g, we define $\mathcal{P}(g)$ to be the set all geodesic factorisations of g; that is,

$$\mathcal{P}(g) := \{(g_1, g_2) \in G^2 : g_1 g_2 = g,\ |g_1| + |g_2| = |g|\}.$$

If m is infinite (that is, the 2-generator group is free) then for any element g we define $\mathcal{P}(g)$ to be its set of geodesic factorisations. From now on we shall assume that $m < \infty$.

Unfortunately, for $m < \infty$, if we adopt the above definition of $\mathcal{P}(g)$ for signed elements g, then D1 does not hold, so we are forced to use a more restrictive definition, which significantly increases the technical complications in the proofs.

For a positive (respectively negative) element g, we define $d(g)$ to be the maximal k such that Δ^k (respectively Δ^{-k}) is a divisor of g. Then we call a geodesic factorisation $g_1g_2 = g$ of g Δ-*decreasing* if $d(g_1) + d(g_2) < d(g)$. We define $\mathcal{P}(g) := \mathcal{P}^1(g) \cup \mathcal{P}^2(g)$, where $\mathcal{P}^1(g)$ is the set of factorisations g_1g_2 for which at least one of g_1, g_2 is represented by geodesic words with at most two syllables (i.e. either g_1 or g_2 equals a^sb^t with $a, b \in A$, $s, t \geq 0$), and $\mathcal{P}^2(g)$ is the set of geodesic factorisations of g that are not Δ-decreasing. That is,

$$\mathcal{P}^2(g) := \{(g_1, g_2) \in G^2 : g_1g_2 = g,\ |g_1| + |g_2| = |g|,\ d(g_1) + d(g_2) = d(g)\}.$$

(In fact we could omit the factorisations in $\mathcal{P}^1$ and still obtain D1 and D2 in the dihedral case, but we will need them in the next section in order to prove the conditions when there are more than two generators; that proof is easier if we include those factorisations already for the dihedral case.) We say that a left (respectively right) divisor h of g lies in $\mathcal{P}^i_l(g)$ (respectively $\mathcal{P}^i_r(g)$) for $i = 1$ or 2 if $(h, h^{-1}g) \in \mathcal{P}^i(g)$ (respectively $(gh^{-1}, h) \in \mathcal{P}^i(g)$), and let $\mathcal{P}_l(g) := \mathcal{P}^1_l(g) \cup \mathcal{P}^2_l(g)$ and $\mathcal{P}_r(g) := \mathcal{P}^1_r(g) \cup \mathcal{P}^2_r(g)$.

Our first aim is to prove Property D1. In order to do that we need to examine the set of geodesic words representing a given element. Let w be a geodesic word with $p(w) + n(w) = m$; we write $p := p(w), n := n(w)$. We have seen that w can be reduced to its shortlex normal form by a sequence of lex-reducing critical reductions. It follows that if w' is another geodesic word with $w =_G w'$, then w can be transformed to w' by a sequence of τ-moves.

First suppose that $p, n > 0$. The word w has the form $\eta_0 w_1 \eta_1 \cdots w_s \eta_s$, where each w_i has the form $(x, y)_p$ or $(x^{-1}, y^{-1})_m$ with $\{x, y\} = \{x_1, x_2\}$, and $p(\eta_i) < p$, $m(\eta_i) < m$ for all i. A τ-move has the effect of changing the signs of w_i, w_j for some $i < j$ for which w_i, w_j have opposite signs, and replacing the subword between them by its image under δ. Denote this τ-move by (i, j). So $w' = \eta'_0 w'_1 \eta'_1 \cdots w'_s \eta'_s$, where each $\eta'_r = \eta_r$ or $\delta(\eta_r)$, and all of the subwords w_r, w'_r are maximal alternating. But note that although $|\eta'_r| = |\eta_r|$, it is not necessarily true that $|w_r| = |w'_r|$.

Lemma 6.4. *Let w, w' be geodesic words as above. If, for some r with $1 \leq r \leq s$, the number of positive w_i with $1 \leq i \leq r$ is equal to the number of positive w'_i with $1 \leq r \leq s$, then $\eta'_0 w'_1 \eta'_1 \cdots w'_r =_G \eta_0 w_1 \eta_1 \cdots w_r$ and $\eta_r = \eta'_r$.*

Proof. This is by induction on r. Since none of the τ-moves changes η_0, we have $\eta_0 = \eta'_0$, and so the result is true for $r = 0$.

So suppose that $r \geq 1$. Since w_1 can only be changed by a transformation $(1, i)$ with $1 < i$, there are only two possible w'_1, one of which is w_1 and the other a word of opposite sign to w_1. So if $\mathrm{Sign}(w_1) = \mathrm{Sign}(w'_1)$, then $w_1 = w'_1$ and the result follows by induction. Otherwise, we have $\{w_1, w'_1\} = \{{}_p(x, y),\ {}_m(y^{-1}, x^{-1})\}$. From the hypothesis, there must be both positive and negative w_i with $i \leq r$, so there is a w'_i with $2 \leq i \leq r$ and $\mathrm{Sign}(w'_i) = -\mathrm{Sign}(w'_1)$. We can do the move $(1, i)$ to w' giving w''. Such a move does not change the group element $\eta'_0 w'_1 \eta'_1 \cdots w'_r$ or η'_r, but it does change w'_1 back to w_1, so the result follows by induction applied to w, w''. $\square$

For a positive element g (the negative case is similar), let $r = d(g)$. Then, for a geodesic factorisation $g = g_1g_2$ in $\mathcal{P}^2(g)$, we have $d(g_1) = s$ and $d(g_2) = r - s$ for some $0 \leq s \leq r$, and then g_1 has Δ^s as a left divisor and g_2 has Δ^{r-s} as a right divisor.

Lemma 6.5. *Let w, w' be positive geodesic words representing $g \in G$ such that w and w' both have Δ^s as a prefix and Δ^{r-s} as suffix for some s, where $r = d(g)$. Then $w = w'$.*

Proof. We have $w = \Delta^s \eta \Delta^{r-s}$, $w' = \Delta^s \eta' \Delta^{r-s}$ with $\eta =_G \eta'$. Since $d(g) = r$, we have $d(\eta) = d(\eta') = 0$ so by [21, Proposition 4.3] η_G has a unique geodesic representative, and hence $\eta = \eta'$ and $w = w'$. □

Lemma 6.6. *Let $g \in G$ be a positive element and let $l \geq 0$. Then, for $a, b \in A$ with $\{a, b\} = \{x_1, x_2\}$, the number of right divisors of g of the form $a^s b^t$ with $s + t = l$ is at most $d(g) + 1$.*

Proof. The proof is by induction on l, the case $l = 0$ being trivial. We may suppose that a^l is a right divisor of g, since otherwise all right divisors of g of the required form end in b, and the result follows from the case $l - 1$ applied to gb^{-1}. Similarly, we may assume that b^l is a right divisor of g, since otherwise the suffixes of the required form all begin with a, and the result follows from the case $l - 1$ applied to g.

Let $d = d(g)$. We may assume that $d > 0$, since otherwise g has a unique geodesic representative. It is straightforward to prove by induction on t that, for any $t > 0$, the group element $g_{a,t} := \Delta^t a^{-t}$ is positive with $d(g_{a,t}) = 0$ and $\mathrm{l}[g_{a,t}] = b$ (that is, the unique geodesic representative ends in b); in fact the unique geodesic representative is a concatenation of positive alternating words each of length $m - 1$. Similarly $g_{b,t} = \Delta^t b^{-t}$ has $d(g_{b,t}) = 0$ with $\mathrm{l}[g_{b,t}] = a$ and, since $g_{b,t}$ is obtained from $g_{a,t}$ by interchanging a and b, we have $\mathrm{f}[g_{a,t}] \neq \mathrm{f}[g_{b,t}]$. We write $g = g'\Delta^d$ with $d(g') = 0$, and observe that the element $g' g_{a,d} = ga^{-d}$ is divisible by Δ if and only if g' ends in a positive letter other than $\mathrm{f}[g_{a,t}]$, and similarly for the element $g' g_{b,d} = gb^{-d}$; hence exactly one of the two group elements $g' g_{a,d}$ and $g' g_{b,d}$ is not divisible by Δ. If $g' g_{a,d} = ga^{-d}$ is not divisible by Δ, then its unique geodesic representative ends in $\mathrm{l}[g_{a,d}] = b$; ga^{-d} cannot have a as a right divisor, and g cannot have a^{d+1} as a right divisor; Similarly we see that if $g' g_{b,d}$ is not divisible by Δ, then g cannot have b^{d+1} as a right divisor. Hence g cannot have both a^{d+1} and b^{d+1} as right divisors. It now follows from the preceding paragraph that $l \leq d$, and then the result follows immediately, by counting the number of pairs (s, t) with $s + t = l$. □

We can now prove D1, which we can express as follows:

Corollary 6.7. *There is a polynomial $P(x)$ with the following property: for any $g \in G$ the number of left divisors of g of length k in $\mathcal{P}_l(g)$ is bounded above by $P(k)$.*

Proof. Suppose that $g \in G$ has length $k + l$. We need to bound the number of group elements u_G for which $w = uv$ is a geodesic word representing g, with $|u| = k, |v| = l$ and $(u_G, v_G) \in \mathcal{P}$.

First suppose that $w = uv$ is unsigned, of the form $\eta_0 w_1 \eta_1 \cdots w_s \eta_s$, as above. If one of the subwords w_i of w intersects both u and v, then let u_1 be the prefix of w that ends at the end of w_i; otherwise let $u_1 = u$.

Suppose that u_1 contains r_p positive subwords w_i and r_n negative subwords w_i. The group element $(u_1)_G$ is determined by those two parameters by Lemma 6.4. Since each of r_p, r_n must be in the range $[0, k]$, there are at most k^2 choices for $(u_1)_G$ corresponding to unsigned geodesics w. Since u differs from

u_1 by an alternating subword of length less than m (and there are less than $4m$ such words), there are at most $4mk^2$ choices for u_G that correspond to unsigned geodesic representatives w.

Now suppose that $w = uv$ is signed. Suppose first that $(u, v) \in \mathcal{P}^1$, so that either u or v has at most 2 syllables. There are at most $2(k + 1)$ possible words u with at most two syllables. By Lemma 6.6, there are at most $2(d(g) + 1)$ possible words v with at most two syllables. If the signed element g has the geodesic factorisation $g = g'a^sb^t$ then, by collecting powers of Δ that divide g to the left, we see that at least $m - 2$ of the letters in each occurrence of Δ in g must come from g', and hence $|g'| \geq (m - 2)d(g) \geq d(g)$. So, if there are factorisations in which v has two syllables, then $k \geq d(g)$. So there are at most $4(k + 1)$ such pairs (u, v).

Now assume that $(u, v) \in \mathcal{P}^2$. Then $u =_G \Delta^s u'$ and $v =_G v'\Delta^{d(g)-s}$ for some s, where $u'v'$ is not divisible by δ, and by Lemma 6.5 $u'v'$ is completely determined by g, s. Further $u'v'$ is the unique geodesic representative of $(u'v')_G$. Hence (given g, k), (u_G) is completely determined by $s \in [0, k]$. The number of such factorisations is therefore at most $k + 1$. □

In order to verify the condition D2, we first describe a process that we call *merging*, which we can use to define the set $S(g, k, l)$ that appears in that condition.

Given elements g_1, g_2 of length k, l whose product g has length less than $k + l$, an application of the merging process results in a triple (f_1, Δ^r, f_2) of elements, such that for some h_1, h_2 with $h_1h_2 =_G \Delta^r$, f_1h_1 and h_2f_2 are geodesic factorisations of g_1, g_2, respectively, and furthermore $(f_1, h_1), (h_2, f_2) \in \mathcal{P}$. We call such a triple a *merger* of g_1 and g_2; we do not claim or need g_1, g_2 to have a unique merger (although we suspect that it does). Then we define the set $S(g, k, l)$ to be the set of all triples (f_1, Δ^r, f_2) that arise as mergers of pairs of elements g_1, g_2 of lengths k, l and with $g_1g_2 =_G g$.

We compute a merger of (g_1, g_2) as the last term of a sequence of triples $(g_1^{(t)}, \Delta^{r_t}, g_2^{(t)})$, defined as follows. When one or both of g_1, g_2 is a signed word, we have to be careful to ensure that the resulting geodesic factorisations of g_1 and g_2 lie in $\mathcal{P}$. We set $g_1^{(1)} := g_1$, $r_1 := 0$, $g_2^{(1)} := g_2$.

Now, the t-th step of the merging process computes $g_1^{(t+1)}, r_{t+1}, g_2^{(t+1)}$ as follows. In each of the three situations below, we choose non-identity group elements h, h', and then define $g_1^{(t+1)} := g_1^{(t)}h^{-1}$ and $g_2^{(t+1)} := h'^{-1}g_2^{(t)}$ such that $g_1^{(t+1)}h$ and $h'g_2^{(t+1)}$ are geodesic factorisations of $g_1^{(t)}$ and $g_2^{(t)}$ respectively, with $g_1^{(t+1)} \in \mathcal{P}_l(g_1)$ and $g_2^{(t+1)} \in \mathcal{P}_r(g_2)$.

(i) If we can choose h, h' with $h\delta^{r_t}(h') = 1$, then we do so with h, h' as long as possible, and put $r_{t+1} := r_t$ (we call this a cancellation move).

(ii) Otherwise, if $g_1^{(t)}$ and $g_2^{(t)}$ are both signed words, and we can choose $h = h' = \Delta^\epsilon$ with $\epsilon = \pm 1$, then do so, and put $r_{t+1} := r_t + 2\epsilon$.

(iii) Otherwise, if we can choose h, h' with $h\delta^{r_t}(h') = \Delta^\epsilon$ with $\epsilon = \pm 1$, then do so and put $r_{t+1} := r_t + \epsilon$ (we call this a Δ-extraction move).

If for some t, no pair of elements h, h' satisfy any of these three conditions, then merging is complete, and we output $(f_1, \Delta^r, f_2) := (g_1^{(t)}, \Delta^{r_t}, g_2^{(t)})$ as a merger of (g_1, g_2). Note that we needed the conditions at each previous step that $g_1^{(t+1)} \in \mathcal{P}_l(g_1)$ and $g_2^{(t+1)} \in \mathcal{P}_r(g_2)$ to ensure that (f_1, Δ^r, f_2) possesses all the required properties of a merger of (g_1, g_2).

Lemma 6.8. *Suppose that g_1, g_2 are elements of length k, l respectively, where $k \leq l$, and that (f_1, Δ^r, f_2) is a merger of (g_1, g_2); Let $h_1 := f_1^{-1} g_1$ and $h_2 := g_2 f_2^{-1}$. Then $r \leq k$ and $|h_1|, |h_2| \leq (m-1)k$.*

Proof. It is clear that the process above completes in at most k stages, giving the required bound on r. At each stage of the merging process, the ratio of the lengths of the elements h, h', stripped from $g_1^{(t)}$ and $g_2^{(t)}$ to give $g_1^{(t+1)}$ and $g_2^{(t+1)}$, is in the interval $[1/(m-1), m-1]$. Hence the same is true for the total lengths of elements stripped off, that is, $1/(m-1) \leq |h_1|/|h_2| \leq m-1$. Since $|h_1| \leq |g_1| = k$ and $h_2 \leq |g_2| = l$, we get $|h_1|, |h_2| \leq (m-1)k$, as claimed. □

The definition of the merging process ensures that, for a merger (f_1, Δ^r, f_2) of (g_1, g_2), we have $(f_1, h_1), (h_2, f_2) \in \mathcal{P}$. So, in order to verify that D2 holds, we just need to find polynomial bounds on $S(g, k, l)$ and the associated set $T(k, l)$ of powers of Δ, and a value for the constant K.

The above lemma already bounds $|T(k,l)|$ by $2k+1$ and K by $m-1$, so it remains to find a polynomial bound on $S(g,k,l)$. In order to do this we need to explore a reduction process, which we call *compression*, that starts with a triple $(f_1, \Delta^r, f_2) \in S(g,k,l)$ and produces a geodesic representative of g. Reversing the compression process will then enable us to estimate the size of $S(g,k,l)$.

So suppose that $(f_1, \Delta^r, f_2) \in S(g,k,l)$. The fact that the merging process has completed at (f_1, Δ^r, f_2) ensures that f_1, f_2 cannot both have powers of Δ as divisors and that, if one of the two elements has such a divisor Δ^{r_0} and $r, r_0 \neq 0$, then r, r_0 cannot have opposite signs. Hence we see that f_1, f_2 have geodesic representatives of the form $u\Delta^{r_0}, v$ or $u, \Delta^{r_0} v$ for some r_0 (possibly zero), where u, v are not divisible by $\Delta^{\pm 1}$. Then $g_1 \Delta^r g_2$ is represented by $u\Delta^{r_1} v =_G u\delta^{r_1}(v)\Delta^{r_1}$, where $r_1 = r + r_0$ has the same sign as r, and (since the merging process terminated), $u\delta^{r_1}(v)$ is freely reduced and also contains no powers of Δ. Then, by Lemma 6.2, if non-geodesic, $u\delta^{r_1}(v)$ can be reduced to a geodesic word using only unsigned length reducing τ-moves.

We describe that reduction more precisely as follows. We define u_1 to be the shortest prefix of $u\delta^{r_1}(v)$ that contains u and ends at the end of a maximal alternating subword, and define v_1 to be the remainder of the word. Then v_1 is geodesic, and we set $v_2 := v_1$. If u_1 is also geodesic, then we set $u_2 := u_1$, but otherwise we set u_2 to be a geodesic word derived from u_1 by a single length reducing τ-move. We set $u_2^{(1)} := u_2, v_2^{(1)} := v_2$.

We can now reduce $u_2^{(1)} v_2^{(1)}$ to geodesic form through a series of words $u_2^{(t)} v_2^{(t)}$ using a sequence of length reducing τ-moves, the i-th of which involves one alternating subword within $u_2^{(t)}$ and one within $v_2^{(t)}$. Specifically, we derive $u_2^{(t+1)}, v_2^{(t+1)}$ from $u_2^{(t)}, v_2^{(t)}$ by either replacing a suffix ${}_p(x,y)\,\xi_1$ in $u_2^{(t)}$ and a prefix $\xi_2\,(z^{-1}, t^{-1})_n$ in $v_2^{(t)}$ by a suffix ${}_{m-p}(y^{-1}, x^{-1})\,\delta(\xi_1)$ in $u_2^{(t+1)}$ and a prefix $\delta(\xi_2)\,(t, z)_{m-n}$ in $v_2^{(t+1)}$ or by replacing a suffix ${}_n(x^{-1}, y^{-1})\,\xi_1$ in $u_2^{(t)}$ and a prefix $\xi_2\,(z, t)_p$ in $v_2^{(t)}$ by a suffix ${}_{m-n}(y, x)\,\delta(\xi_1)$ in $u_2^{(t+1)}$ and a prefix $\delta(\xi_2)\,(t^{-1}, z^{-1})_{m-p}$ in $v_2^{(t+1)}$. Eventually this process produces a geodesic word $u_3 v_3$.

Next we need to consider the reduction to geodesic form of $u_3 v_3 \Delta^{r_1}$. We do this in two stages. First we reduce $v_3 \Delta^{r_1}$ to $v_4 \Delta^{r'}$, using a sequence of length reducing τ-moves, each of which involves a single $\Delta^{\pm 1}$ and an alternating subword of the opposite sign within v_3 (or the word derived from it); at the end of this stage, either $v_4\Delta^{r'}$ is signed or $r' = 0$.

Now if $r' = 0$ we set $u_4 := u_3$, but otherwise, we apply a further sequence of length reducing τ-moves to $u_3\Delta^{r'}$, each move involving a single $\Delta^{\pm 1}$ and an alternating subword of the opposite sign in the word derived from u_3', and hence reduce $u_3\Delta^{r'}$ to $u_4\Delta^s$. The fact that $u_4\delta^{r'-s}(v_4)\Delta^s$ is now a geodesic representative of $u_3v_3\Delta^{r_1}$ follows from the two equations $u_3v_4\Delta^{r'} =_G u_3\Delta^{r'}\delta^{r'}(v_4)$ and $u_4\delta^{r'-s}(v_4)\Delta^s =_G u_4\Delta^s\delta^{r'}(v_4)$. Our construction ensures that $u_4\delta^{r'-s}(v_4)$ has no divisor of the form $\Delta^{\pm 1}$. This is the end of compression; we use the name $\kappa(f_1, \Delta^r, f_2)$ for the group element $(u_4)_G$.

We now want to estimate both the number of group elements that can arise as $\kappa(f_1, \Delta^r, f_2)$ out of the compression of at least one element (f_1, Δ^r, f_2) of $S(g, k, l)$, and also the number of triples $(f_1, \Delta^r, f_2) \in S(g, k, l)$ for which $\kappa(f_1, \Delta^r, f_2) = g'$ for a particular group element g'. The product of these two values will give a bound on $|S(g, k, l)|$.

Lemma 6.9. *The size of $\kappa(S(g, k, l))$ is bounded by a polynomial in k.*

Proof. Suppose that compression of (f_1, Δ^r, f_2) leads to $u_4\delta^{r'-s}(v_4)\Delta^s$. We first need to relate $|u_4|$ to $k = |g_1|$. We recall that $|u| \leq |f_1| \leq k$, $|u_1| \leq |u| + m - 1$, and $|u_2| \leq |u_1|$. Then each move from $u_2^{(t)}$ to $u_2^{(t+1)}$ replaces a distinct alternating subword by an alternating subword that is longer by a factor of at most $m - 1$, and doesn't alter the lengths of the subwords before or after that alternating subword. Hence $|u_3| \leq (m-1)|u_2|$. By the same argument $|u_4| \leq (m - 1)|u_3|$. So $|u_4| \leq (m - 1)^2(k + m - 1)$.

Then u_4 is a prefix of length $k' \leq (m - 1)^2(k + m - 1)$ of $w' := u_4\delta^{r'-s}(v_4)$, for which $w'\Delta^s$ is a geodesic representing g and no power of Δ divides w'; it follows that $u_4 \in \mathcal{P}_l(g)$. By Corollary 6.7 we know that, for some polynomial $P_1(x)$, $P_1(k')$ bounds the number of elements of $\mathcal{P}_l(g)$ represented by a word that is a prefix of length k' of some such w. Hence, since k' is bounded by a polynomial in k, the number of group elements represented by any such u_4 with k' in the appropriate range is bounded by a polynomial in k. ☐

Lemma 6.10. *The number of triples (f_1, Δ^r, f_2) in $S(g, k, l)$ for which $\kappa(f_1, \Delta^r, f_2)$ is a particular element $g' \in G$ is bounded by a polynomial in k.*

Proof. We recall that for elements of $S(g, k, l)$, f_1, f_2 are represented either by words $u\Delta^{r_0}, v$ or by words $u, \Delta^{r_0}v$, where u, v have no divisors of the form $\Delta^{\pm 1}$. The triple (f_1, Δ^r, f_2) is completely determined by the above binary choice together with u_G, r, and r_0 in the first case. But r (and r_0 in the first case) is bounded above by k. Hence it is now sufficient to bound the number of choices of u_G. Hence the proof of this lemma is completed by application of the lemma that follows. ☐

Lemma 6.11. *Given $g' \in G$, a polynomial in k bounds the number of group elements u_G associated as above with triples (f_1, Δ^r, f_2) in $S(g, k, l)$ for which $\kappa(f_1, \Delta^r, f_2) =_G g'$.*

Proof. We reverse the steps of compression. Suppose that $w = u_4\delta^{r'-s}(v_4)\Delta^s$ is a word resulting from compression of some element of $S(g, k, l)$, and that $(u_4)_G = g'$.

We first examine the possible geodesics of the form $u_3v_3\Delta^{r'}$ from which w might have been derived during compression by length reducing moves.

If $v_4\Delta^s$ is unsigned, then we must have $u_3 = u_4$, $r' = s$, and $(u_3)_G = (u_4)_G$ Otherwise, if $\delta^{r'-s}(v_4)\Delta^s$ is positive, $r' - s$ might be any non-negative integer bounded above by the number of maximal positive alternating subwords in u_4, and u_3 could be any word derived from u_4 by reversing $r' - s$ length reducing rules using Δ, while if $\delta^{r'-s}(v_4)\Delta^s$ is negative, $s-r'$ might be any non-negative integer bounded above by the number of maximal negative alternating subwords in u_4, u_3 could be a word derived from u_4 by reversing $s - r'$ length reducing rules using Δ^{-1}. We can check that the value of $(u_3)_G$ is determined by $(u_4)_G$ and $r' - s$.

Hence there are at most $(k+1)$ possibilities for $(u_3)_G$ for which $(u_4)_G =_G g'$.

We saw above that if v_3 is unsigned or if it has the opposite sign to r', then $|r'| \leq k$, and so can take up to $2k+1$ values. Otherwise, $v_3\Delta^{r'}$ is a signed word, and then r' is uniquely determined by $(v_3\Delta^{r'})_G$.

We next reverse the length reducing τ-moves that transformed u_2v_2 to u_3v_3. Each reversed move involves replacing a subword ${}_p(x,y)\xi(z^{-1},t^{-1})_n$ or ${}_n(x^{-1},y^{-1})\xi(z,t)_p$ with $0 < p, n < m$ and $p+n < m$ by ${}_{m-p}(y^{-1},x^{-1})\,\delta(\xi)\,(t,z)_{m-n}$ or ${}_{m-n}(y,x)\,\delta(\xi)\,(t^{-1},z^{-1})_{m-p}$, where the left maximal alternating subword is in some $u_2^{(t+1)}$ and the right one in $v_2^{(t+1)}$. Call these transformations of type $(p,-n)$ or $(-n,p)$. Then there are at most m^2 such possible types. The element $(u_2)_G$ depends only on $(u_3)_G$ and the total number of τ-moves of each type. Since there are at most k such τ-moves in total, there are at most k^{m^2} possible elements $(u_2)_G$ for a given $(u_3)_G$.

And u_1 represents the same element as u_2. A final reversal of a τ-move on the final maximal alternating suffix of u_2 recovers the word u_1, which represents the same group element as u_2. And the deletion of an alternating suffix of length at most $m-1$ from u_1 yields u; hence each choice of $(u_1)_G$ yields at most m choices of u. □

The combination of Lemma 6.9 and Lemma 6.10 gives the required polynomial bound on $S(g,k,l)$. Hence the proof of D2 for dihedral Artin groups is complete.

7 Artin groups of large type

Our aim in this section is to prove that Artin groups satisfying the hypotheses of Theorem 1.1 satisfy D1 and D2, and hence have rapid decay. Much of what we say is true for any large-type Artin group. But occasionally we shall need to restrict the groups we consider to those satisfying the conditions of Theorem 1.1. For those results where this is the case, we shall make it clear in the statement of the result; the remaining results are proved for all Artin groups of large type.

We assume throughout this section that G is an Artin group of large type, with notation as defined in Section 5. We may assume that not all m_{ij} are infinite, for otherwise the group is free, and rapid decay is known. For any distinct pair of generators x_i, x_j, we let $G(i,j) = G(j,i)$ be the subgroup of G generated by x_i and x_j. We use Δ_{ij} to denote $(x_i,x_j)_{m_{ij}}$, and δ_{ij} to denote the permutation of $\{x_i, x_i^{-1}, x_j, x_j^{-1}\}$ induced by conjugation by Δ_{ij}.

The process of reducing words in G to shortlex minimal form is described in [17, Proposition 3.3]. In [17, Section 3], a leftward or rightward *critical sequence* is defined as a sequence of τ-moves applied to a word, in which successive moves overlap in a single letter. For a geodesic word whose maximal proper prefix is already shortlex minimal, but which is not itself lexicographically minimal, a lexicographic reduction to a shortlex minimal word can be achieved by a single leftward critical sequence, known as a *leftward lex reducing sequence*. Similarly, for a non-geodesic word whose maximal proper prefix is shortlex minimal, a length reduction to a shortlex minimal word can be achieved by a single rightward critical sequence followed by a single free cancellation; the combination is known as a *rightward length reducing sequence*.

For example, with $m_{12}, m_{13}, m_{23} = 3, 4, 5$ and writing a, b, c for x_1, x_2, x_3:

$$\begin{gathered}\alpha(\mathbf{ab^{-2}a^{-1}})cb^2c^{-1}b^{-1}aca^2ca^{-1},\\ \alpha b^{-1}a^{-2}(\mathbf{bcb^2c^{-1}b^{-1}})aca^2ca^{-1},\\ \alpha b^{-1}a^{-2}c^{-1}b^{-1}c^2b(\mathbf{caca^2c})a^{-1},\\ \alpha b^{-1}a^{-2}c^{-1}b^{-1}c^2bac^2ac(\mathbf{aa^{-1}}),\\ \alpha b^{-1}a^{-2}c^{-1}b^{-1}c^2bac^2ac,\end{gathered}$$

where α is an arbitrary word, is a rightward length reducing sequence in which the critical words to which τ-moves are applied are bracketed and printed in boldface, and the final move is the free reduction of aa^{-1}.

7.1 The geodesics in Artin groups of large type

We shall need some results about geodesics in Artin groups of large type. The first of these is proved in [17].

Proposition 7.1 ([17, Proposition 4.5])**.** *Suppose that v, w are any two geodesic words representing the same group element, and that $\mathrm{l}[v] \neq \mathrm{l}[w]$. Then:*

(1) $\mathrm{l}[v]$ and $\mathrm{l}[w]$ have different names;

(2) The maximal 2-generator suffixes of v and w involve generators with names equal to those of $\mathrm{l}[v]$ and $\mathrm{l}[w]$;

(3) Any geodesic word equal in G to v must end in $\mathrm{l}[v]$ or in $\mathrm{l}[w]$.

Corresponding results apply if $\mathrm{f}[v] \neq \mathrm{f}[w]$.

Corollary 7.2. *If wa is a geodesic word for some $a \in A$ and $k > 1$, then wa^k is a geodesic word for all $k > 1$.*

Proof. Otherwise, choosing k minimal with wa^k non-geodesic, the group element wa^{k-1} has geodesic representatives ending both in a and in a^{-1}, contradicting Proposition 7.1 (1). □

Proposition 7.3. *Let $g \in G$ and $x_i, x_j \in X$ with $i \neq j$. Then g has a unique left divisor $\mathrm{LD}_{ij}(g) \in G(i,j)$ of maximal length. Furthermore, if w is any geodesic word representing g, and u is the maximal $\{x_i, x_j\}$-prefix of w, then $\mathrm{LD}_{ij}(g) =_G ua^r$ for some $r \neq 0$ with $a \in \{x_i^{\pm 1}, x_j^{\pm 1}\}$ and $|\mathrm{LD}_{ij}(g)| = |u| + r$.*

Similarly, g has a unique right divisor $\mathrm{RD}_{ij}(g) \in G(i,j)$ of maximal length, to which the corresponding results apply.

Proof. Order the monoid generators of G such that $x_i^{\pm 1}, x_j^{\pm 1}$ (in some order) come first, and let w' be the shortlex least representative of g, using this ordering, with maximal $\{x_i, x_j\}$-prefix u'.

Let w be an arbitrary geodesic representative of g and let u be the maximal $\{x_i, x_j\}$-prefix of w. We consider the process of reducing w to its shortlex form w' by considering each letter of w in turn, and reducing the prefix ending in that letter to shortlex form. Suppose that $w_0 = w$, that w reduces through the sequence of words $w_1, w_2, \ldots$ to $w_n = w'$ and that $u_0 = u, u_1, \ldots, u_n = u'$ is the corresponding sequence of maximal $\{x_i, x_j\}$-prefixes.

By [17, Proposition 3.3], the prefix of length k in w_k is either already shortlex reduced, or can be reduced to shortlex form with a single leftward lex reducing sequence. Such a reduction cannot change a letter of u_k to a letter with name not in $\{x_i^{\pm 1}, x_j^{\pm 1}\}$, since that would be shortlex increasing. So either $u_{k+1} = u_k$ or u_{k+1} is the shortlex reduction of $u_k a^s$ for some $a \in \{x_i^{\pm 1}, x_j^{\pm 1}\}$ and $s > 0$. By [17, Lemma 3.8(2)], after such a reduction, the names of the next two letters in w_{k+1} after u_{k+1} are the name of a and a generator not in $\{x_i, x_j\}$. So if subsequent reductions occur that increase the length of the maximal i,j-prefix, then they all involve adjoining the same letter a. (We can't adjoin a and then a^{-1} or the word would not be geodesic!) So we have $u' =_G ua^r$ with $|u'| = |u|+r$. as claimed. Since the equation holds for any choice of w, we have proved that $(u')_G$ is the unique longest left divisor of g in $G(i,j)$.

The proof for the maximal right divisor is similar. ☐

Proposition 7.4. *(1) Let v, w be two geodesic words representing the same group element g, with $\mathrm{l}[v] \neq \mathrm{l}[w]$. Then a single rightward critical sequence can be applied to v to yield a word ending in $\mathrm{l}[w]$.*

(2) Let v be a freely reduced non-geodesic word with $v = wa$ with $a \in A$ and w geodesic. Then v admits a rightward length reducing sequence.

Proof. The proof of (1) is by induction on $|v|$, the result being vacuously true when $|v| = 1$. Let x_i and x_j be the names of $\mathrm{l}[v]$ and $\mathrm{l}[w]$, and let $v = v'u$, where u is the longest 2-generator suffix of v. If u has a geodesic representative ending in $\mathrm{l}[w]$, then the result follows from [17, Corollary 2.6], so suppose not. By Proposition 7.3, we have $\mathrm{RD}_{ij}(g) =_G a^r u$ for some $a \in \{x_i^{\pm 1}, x_j^{\pm 1}\}$, and also $\mathrm{RD}_{ij}(g)$ has a geodesic representative ending in $\mathrm{l}[w]$. So, by [17, Lemma 2.8], a single τ-move can be applied to au to give a word ending in $\mathrm{l}[w]$. Now, by inductive hypothesis, a single rightward critical sequence can be applied to v' to yield a word ending in a, and following this by the above τ-move results in the required critical sequence, which completes the proof of (1).

If v is as in (2), then w has a geodesic representative ending in a^{-1}, and the result follows from (1). ☐

In fact the above proposition provides a slightly shorter proof that Artin groups of large type have FFTP than the one in [17].

The next proposition implies that for the groups in Theorem 1.1, for a given g, i and j, the letter a arising in Proposition 7.3 is unique.

Proposition 7.5. *Assume that G satisfies the hypotheses of Theorem 1.1, and let $g \in G$ and $x_i, x_j \in X$ with $i \neq j$. Then either*

(1) every geodesic word representing g has a prefix that represents $LD_{ij}(g)$; or

(2) *there is a unique letter* $a \in \{x_i^{\pm 1}, x_j^{\pm 1}\}$*, and an element* $\mathrm{LD}'_{ij}(g) \in G(i,j)$ *such that, for any geodesic representative* w *with maximal* $\{x_i, x_j\}$*-prefix* u*, we have* $\mathrm{LD}_{ij}(g) =_G ua^r$ *and* $u =_G \mathrm{LD}'_{ij}(g)a^s$*, for* $r, s \geq 0$*.*

In case (1), we define $\mathrm{LD}'_{ij}(g) = \mathrm{LD}_{ij}(g)$*, for future reference.*

Proof. Let $h := \mathrm{LD}_{ij}(g)$ and $h' := h^{-1}g$, and suppose that case (1) does not hold. Then g has at least one geodesic representative whose maximal $\{x_i, x_j\}$-prefix u is a proper left divisor of $\mathrm{LD}_{ij}(g)$. Proposition 7.3 implies that $h =_G ua^r$ with $r > 0$.

If a is not uniquely determined, then g has another geodesic representative with maximal $\{x_i, x_j\}$-prefix v such that $h =_G vb^s$, with $s > 0$ and $b \neq a$. We may assume that the names of a and b are x_i, x_j respectively. Note that, since h has more than one geodesic representative, the vertices corresponding to x_i, x_j in the graph Γ defining G must be joined by an edge (that is $m_{ij} < \infty$).

Now $a^r h' = u^{-1}g$ has a geodesic representative beginning with a and another geodesic representative beginning with a letter with name $x_{i'}$ for some $i' \neq i, j$. By Proposition 7.1, all geodesic representatives of $a^r h'$ have their longest 2-generator prefix in $G(i, i')$, and hence all geodesic representatives of h' must begin with the same letter c, of which the name is $x_{i'}$. Furthermore, any geodesic representative of $\mathrm{LD}_{ii'}(a^r h')$ must contain a critical subword, and similarly $\mathrm{LD}_{ji'}(b^s h')$ contains a critical subword.

From the hypothesis of Theorem 1.1 and the fact that the vertices x_i and x_j are joined by an edge in Γ, it follows that the edges joining vertices $x_i, x_{i'}$ and vertices $x_j, x_{i'}$ cannot both have the label 3. So, for at least one of $\mathrm{LD}_{ii'}(a^r h')$ and $\mathrm{LD}_{ji'}(b^s h')$, it is true that any geodesic representative has at least four syllables. Assume without loss that this is true for $\mathrm{LD}_{ii'}(a^r h')$. So h' has geodesic representatives u_1, u_2, where u_1 has a prefix in $G(i, i')$ with at least three syllables and u_2 has a prefix in $G(j, i')$ with at least two syllables. But then, if c^s is the highest power of c that is a left divisor of h', $c^{-s}h'$ has a geodesic representative beginning with x_i with a 2-generator prefix in $G(i, i')$ and another geodesic representative beginning with x_j, which contradicts Proposition 7.1 (as applied to prefixes of u_1 and u_2).

So a is indeed uniquely determined. We define $\mathrm{LD}'_{ij}(g) := \mathrm{LD}_{ij}(g)a^{-s}$, where a^s is the maximal power of a that is a right divisor of $\mathrm{LD}_{ij}(g)$. ☐

The following example shows that Proposition 7.5 fails without the $(3, 3, m)$-hypothesis. Let $m_{12}, m_{13}, m_{23} = 4, 3, 3$ and write a, b, c for x_1, x_2, x_3. Let g be the group element represented by the geodesic word $w := babacabab$. Then, since the only two geodesic representatives of the suffix $cabab$ of w are $cabab$ and $cbaba$, neither of which starts with $a^{\pm 1}$ or $b^{\pm 1}$, we see that $\mathrm{LD}_{1,2}(g) = baba$. But, by replacing aca by cac, we have $g =_G w_1$ with $w_1 = babcacbab$, and also $g =_G (baba)c(abab) =_G aba(bcb)aba =_G w_2$ with $w_2 = abacbcaba$, so $\mathrm{LD}_{1,2}(g) =_G v_1 a =_G v_2 b$, where v_1 and v_2 are the maximal $\{a, b\}$-prefixes of w_1 and w_2, respectively. A corresponding example can be constructed with $m_{12} = m$ for any m with $3 \leq m < \infty$.

7.2 Verifying D1 and D2 for the Artin groups in Theorem 1.1

We assume for the remainder of the paper that G satisfies the hypotheses of Theorem 1.1: G is an Artin group of large type that satsifies the $(3,3,m)$-hypothesis.

We must first define our set $\mathcal{P}$ of permissible geodesic factorisations (g_1, g_2). We use the notation $\mathcal{P}_{ij}$ to denote the set of permissible geodesic factorisations in the dihedral Artin group on $G(i,j)$, as defined in Subsection 6.2. Recall that $\mathrm{LD}_{ij}(g)$ and $\mathrm{RD}_{ij}(g)$ were defined in Proposition 7.3 as the longest left and right divisors of g in $G(i,j)$. Then we define $\mathcal{P}$ to be the set of geodesic factorisations (g_1, g_2) in G such that for every pair of generators x_i, x_j, $(\mathrm{RD}_{ij}(g_1), \mathrm{LD}_{ij}(g_2)) \in \mathcal{P}_{ij}$.

Our proof of D1 will involve an inductive argument. To make this work properly, we have to investigate in what circumstances we can have a geodesic factorisation $(ag_1, g_2) \in \mathcal{P}$ for some $a \in A$, with $(g_1, g_2) \notin \mathcal{P}$.

Lemma 7.6. *Suppose that $g \in G$, $a \in A$, $|ag| = |g| + 1$, $(ag_1, g_2) \in \mathcal{P}(ag)$ but $(g_1, g_2) \notin \mathcal{P}(g)$.*

(1) Then ag has a geodesic representative that begins with a letter $b \neq a$.

(2) If ag_1 does not lie in any 2-generator subgroup of G, then all geodesic representatives of $\mathrm{LD}'_{ij}(ag)$ (as defined in Proposition 7.5) begin with b, where x_i and x_j are the names of a and b.

Proof. Suppose first that ag_1 lies in a 2-generator subgroup $G(i,j)$. Then $(ag_1, g_2) \in \mathcal{P}_{ij}$ and $(g_1, g_2) \notin \mathcal{P}_{ij}$ implies that the factorisation (g_1, g_2) of g must be Δ_{ij}-decreasing (as defined in Section 6.2), while the factorisation (ag_1, g_2) of ag is not. So ag_1 is divisible by Δ_{ij}, and conclusion (1) is immediate.

So now suppose that ag_1 does not lie in a 2-generator subgroup of G. Then there exist distinct i', j' such that $(\mathrm{RD}_{i'j'}(ag_1), \mathrm{LD}_{i'j'}(g_2)) \in \mathcal{P}_{i'j'}$ and $(\mathrm{RD}_{i'j'}(g_1), \mathrm{LD}_{i'j'}(g_2)) \notin \mathcal{P}_{i'j'}$. Let $h := \mathrm{RD}_{i'j'}(ag_1)$. Since, by Proposition 7.3, $\mathrm{RD}_{i'j'}(g_1)$ is a right divisor of h, we have $|h| > |\mathrm{RD}_{i'j'}(g_1)|$. We are assuming that $ag_1 \notin G(i', j')$, so $|h| < |ag_1|$, and ag_1h^{-1} cannot have a geodesic representative beginning with a, as that would imply that h was a right divisor of g_1. Hence all geodesic representatives of ag_1h^{-1} begin with some letter b with $b \neq a$, and hence ag has a geodesic representative beginning with b, which proves (1).

We observe also that $(\mathrm{RD}_{i'j'}(g_1), \mathrm{LD}_{i'j'}(g_2))$ must be a $\Delta_{i'j'}$-decreasing factorisation, whereas $(h, \mathrm{LD}_{i'j'}(g_2))$ is not, so h must be a signed element divisible by $\Delta_{i'j'}^{\pm 1}$. Hence h has geodesic representatives beginning with letters with names $x_{i'}$ and $x_{j'}$; these are suffices u', v' of geodesic representatives u, v of ag_1. Now if both u' intersects the maximal $\{x_i, x_j\}$-prefix of u and v' intersects the maximal $\{x_i, x_j\}$-prefix of v, we have $\{i, j\} = \{i', j'\}$ and hence $ag_1 \in G(i,j)$, a contradiction. This implies that ag has a geodesic representative in which the maximal $\{x_i, x_j\}$-prefix is a left divisor of ag_1h^{-1}. Then, by Proposition 7.5, $\mathrm{LD}'_{ij}(ag)$ is a left divisor of ag_1h^{-1}; hence, as we showed in the proof of (1), all of its geodesic representatives begin with b, and (2) is proved.

□

In Corollary 6.7, we proved D1 for 2-generator Artin groups. Let P' be a polynomial such that D1 holds with P' for each of the 2-generator subgroups $G(i,j)$ of G.

Lemma 7.7. *Let $g \in G$ with $|g| = m$ and suppose that v, w are geodesic representatives of g with $\mathrm{f}[v] \neq \mathrm{f}[w]$. Suppose that $0 \leq k \leq m$. Let $\mathcal{D}$ be the set of left divisors g_1 of g in $\mathcal{P}_l(g)$ of length k for which all geodesic representatives begin with $\mathrm{f}[v]$. Then $|\mathcal{D}| \leq Q(k) := \sum_{j=1}^{k} P'(j)$.*

Proof. The proof is by induction on m, the result being vacuously true for $m = 1$. Let the names of $\mathrm{f}[v]$ and $\mathrm{f}[w]$ be x_i, x_j, let $h := \mathrm{LD}_{ij}(g)$, and let $h' := \mathrm{LD}'_{ij}(g)$, as defined in Proposition 7.5. Suppose that $g_1 \in \mathcal{D}$, with $g = g_1 g_2$.

If $g_1 \in G(i,j)$, then $g_1 \in \mathcal{P}_l(h)$. It follows immediately from Corollary 6.7 that there are at most $P'(k)$ such choices for g_1.

So now suppose that $g_1 \notin G(i,j)$. Since g_1 has $\mathrm{f}[v]$ as unique left divisor of length 1, whereas g and hence h, by Proposition 7.3, has more than one such left divisor, h cannot be a left divisor of g_1.

Since $g_1 \notin G(i,j)$, Proposition 7.5 implies that h' is a left divisor of g_1, and so a proper divisor of h, and that $h = h'a^r$ for some $r > 0$, $a \in \{x_i^{\pm 1}, x_j^{\pm 1}\}$. But now, since $\mathrm{LD}_{ij}(g_1)$ is certainly a left divisor of g, Proposition 7.5 also implies that $h = \mathrm{LD}_{ij}(g_1)a^s$, for some $s > 0$. We deduce that $\mathrm{LD}_{ij}(g_1) = h'a^t$, with $t \geq 0$. By [17, Lemma 2.8], $h'a$ has more than one left divisor of length 1, and so cannot be a left divisor of g_1, so $t = 0$. Hence $\mathrm{LD}_{ij}(g) = h'$.

Now $h'^{-1}g$ has a geodesic representative with first letter a, whereas $g'_1 := h'^{-1}g_1$ does not. So by Proposition 7.1, all geodesic representatives of g'_1 have the same first letter.

We claim that $g'_1 \in \mathcal{P}_l(h'^{-1}g)$. If not then for some i', j', (g'_1, g_2) is $\Delta_{i'j'}$-decreasing whereas (g_1, g_2) is not, and hence $\mathrm{RD}_{i'j'}(g_1)$ is longer than $\mathrm{RD}_{i'j'}(g'_1)$, Now suppose that vw is a geodesic representative of g_1, with w representing $\mathrm{RD}_{i'j'}(g_1)$. Then by Proposition 7.5 (2) some prefix u of vw represents h'. If u is a prefix of v, then w is a suffix of a representative of g'_1, and $\mathrm{RD}_{i'j'}(g_1)$ is a right divisor of g'_1, yielding a contradiction. So u and w must intersect non-trivially, and hence $|\{i,j\} \cap \{i',j'\}| = 1$. We may suppose that $i = i'$ and $j \neq j'$. Then since (g_1, g_2) is not $\Delta_{i'j'}$-decreasing, the element $\mathrm{RD}_{i'j'}(g_1)$ is divisible by $\Delta_{i'j'}$, and so has a representative that begins with $x_{j'}^{\pm 1}$; hence we can find a geodesic representative of g_1, and then one of g, in which the longest $\{x_i, x_j\}$-prefix is shorter than $|h'|$, contradicting Proposition 7.5 (2).

Now by the inductive hypothesis applied to $h'^{-1}g$, there are at most $Q(k - |h'|) \leq Q(k-1)$ possible g'_1 and hence at most $Q(k-1)$ possible g_1 of length k not in $G(i,j)$. This bounds the total number of g_1 of length k by $Q(k-1) + P'(k) = Q(k)$ for any $k \leq m$, and so completes the inductive step. ☐

Proposition 7.8. *D1 holds with the polynomial P_1 defined by $P_1(n) = 2\sum_{j=0}^{n} Q(j)$, where the set $\mathcal{P}$ and the polynomial Q are as defined above.*

Proof. Let g be an element of length m and choose k with $0 \leq k \leq m$. The proof is by induction on m, the case $m = 1$ being clear.

If all geodesic representatives of g begin with the same letter, then the result follows immediately from the inductive hypothesis and Lemma 7.6 (1). So suppose that g has geodesic representatives v, w with $\mathrm{f}[v] \neq \mathrm{f}[w]$, and let x_i, x_j be the names of $\mathrm{f}[v], \mathrm{f}[w]$. Assume without loss that $\mathrm{LD}'_{ij}(g)$ has a geodesic representative beginning with $\mathrm{f}[w]$.

Let $\mathcal{D}$ be the set of elements in $\mathcal{P}_l(g)$ of length k. Then $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$, where $\mathcal{D}_1$ consists of those elements of $\mathcal{D}$ that lie in $G(i,j)$, $\mathcal{D}_2$ consists of those elements of $D \setminus \mathcal{D}_1$ with at least one geodesic representative beginning with $\mathrm{f}[w]$, and $\mathcal{D}_3$ consists of those elements of $\mathcal{D} \setminus \mathcal{D}_1$ all of whose geodesic representatives begin with $\mathrm{f}[v]$. Then $|\mathcal{D}_1| \le Q(k)$ from the 2-generator case and $|\mathcal{D}_3| \le Q(k)$ by Lemma 7.7.

Suppose that $g_1 \in \mathcal{D}_2$. If $\mathrm{f}[w]^{-1}g_1 \notin \mathcal{P}_l(\mathrm{f}[w]^{-1}g)$, then by Lemma 7.6 (2), all geodesic representatives of $\mathrm{LD}'_{ij}(g)$ begin with $\mathrm{f}[v]$, contrary to assumption; so $\mathrm{f}[w]^{-1}g_1 \in \mathcal{P}_l(\mathrm{f}[w]^{-1}g)$, and the inductive hypothesis applied to $\mathrm{f}[w]^{-1}g_1$ gives $|\mathcal{D}_2| \le P_1(k-1)$. So $|\mathcal{D}| \le P_1(k-1) + 2Q(k) = P_1(k)$ as required. □

The following lemma is of the same type as Lemma 6.3. It holds for all Artin groups of large type.

Lemma 7.9. *For some $i \neq j$, suppose that no nontrivial element of $G(i,j)$ is either a left divisor of g_1 or a right divisor of g_2. If $g_1g_2 \in G(i,j)$, then $g_1g_2 = 1$.*

Proof. Suppose $g_1g_2 = h$ with $1 \neq h \in G(i,j)$. Since g_1 has no element of $G(i,j)$ as left divisor, hg_2^{-1} cannot be a geodesic factorisation. Let u, v be geodesic words representing h, g_2^{-1}. Let $u' = au''$ with $a \in \{x_i^{\pm 1}, x_j^{\pm 1}\}$ be the shortest suffix of u such that $u'v$ is non-geodesic. Then $u''v$ has a geodesic representative beginning with a^{-1} whereas u'' does not. So the maximal $\{x_i, x_j\}$-prefix of that geodesic representative of $u''v$ cannot represent a left divisor of $(u'')_G$, and hence by Lemma 7.3 $(u'')_G \neq \mathrm{LD}_{ij}((u''v)_G)$. But then g_2 must have a nontrivial right divisor in $G(i,j)$, contrary to assumption. □

It remains to verify D2. We start by defining a merging process for elements $g_1 \in C_k$ and $g_2 \in C_l$ which, as in the 2-generator case, will result in a merger $(f_1, \Delta_{ij}^r, f_2)$ for some i, j where, for $h_1 := f_1^{-1}g_1$, $h_2 := g_2f_2^{-1}$, we have $h_1h_2 = \Delta_{ij}^r$, $(f_1, h_1) \in \mathcal{P}(g_1)$ and $(h_2, f_2) \in \mathcal{P}(g_2)$.

As in the 2-generator case, the merging process proceeds through a series of steps, starting with $g_1^{(1)} := g_1$, $r_1 := 0$, $g_2^{(1)} := g_2$, and in the t-th step we compute $g_1^{(t+1)}, r_{t+1}, g_2^{(t+1)}$, by choosing geodesic factorisations $g_1^{(t+1)}h$ and $h'g_2^{(t+1)}$ of $g_1^{(t)}$ and $g_2^{(t)}$ such that $g_1^{(t+1)} \in \mathcal{P}_l(g_1)$ and $g_2^{(t+1)} \in \mathcal{P}_r(g_2)$. But there is an additional complication in that the i, j in the term $\Delta_{ij}^{r_t}$ may change during the process.

Recall from Subsection 6.2 that there are three types of merging steps, the first being free cancellation, and the other two involving a specific Δ_{ij}. In general, they are defined in the same way as in the 2-generator case subject to the condition that, if $r_t \neq 0$, then we must choose $h, h' \in G(i,j)$. If $r_t = 0$, then there is no such restriction on free cancellation moves whereas, for the other two types, we have $h, h' \in G(i,j)$ for the new values of i, j.

Again as in the 2-generator case, for $g \in G$, $k, l \in \mathbb{N}$ $k + l = m := |g|$ we define the set $S(g,k,l)$ to be the set of all triples (f_1, Δ^r, f_2) that arise as mergers of pairs of elements g_1, g_2 of lengths k, l and with $g_1g_2 = g$. For each such triple, we have $\Delta = \Delta_{ij}$ for some i, j; if $r = 0$, then $\{i,j\}$ is not necessarily uniquely defined. There exist $h_1, h_2 \in G$ with $h_1h_2 = \Delta^r$ where $g_1 = f_1h_1$ and $g_2 = h_2f_2$, and the restrictions on the merging steps described above ensure that $(f_1, h_1) \in \mathcal{P}(g_1)$ and $(h_2, f_2) \in \mathcal{P}(g_2)$.

In order to verify D2, we have to show that $|S(g,k,l)|$ is bounded by $P(k)$ for some polynomial P. We decompose S as a disjoint union $S_0 \cup S_1 \cup S_2$, and establish polynomial bounds for each of those subsets.

We define S_0 to be the subset of elements of $S(g,k,l)$ for which $r=0$ and $f_1, f_2 \in \langle x_i \rangle$ for some generator x_i. Since $|f_1| \le k$, $f_2 = f_1^{-1}g$, and x_i is determined by g, we have $|S_0| \le 2k+1$.

The sets S_1, S_2 are defined with respect to the elements $f_1'', \hat{f}, f_2''$ that are defined in Proposition 7.10 below. We define S_1 to be the set of triples in $S(g,k,l) \setminus S_0$ for which $f_1''\hat{f}f_2''$ is a geodesic factorisation of g, and S_2 to be the others.

Proposition 7.10. *Let $(f_1, \Delta^r, f_2) \in S(g,k,l) \setminus S_0$. Then the following is true for some $i \ne j$ with $1 \le i,j \le n$ where, if $r \ne 0$, i,j are determined by $\Delta = \Delta_{ij}$. There exist $f_1'', f_1', f_2'', f_2', \hat{f} \in G$ such that*

(1) $f_1', f_2', \hat{f} \in G(i,j)$, but f_1'' has no right divisor in $G(i,j)$ and f_2'' has no left divisor in $G(i,j)$,

(2) $g = f_1''\hat{f}f_2''$, $f_1 = f_1''f_1'$, $f_2 = f_2'f_2''$, and

(3) $(f_1', \Delta^r, f_2') \in S(\hat{f}, k', l')$ for some $k' \le k$ and $l' \le l$.

Furthermore, $\hat{f}$ does not lie in $\langle x_i \rangle$ for any i.

Proof. Since $(f_1, \Delta^r, f_2) \in S(g,k,l)$, it is the merger of elements $g_1 \in C_k$ and $g_2 \in C_l$, and so there exist $h_1, h_2 \in G$ with $g_1 = f_1h_1$, $g_2 = h_2f_2$ and $h_1h_2 = \Delta^r$.

Suppose first that $r \ne 0$, and hence that i,j are uniquely specified, with $m_{ij} < \infty$. Define $f_1' := \mathrm{RD}_{ij}(f_1)$, $h_1' := \mathrm{LD}_{ij}(h_1)$, $f_2' := \mathrm{LD}_{ij}(f_2)$, $h_2' := \mathrm{RD}_{ij}(h_2)$, $k' = |f_1'h_1'|$, $l' = |h_2'f_2'|$ and $\hat{f} = f_1'h_1'h_2'f_2'$. Now Lemma 7.9 applied to $h_1'^{-1}h_1$ and $h_2h_2'^{-1}$ tells us that $h_1'h_2' = h_1h_2 = \Delta^r$. In other words, during the merging the right divisor $(h_1')^{-1}h_1$ of h_1 has cancelled with the left divisor $h_2(h_2')^{-1}$ of h_2. We see also that $(f_1', \Delta^r, f_2') \in S(\hat{f}, k', l')$ is the result of the merging of $f_1'h_1'$ and $h_2'f_2'$ in $G(i,j)$. Suppose that $\hat{f} \in \langle x_{i'} \rangle$ for $i' = i$ or j. Then the application of Lemma 6.3 to $f_1'h_1'$ and $h_2'f_2'$ shows that free cancellation of w with $w \in \mathcal{P}_r(f_1'h_1')$, $w^{-1} \in \mathcal{P}_l(h_2'f_2')$ can be used to merge $f_1'h_1'$ and $h_2'f_2'$ to give just a power of $x_{i'}$, contradicting the fact that $r \ne 0$. (Here we are using the factorisations in $\mathcal{P}^1$, to ensure that such a cancellation would be permissible.)

Suppose instead that $r=0$. Then $h_1 = h_2^{-1}$. Since we are assuming that $(f_1, \Delta^r, f_2) \notin S_0$, we can choose distinct i,j such that $f_1' := \mathrm{RD}_{ij}(f_1)$ and $f_2' := \mathrm{LD}_{ij}(f_2)$ do not both lie in the same cyclic subgroup $\langle x_{i'} \rangle$ for $i' = i$ or j. If possible, we choose i,j such that the vertices corresponding to i,j in the graph Γ defining G are joined by an edge (equivalently, $m_{ij} < \infty$). Again we define $h_1' := \mathrm{LD}_{ij}(h_1)$, $h_2' := \mathrm{RD}_{ij}(h_2)$, $k' = |f_1'h_1'|$, $l' = |h_2'f_2'|$ and $\hat{f} = f_1'h_1'h_2'f_2'$. Since $h_1 = h_2^{-1}$, we have $h_1' = (h_2')^{-1}$, so in this case too $(f_1', \Delta_{ij}^r, f_2') \in S(\hat{f}, k', l')$ is a merger of $f_1'h_1'$ and $h_2'f_2'$ in $G(i,j)$ and, by Lemma 6.3, $f_1'h_1'h_2'f_2'$ does not lie in a cyclic subgroup $\langle x_{i'} \rangle$ for $i' = i$ or j. □

For triples in S_1, we claim that $f_1'' \in \mathcal{P}_l(g)$. Since f_1'' has no right divisor in $G(i,j)$ the only way that the factorisation $(f_1'', \hat{f}f_2'')$ could be Δ-decreasing for some Δ would be with $\Delta = \Delta_{ii'}$, where $x_{i'}$ is the name of both $\mathrm{l}[f_1'']$ and $\mathrm{f}[f_2'']$ and $\hat{f} = x_i^s x_j^t$ for some $s,t > 0$. But then $\mathrm{RD}_{ii'}(\hat{f}f_2'')$ would have at most two syllables, and so the factorisation would still be permissible. So the claim is true.

Hence, since $|f_1''| \le k$, the number of choices of f_1'' is bounded by $P_1(k)$. For given f_1'' and a pair i,j, we have $\hat{f} = LD_{ij}(f_1''^{-1}g)$ and $f_2'' = \hat{f}^{-1}f_1''^{-1}g$. Now

$(f_1', \Delta_{ij}^r, f_2') \in S(\hat{f}, |f_1'h_1'|, |h_2'f_2'|)$, which we know from the dihedral case has size bounded by $P_2^{ij}(\min |f_1'h_1'|, |h_2'f_2'|)) \le P_2^{ij}(k)$. So $|S_1| \le P_1(k) \sum_{i,j} P_2^{ij}(k)$.

In order to bound $|S_2|$, we need some further technical lemmas.

Lemma 7.11. *Suppose that $f_1''\hat{f}f_2''$ is not a geodesic factorisation of g. Then $\hat{f} = a^s b^t$ for some $s, t > 0$ where $a, b \in \{x_i^{\pm 1}, x_j^{\pm 1}\}$ and a, b have different names.*

Proof. Let $u_1'', \hat{u}, u_2''$ be geodesic words representing $f_1'', \hat{f}, f_2''$, respectively. Suppose (for a contradiction) that $\hat{u}$ has at least three syllables. Since $u_1''\hat{u}u_2''$ is not geodesic, we can by Proposition 7.4 (2) apply a rightward length reducing sequence to a prefix of it containing u_1''.

If the left-hand end of the first critical subword in the sequence were to the left of $\hat{u}$, then some τ-move in the sequence would replace a letter in the first syllable of $\hat{u}$ by a letter c with name not equal to x_i or x_j. But since c would then be followed by at least two syllables with names x_i and x_j, c would not be the leftmost letter in a critical subword, and so this would be the last τ-move in the sequence, and would not provoke a free reduction. So the left hand end of the first critical subword must be within $\hat{u}$.

We may assume that the first τ-move of the sequence is not completely within $\hat{u}$, since otherwise we could just replace $\hat{u}$ by the result of this move. So the first critical subword overlaps the right hand end of $\hat{u}$ and hence has its left hand end in the final syllable, a^s say, of $\hat{u}$. But then $a^s u_2''$ is not geodesic and so, by Corollary 7.2, neither is au_2''. But then f_2'' has a geodesic representative beginning with a^{-1}, contradicting $f_2 = f_2'f_2''$ with $f_2' = \mathrm{LD}_{ij}(f_2)$. ☐

Lemma 7.12. *Suppose that $\hat{f} = a^s b^t$ for some $s, t > 0$, where a, b have distinct names x_i, x_j, and that $f_1''\hat{f}f_2''$ is not a geodesic factorisation of g. Then there exists a letter c with name $x_{i'}$ not equal to x_i or x_j, $q > 0$, and $e_1, e_2 \in G$, such that $f_1''a^s = e_1c^q$ and $b^t f_2'' = c^{-q}e_2$, where e_1c^q, $c^{-q}e_2$ and e_1e_2 are all geodesic factorisations. Furthermore, the vertices corresponding to x_i, x_j in the graph Γ are joined by an edge; that is $m_{ij} < \infty$.*

Proof. Let u_1'', u_2'' be geodesic representatives of f_1'', f_2''. Using similar reasoning to that of the proof of the previous lemma, we see that a rightward length reducing sequence applied to the word $u_1''a^s b^t u_2''$ would consist of a sequence of moves that replaced $u_1''a^s$ by a word ending in a letter c with name $x_{i'}$ not equal to x_i or x_j, followed by a length reducing sequence applied to cb^tu_2''. But then b^tu_2'' would have a geodesic representative starting with c^{-1}. Let q be maximal such that $f_1''a^s$ and b^tf_2'' have geodesic factorisations e_1c^q and $c^{-q}e_2$ respectively.

If m_{ij} were infinite, then we would have $r = 0$ in Proposition 7.10, in which case we could and would have chosen one of the pairs i, i' or j, i', depending on which of $m_{ii'}$ and $m_{ji'}$ was finite rather than i, j in the proof of that proposition. So $m_{ij} < \infty$, which proves the final assertion in the lemma. Hence the hypothesis of Theorem 1.1 implies that at least one of the $m_{ii'}$ and $m_{ji'}$ is at least 4.

Let h_1 and h_2 be the group elements represented by $e_1c^q =_G u_1''a^s$ and $c^{-q}e_2 =_G b^tu_2''$, respectively. Applying Propositions 7.1 and 7.3 to these elements, we see that $\mathrm{RD}_{ii'}(h_1)$ and $\mathrm{LD}_{ji'}(h_2)$ are represented by w_1 and w_2, ending in c^q and beginning with c^{-q}, respectively. Since w_1 and w_2 are not the unique geodesic representatives of $\mathrm{RD}_{ii'}(h)$ and $\mathrm{LD}_{ji'}(h_2)$, the above condition on the edge labels implies that at least one of w_1 and w_2 has at least four syllables. Hence either e_1 has a geodesic representative v_1 in which at least the final three syllables are powers of a and c, or e_2 has a geodesic representative v_2 in which at least the first three syllables are powers b and c. We see that no rightward length reducing sequence can pass through this middle section, and hence v_1v_2 is geodesic, which completes the proof. $\square$

By Lemma 7.11, each element $(f_1, \Delta_{ij}^r, f_2)$ of S_2 determines $f'', \hat{f}, f_2''$ for which the hypotheses and conclusion of Lemma 7.12 hold. Since $|g| \geq l - k$, we must have $q \leq k$, so there are at most $2nk$ possibilities for c^q. So assume that c^q is fixed. Unfortunately we have no bound on $|e_1|$ as a polynomial in k. Let $e_1 = e_1''e_1'$ with $e_1' = \mathrm{RD}_{ii'}(e_1)$. Then $|e_1''| < |f_1''| < k$ and $e_1'' \in \mathcal{P}_l(e_1)$, so there are at most $P_1(k)$ possible e_1''.

We claim that the triple $(f_1, \Delta_{ij}^r, f_2)$ is uniquely determined by g, c^q and e_1''. To see this note that $\mathrm{LD}_{ii'}(e_1''^{-1}g) = e_1c^{-t}$ for some $t \geq 0$ and, since (e_1, c^q) is a geodesic factorisation, e_1 does not have c^{-1} as right divisor. Hence, given g, c^q and e_1'', e_1' is determined as $\mathrm{LD}_{ii'}(e_1''^{-1}g)c^t$, where c^{-t} is the longest right divisor of $\mathrm{LD}_{ii'}(e_1''^{-1}g)$ of that form. Then e_1 and f_1'' are determined as $e_1''e_1'$ and $e_1c^qa^{-s}$, where a^s is the highest power of a that is a right divisor of e_1c^q. Then, as in the previous case, we have $\hat{f} = LD_{ij}(f_1''^{-1}g)$ and $f_2'' = \hat{f}^{-1}f_1''^{-1}g$, which establishes the claim.

So the number of triples $(f_1, \Delta_{ij}^r, f_2) \in S(g, k, l)$ that satisfy the conclusion of Lemma 7.12, and hence also $|S_2|$, is bounded by $2nkP_1(k)\sum_{i,j} P_2^{ij}(k)$.

This completes the proof that Artin groups satisfying the hypotheses of Theorem 1.1 satisfy D1 and D2, and hence also the proof of Theorem 1.1.

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References

[1] K I Appel, On Artin groups and Coxeter groups of large type, Contemporary Mathematics 33 (1984) 50–78.

[2] K I Appel and P E Schupp, Artin groups and infinite Coxeter groups, Invent. Math. 72 (1983) 201–220.

[3] W. Ballmann, Singular spaces of non-positive curvature, *Sur les groupes hyperboliques d'après Mikhail Gromov*, Progr. Math. 83, Birkhäuser Boston, Boston MA, 1990

[4] W. Ballmann, Lectures on spaces of non-positive curvature, DMV seminar 25, Birkhäuser, 1995.

[5] J. Behrstock and Y. Minsky, Centroids and the rapid decay property in mapping class groups, J. Lon. Math. Soc. 84 (2011) 765–784.

[6] T. Brady and J. McCammond, Three generator Artin groups of large type are biautomatic, J. Pure Appl. Alg. 151 (2000) 1–9.

[7] Indira Chatterji, On property RD for certain discrete groups, PhD thesis, ETH Zürich, September 2001.

[8] I. Chatterji and K. Ruane, Some geometric groups with rapid decay, Geom. Funct. Anal. 15 (2005) 311–339.

[9] P.A. Cherix, M. Cowling, P. Jolissaint, P. Julg and A. Valette, Groups with the Haagerup property (Gromov's a-T-menability), Progr. Math. 197, Birkhäuser, Basel, 2001.

[10] L. Ciobanu, D.F. Holt and S. Rees, Rapid decay is preserved by graph products, `http://arxiv.org/abs/1110.6831`, preprint 2011.

[11] J. Crisp and L. Paris, Artin groups of type B and D, Adv. Geom. 5 (2005) 607–636.

[12] C. Drutu and M. Sapir. Relatively Hyperbolic Groups with Rapid Decay Property, Int. Math. Res. Notices 19 (2005) 1181–1194.

[13] U. Haagerup, An example of a non-nuclear C^*-algebra, which has the metric approximation property, Invent. Math. 50 (1978/9) 278–293.

[14] P. de la Harpe. Groupes hyperboliques, algèbres d'opérateurs et un théorème de Jolissaint, C. R. Acad. Sci. Paris Sér. I. Math. 307 (1988) 771–774.

[15] N. Higson and G. Kasparov, E-theory and KK-theory for groups which act properly and isometrically on Hilbert space, Invent. Math. 144 (2001) 23–74.

[16] N. Higson, V. Lafforgue, G. Skandalis, Counter-examples to the Baum-Connes conjecture, Geom. Funct. Anal. 12 (2002) 330–354.

[17] D.F. Holt and S. Rees, Artin groups of large type are shortlex automatic with regular geodesics, Proc. Lon. Math. Soc. 104 (2012) 486–512.

[18] P Jolissaint, Rapidly decreasing functions in reduced C^*-algebras of groups, Trans. Amer. Math. Soc. 317 (1990) 167–196.

[19] V. Lafforgue, K-théorie bivariante pour les algègbres de Banach et conjecture de Baum-Connes, Invent. Math. 149 (2002) 1–95.

[20] R.C. Lyndon and P.E. Schupp, *Combinatorial Group Theory*, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[21] J. Mairesse and F. Mathéus, Growth series for Artin groups of dihedral type, Int. J. Alg. Comp. 16 (2006) 1087–1107.

[22] P. Z. Ong, Rapid decay property and small cancellation groups. Ph.D. Thesis, University of California, San Diego, 1991.

[23] T. Schick, Finite group extensions and the Baum-Connes conjecture, Geom. Topol. 11 (2007) 1767-1775.

[24] A. Valette, Introduction to the Baum-Connes conjecture. From notes taken by Indira Chatterji. With an appendix by Guido Mislin. Lectures in Mathematics ETH Zürich. Birkhäuser, Basel, 2002.

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