Groups of automorphisms of rooted trees II

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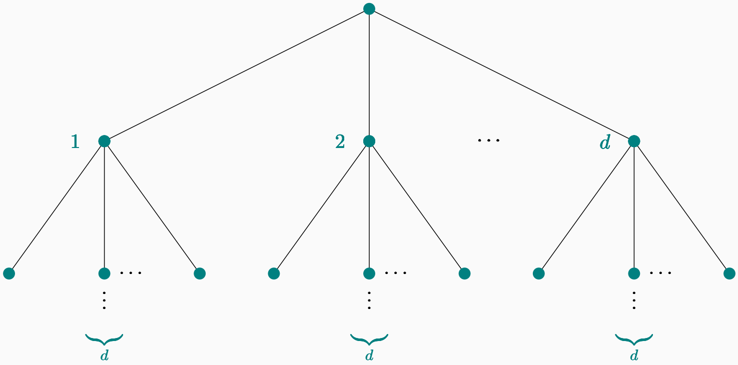
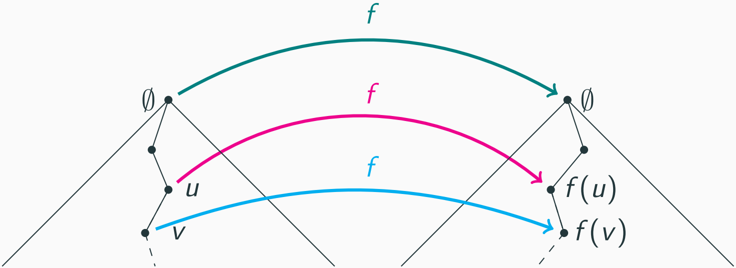
Regular rooted trees



Regular rooted trees $\mathcal{T}_d$ and automorphisms of $\mathcal{T}_d$

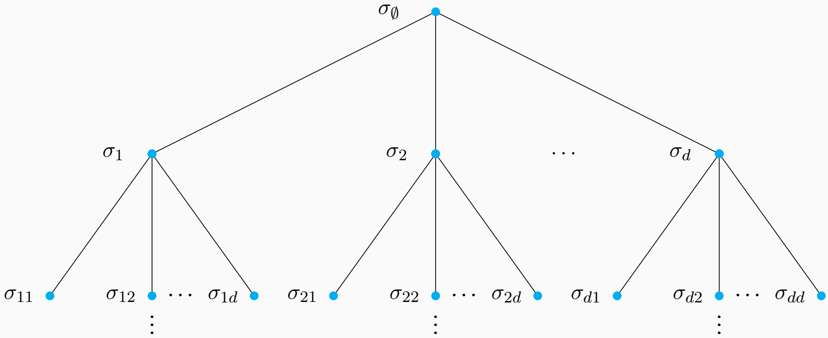
Automorphisms of $\mathcal{T}_d$

Bijections of the vertices that preserve incidence.



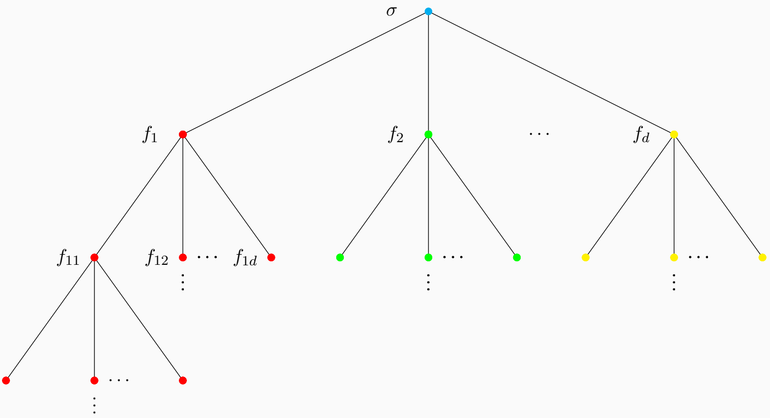
Describing elements of $\mathrm{Aut}\,\mathcal{T}$ I

An automorphism $f \in \mathrm{Aut}\,\mathcal{T}_d$ can be represented by writing in each vertex v a permutation $\sigma_v \in \mathrm{Sym}(d)$ which represents the action of f on the descendants of v.



Describing elements of $\mathrm{Aut}\,\mathcal{T}$ II

Let $f \in \mathrm{Aut}\,\mathcal{T}_d$ with $f = (f_1, f_2, \ldots, f_d)a$, where $f_i \in \mathrm{Aut}\,\mathcal{T}_d$ and a is rooted corresponding to σ.



Self-similar groups

We define the isomorphism

$$\psi : \mathrm{st}(1) \longrightarrow \mathrm{Aut}\,\mathcal{T} \times \overset{d}{\cdots} \times \mathrm{Aut}\,\mathcal{T}$$
$$g \longmapsto (g_1, \ldots, g_d).$$

If $G \leq \mathrm{Aut}\,\mathcal{T}$

$$\psi_G : \mathrm{st}_G(1) \longrightarrow \mathrm{Aut}\,\mathcal{T} \times \overset{d}{\cdots} \times \mathrm{Aut}\,\mathcal{T}$$
$$g \longmapsto (g_1, \ldots, g_d)$$

Additionally, if G is self-similar,

$$\psi_G : \mathrm{st}_G(1) \longrightarrow G \times \overset{d}{\cdots} \times G$$
$$g \longmapsto (g_1, \ldots, g_d).$$

We denote $\psi = \psi_G$, and sometimes, to define automorphisms we omit ψ.

(Regular) Branch groups

Recall that in $\operatorname{Aut}\mathcal{T}_d$ we have:

$$\operatorname{Aut}\mathcal{T}_d \times \overset{d}{\ldots} \times \operatorname{Aut}\mathcal{T}_d \cong \mathrm{st}(1) \leq \operatorname{Aut}\mathcal{T}_d. \tag{1}$$

- Let $G \leq \operatorname{Aut}\mathcal{T}_d$ spherically transitive and self-similar.
- In this case it is too much to ask that (1) holds.
- We content ourselves if (1) holds for a (finite index) subgroup K of G:

$$K \times \overset{d}{\ldots} \times K \leq K.$$

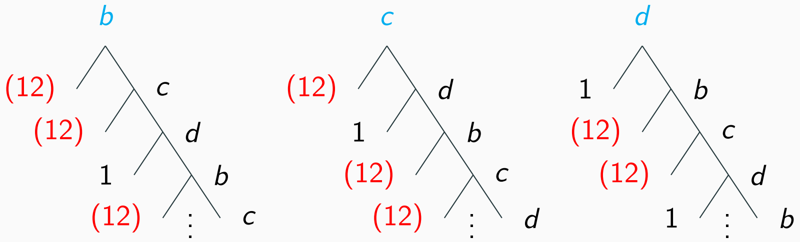
- If this is the case, then G is regular branch over K.
- If the index of K in G is infinite then G is *weakly* regular branch over K.

The Grigorchuk groups

The (first) Grigorchuk group

$$\Gamma = \langle a, b, c, d \rangle$$

$$a = (1, 1)(12) \qquad b = (a, c) \qquad c = (a, d) \qquad d = (1, b)$$



Γ as a counterexample to the GBP

The group Γ is an infinite 2-group $\longrightarrow$ it is a counterexample to the General Burnside Problem (GBP).

- Proof that Γ is finitely generated: ✓
- Proof that Γ is infinite:
  - Idea: find a proper subgroup of Γ that projects surjectively onto Γ
  - Note that $a \notin \mathrm{st}_\Gamma(1)$ $(\star)$
  - Consider the map $\rho = \pi_1(\psi(\mathrm{st}_\Gamma(1))$:

$$\begin{aligned} \rho : \mathrm{st}_\Gamma(1) \to \Gamma \times \Gamma &\to \Gamma \\ b \to (a, c) &\to a \\ d^a \to (b, 1) &\to b \\ b^a \to (c, a) &\to c \\ c^a \to (d, a) &\to d \end{aligned}$$

  - Then $\mathrm{st}_\Gamma(1)$ is onto Γ $(\star)$

Γ as a counterexample to the GBP

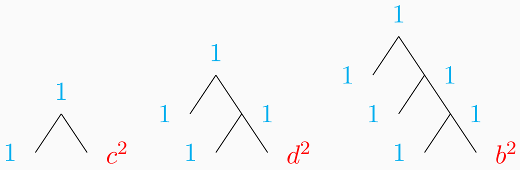
- Proof that Γ is torsion:
    - First step: prove that $a^2 = b^2 = c^2 = d^2 = 1$.
        - $a^2 = 1$ ✓
        - What about b, c and d?
    - General case: ... more technical.

Proof that $b^2 = c^2 = d^2 = 1$

Let us prove that $b^2 = 1$. Recall that

$$a = (1,1)(12) \quad b = (a, c) \quad c = (a, d) \quad d = (1, b).$$

- We have $b^2 = (a^2, c^2) = (1, c^2)$.
- Also $c^2 = (a^2, d^2) = (1, d^2)$ and $d^2 = (1, b^2)$.

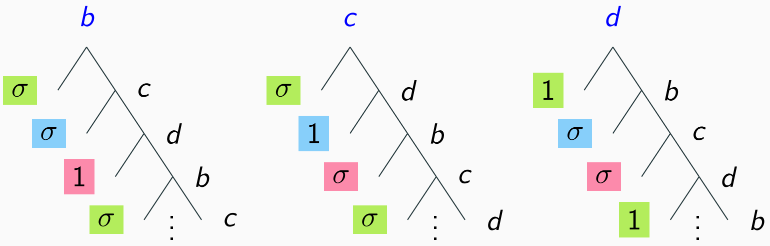


- Then the only possibility is that $b^2 = 1$.
- As a consequence, $c^2 = d^2 = 1$.

Summarizing some properties of Γ

- It is a self-similar group.
- It is a torsion 2-group.
- It is just-infinite.
- It is a regular branch group over the subgroup $K = \langle (ab)^2 \rangle^\Gamma$.
- It has intermediate word growth.
- It is amenable but not elementary amenable.
- Many other exotic properties . . . .

Grigorchuk groups

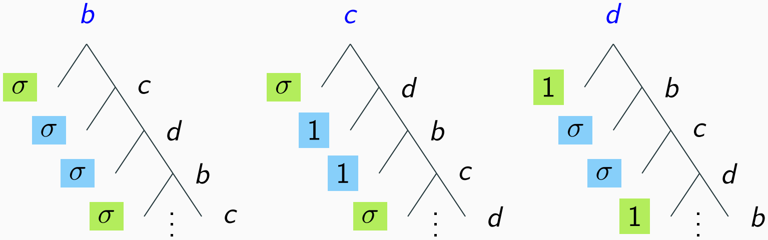


where $\sigma = (1\ 2)$.

Let **0, 1, 2** be the three non-trivial homomorphisms from $C_2 \times C_2 = \{1, b, c, d\}$ to $C_2 = \{1, \sigma\}$ such that:

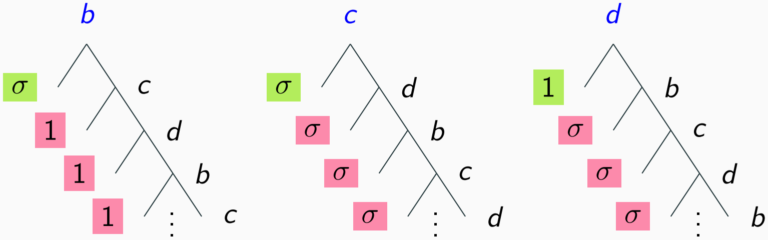
$$\mathbf{0} : b \mapsto \sigma,\quad c \mapsto \sigma,\quad d \mapsto 1 \qquad \mathbf{1} : b \mapsto \sigma,\quad c \mapsto 1,\quad d \mapsto \sigma \qquad \mathbf{2} : b \mapsto 1,\quad c \mapsto \sigma,\quad d \mapsto \sigma.$$

Example 1: **0**, **1**, **1**, **0**...

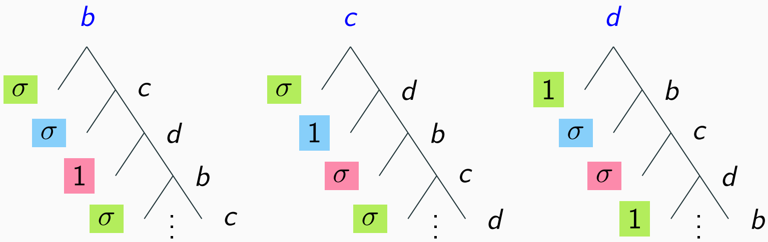


Grigorchuk groups

Example 2: **0**, **2**, **2**, **2**...



Grigorchuk groups



- Let $\Omega = \{\mathbf{0},\ \mathbf{1},\ \mathbf{2}\}^\infty$ be the space of infinite sequences over letters $\{\mathbf{0},\ \mathbf{1},\ \mathbf{2}\}$.

  Given $\omega \in \Omega$ the Grigorchuk group is $G_\omega = \langle a, b_\omega, c_\omega, d_\omega \rangle$.

Grigorchuk groups: properties

- The first Grigorchuk group corresponds to the periodic sequence $\omega = \mathbf{012012}\ldots$.
- If ω is eventually constant then G_ω is virtually abelian.
- Otherwise, G_ω is of intermediate growth.
- The group G_ω is periodic if and only if ω contains all three letters $\mathbf{0}$, $\mathbf{1}$, $\mathbf{2}$ infinitely often.

The GGS-groups

The GGS-groups

Let p be an odd prime and $\mathcal{T}_p$ the p-adic tree.
The Grigorchuk-Gupta-Sidki group (GGS for short) is defined by

- $a = (1, \dots, 1)(1\ 2 \dots\ p)$
- $b = (a^{e_1}, a^{e_2}, \dots, a^{e_{p-1}}, b)$

where $\mathbf{e} = (e_1, \dots, e_{p-1}) \in (\mathbb{Z}/p\mathbb{Z})^{p-1}$ is its defining vector.

The group $G_{\mathbf{e}} = \langle a, b \rangle$ is the GGS-group corresponding to the defining vector e.

A GGS-group is torsion if and only if $\sum_{i=1}^{p-1} e_i \equiv 0 \mod p$.

When a GGS-group is a branch group?

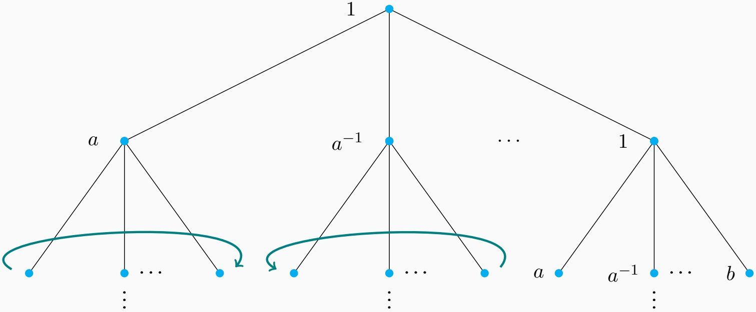
Note that any GGS-group is spherically transitive a self-similar group.

- If $\mathbf{e}$ is the constant vector, then G is weakly regular branch over $(\langle ba^{-1}\rangle^G)'$.
- If $\mathbf{e}$ is non-constant then G is regular branch over $\gamma_3(G)$.
- Moreover, if the defining vector is non-symmetric (symmetric means that $e_i = e_{p-i}$ for all $i = 1, \dots p-1$) then G is regular branch over its derived subgroup G'.

A specific example: the Gupta-Sidki p-group

Let $\mathbf{e} = (1, -1, 0, \ldots, 0)$. The Gupta-Sidki group $\mathcal{G} = G_{(1,-1,0,\ldots,0)}$ is generated by a, b, where

- $a = (1, \ldots, 1)(1\ 2 \ldots\ p)$
- $b = (a, a^{-1}, 1, \ldots, 1, b)$



- Can you prove that this group is infinite and generated by elements of order p? And that is a p-group?

Gupta-Sidki is regular branch over its derived subgroup

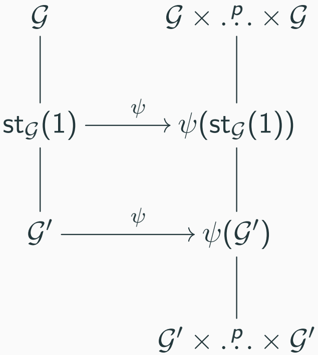
For simplicity, let $p \geq 5$.

- As $\mathcal{G} = \langle a, b \rangle$, then $\mathcal{G}' = \langle [a, b]^m \mid m \in \mathcal{G} \rangle$.
- Take

$$b = (a, a^{-1}, 1, \ldots, 1, b) \quad b^a = (b, a, a^{-1}, 1, \ldots, 1)$$

- Then $[b, b^a] = ([a, b], 1, \ldots, 1)$.
- Since $\mathcal{G}$ is *fractal*, for any $h \in \mathcal{G}$ there exists $g \in \mathrm{st}_{\mathcal{G}}(1)$ such that $g = (h, \star, \ldots, \star)$.
- Then $[b, b^a]^g = ([a, b]^h, 1, \ldots, 1)$.
- This implies that $\mathcal{G}' \times \{1\} \times \cdots \times \{1\} \subseteq \mathcal{G}' \cap \mathrm{st}_{\mathcal{G}}(1) = \mathrm{st}_{\mathcal{G}'}(1) \subseteq \mathcal{G}'$.
- For level transitivity we can have $\mathcal{G}'$ in each component.

...continue: do you remember the general "picture"?



- Also, $|\mathcal{G}/\mathcal{G}'| = p^2$.
  (Can you prove it?)
- The Gupta-Sidki p-group $\mathcal{G}$ is regular branch over its derived subgroup $\mathcal{G}'$.

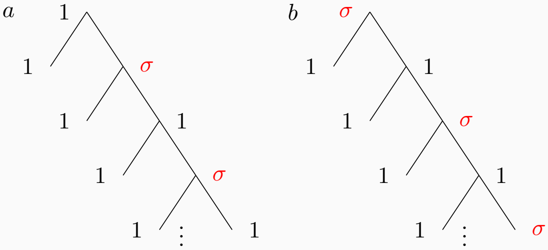
Grigorchuk group vs GGS-groups

Counterexamples to the General Burnside Problem (GBP):

- GGS-groups: $G = \langle a, b \rangle$, with a, b of order p, with p odd.
- What if $p = 2$?
- $G = \langle a, b \rangle$ is generated by elements of order 2.
- Either G is a <u>finite</u> dihedral group or the infinite dihedral group.
- In both cases G is not a counterexample to the GBP.
- Then if you want a group generated by elements of order 2, you must add generators: $\Gamma = \langle a, b, c, d \rangle$.

The Basilica group

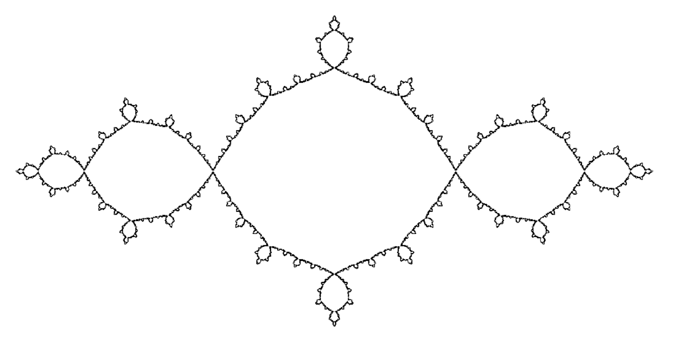
Let $\mathcal{T}_2$ be the binary tree. Define a and b as follows:



- Can you define a and b from their portraits above?
- $a = (\ ,\ )\epsilon \quad b = (\ ,\ )\sigma$ $a = (1, b) \quad b = (1, a)\sigma$

A curiosity about the name

First: the Basilica group is $B = \langle a, b \rangle$.



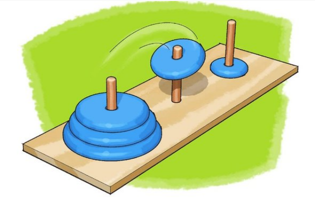
Some properties of the Basilica group

- It is torsion-free (Can you prove that a and b have infinite order?)
- It is weakly regular branch over its derived subgroup B'.
- It has exponential word growth.
- Basilica the first example of an amenable but not subexponentially amenable group.

The Hanoi Tower group

The Hanoi tower game

The tower of Hanoi was invented by a French mathematician Édouard Lucas in the 19th century.



- The goal: to move the entire stack to another peg.
- The rules:
    - One disk can be moved at a time;
    - Each move consists of taking the upper disk from one of the stacks and placing it on top of another or on an empty peg;
    - No disk may be placed on top of a smaller disk.

The Hanoi towers game

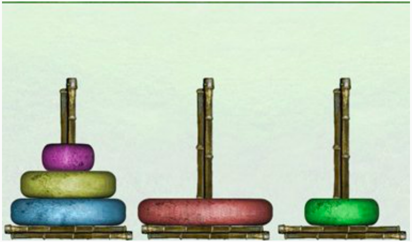
- Let 3 be the number of pegs, then consider $X = \{1, 2, 3\}$. A word in X is a configuration of the disks and the length of the word is the number of disks.
- Each number represents the peg in which the disk lie.
- We "read" from the smallest to the bigger disk.
- Example:

$$23112.$$

- The length of the word above is 6 $\longrightarrow$ 6 disks.
- This means that the smaller disk is in the 2nd position, the second smaller disk is in the 3rd position, the third smaller disk is in the 1st position, and so on.

The Hanoi towers game II

- Other example: can you guess how to write the configuration below?



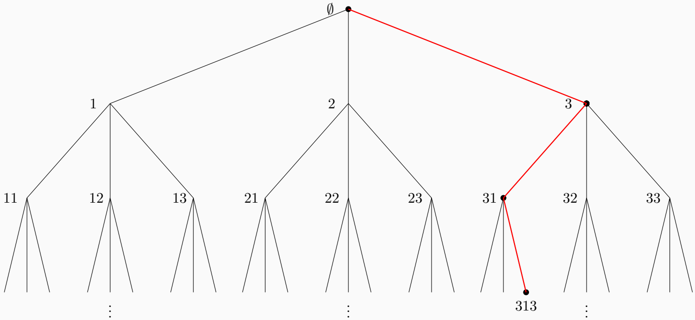
- The configuration is:

$$13112.$$

- Goal: to send $11\ldots1$ to $33\ldots3$.

The Hanoi towers game

- Configurations (sequences of length n of $1, 2, 3$) can be seen as vertices on the n-th level in a rooted ternary tree.



- Any move takes one vertex on the n-th level on the tree to another vertex on the n-th level. Then each move can be thought of as an automorphism of the rooted ternary tree.

The Hanoi towers ~~game~~ group

Move a:

- Search for the first time a 2 or 3 appears in the configuration
- Switch them
- Apply the identity
- This means that a does the only movement we are allowed to do between pegs 2 and 3
- Example: $a(21322) = (31322)$.

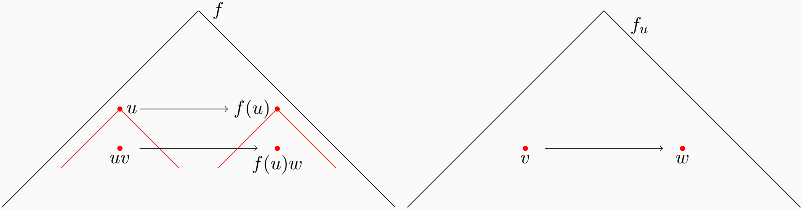
One can define elements a, b and c acting on the whole ternary tree.

$$\mathcal{H} = \langle a, b, c \rangle$$

where $a = (a, 1, 1)(23), b = (1, b, 1)(13), c = (1, 1, c)(12)$

Do you remember the section of an automorphism?

Let u be a vertex of $\mathcal{T}$, and $g \in \operatorname{Aut}\mathcal{T}$. We denote with g_u the *section* of g at the vertex u, that is the action of g on the subtree $\mathcal{T}_u$ that hangs from the vertex u. If $f \in \operatorname{Aut}\mathcal{T}$ and u, v are vertices of the tree, we can define the section f_u by the formula
$f(uv) = f(u)f_u(v)$.



Example: since $a = (a, 1, 1)(23)$, we have

- $a(21322) = a(2)a_2(1322) = 31322$
- $a(1321) = a(1)a_1(321) = 1a(321) = 1a(3)a_3(211) = 1211$


The
End

Conclusions

To conclude . . .

Groups of automorphisms of rooted trees play an important role in group theory.

Some questions:

- What is your favourite group...? :)
- What is the word growth of the GGS-groups?
- Nice topic: study algorithmic problems in branch groups.
- Do there exist finitely presented branch groups?

References

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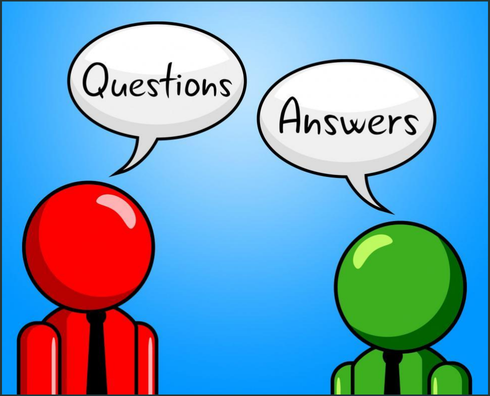
[2] L. Bartholdi, R.I. Grigorchuk, Z. Sunik *Branch groups* - Handbook of Algebra, Volume 3, North-Holland (2003), 989-1112.

[3] R.I. Grigorchuk *Just infinite branch groups* - New Horizons in pro-p Groups, Progress in Mathematics, Volume 184 (2000), 121-179.

[4] P. de la Harpe *Topics in Geometric Group Theory* - Chicago Lectures in Mathematics (2000).


Questions
Answers

Thank you :)

Stay safe!