Groups of automorphisms of rooted trees I

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Motivation: famous problems in group theory

- Milnor's Problem $\Longrightarrow$ growth of a group.
- General Burnside Problem $\Longrightarrow$ finiteness properties of a group.

Some facts (no spoiler: see Alex's talk) about growth of groups

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Milnor's question (1960):

Are there groups of intermediate growth between polynomial and exponential?

Grigorchuk's answer (1980):

Yes, the first ... Grigorchuk group.

About the General Burnside Problem

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In modern terminology the general Burnside problem asks:

can a finitely generated periodic group be finite?

Are finitely generated periodic groups finite?

- Yes, for nilpotent groups.
- Yes, finitely generated periodic subgroups of the general linear group of degree $n > 1$ over the complex field.
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- Counterexample: the first Grigorchuk group.

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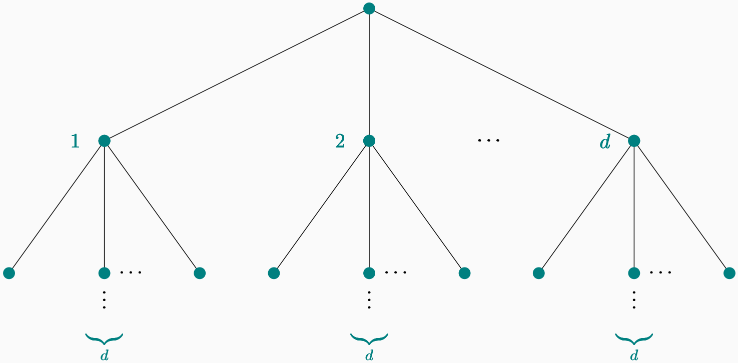
$$\text{Milnor's question} \bigcap \text{General Burnside Problem}$$

$$= \text{the first Grigorchuk group}, \ldots$$

Automorphisms of regular rooted trees

Regular rooted trees

Seriously: the regular rooted tree $\mathcal{T}_d$



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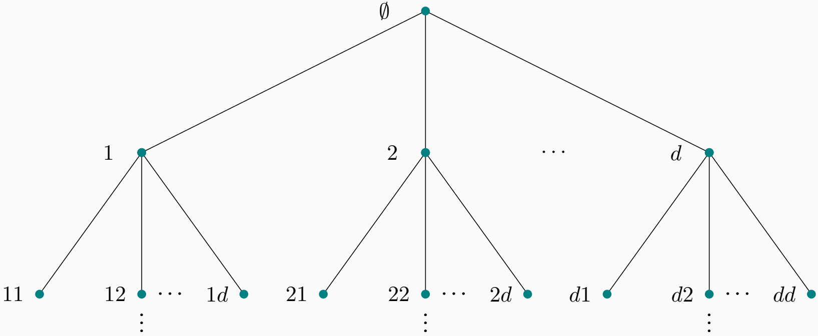

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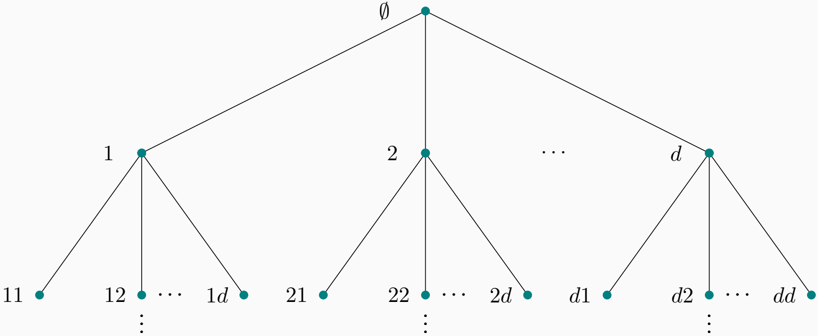
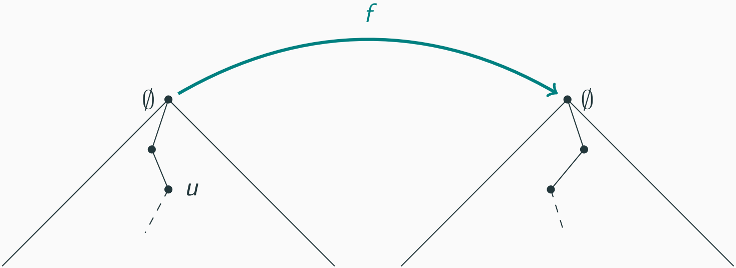


- X^n denotes the nth level of the tree.

Automorphisms of rooted trees

Automorphisms of $\mathcal{T}_d$

Bijections of the vertices that preserve incidence.



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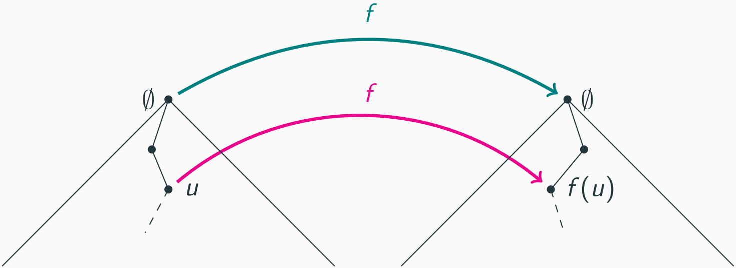

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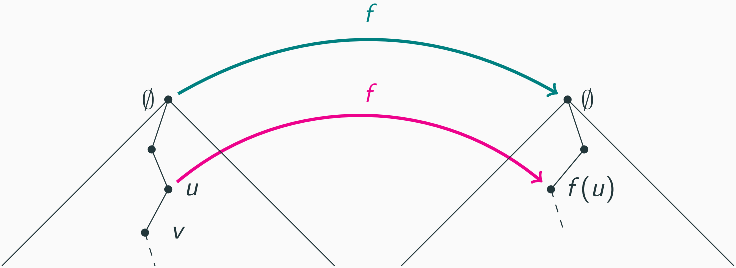

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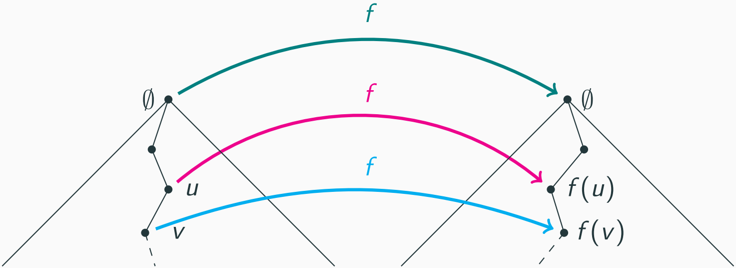

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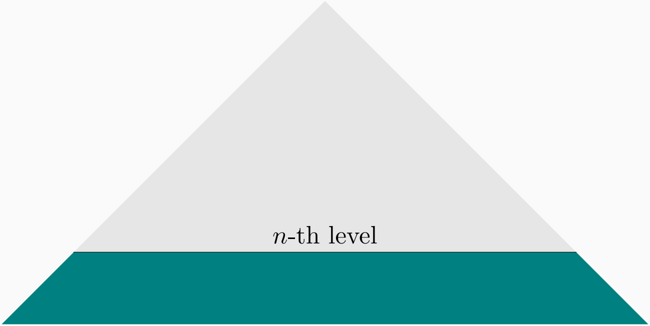
Sometimes we write $\mathcal{T}$ for $\mathcal{T}_d$, and, consequently, Aut $\mathcal{T}$ for Aut $\mathcal{T}_d$.

A subgroup of $\mathrm{Aut}\,\mathcal{T}$: the stabilizer



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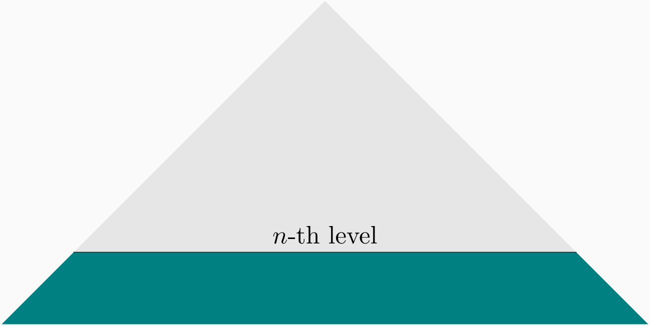

- The nth level stabilizer $\mathrm{st}(n)$ fixes all vertices up to level n.
- If $H \leq \operatorname{Aut}\mathcal{T}$, we define $\mathrm{st}_H(n) = H \cap \mathrm{st}(n)$.

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  where $\bigcap_{n\in\mathbb{N}} \mathrm{st}(n) = 1$.
- Hence $\operatorname{Aut}\mathcal{T}$ is a residually finite group (i.e. a group in which the intersection of all its normal subgroups of finite index is trivial).

Describing elements of $\operatorname{Aut} \mathcal{T}$

An automorphism $f \in \operatorname{Aut} \mathcal{T}_d$ can be represented by writing in each vertex v a permutation $\sigma_v \in \operatorname{Sym}(d)$ which represents the action of f on the descendants of v.

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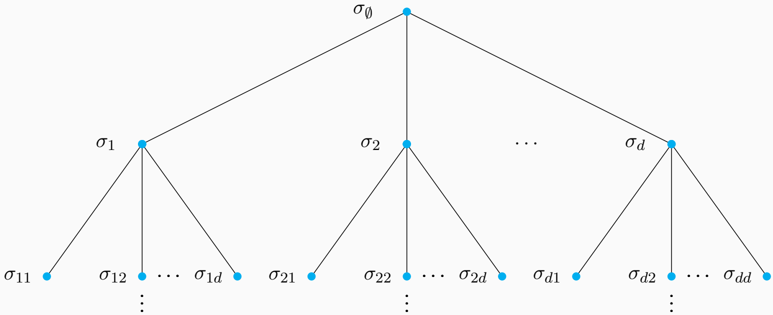

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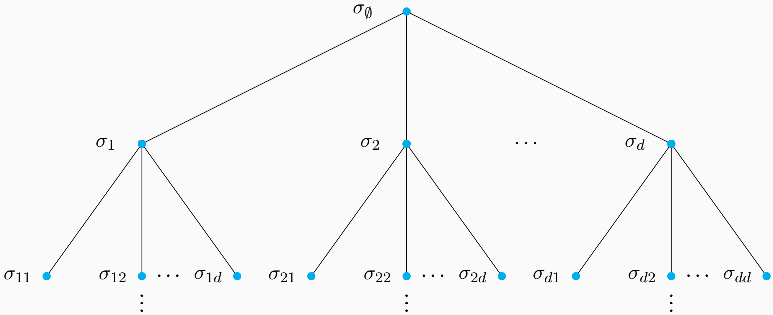

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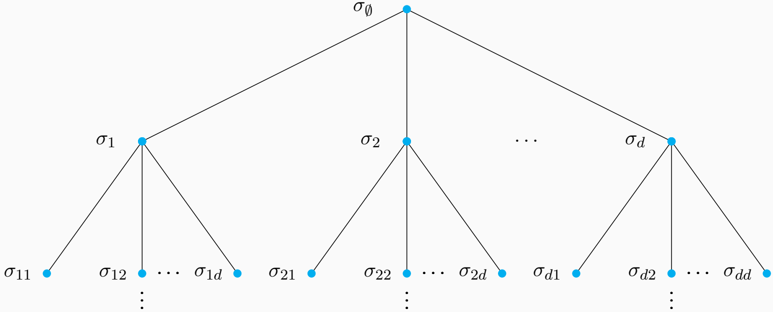
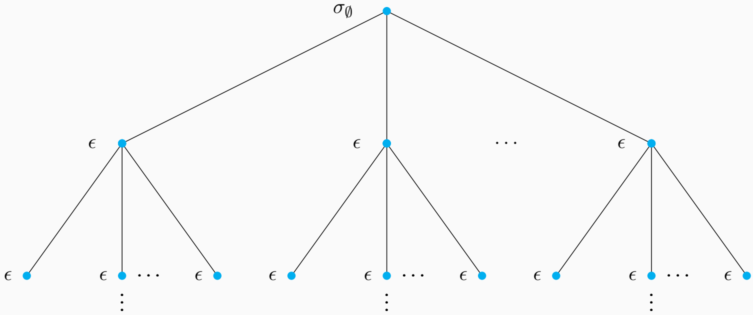


We say that $\sigma_v \in \mathrm{Sym}(d)$ is the *label* of f at the vertex v. The set of all labels is the *portrait* of f.

Describing elements of $\mathrm{Aut}\,\mathcal{T}$

The simplest type are rooted automorphisms: given $\sigma \in \mathrm{Sym}(d)$, they simply permute the d subtrees hanging from the root according to σ.



We denote with ϵ the identity element of $\mathrm{Sym}(d)$.

Example of a rooted automorphism

Let $\mathcal{T}_3$ be the ternary tree, and a the rooted automorphism corresponding to the cycle $\sigma = (1\ 2\ 3)$.

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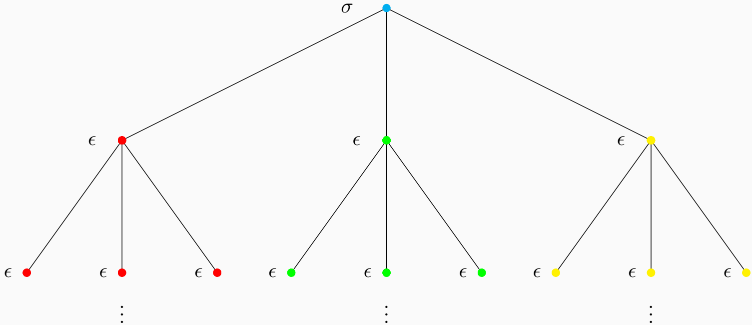

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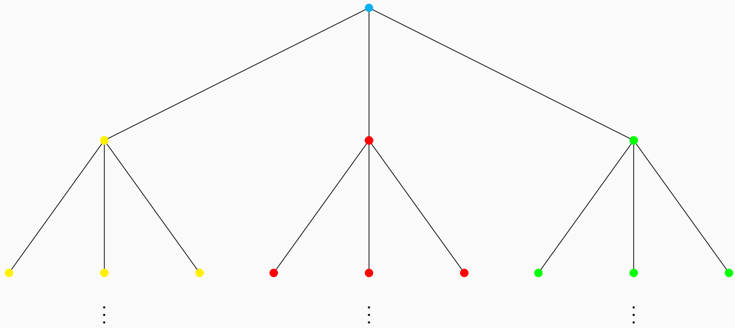

Note: sometimes we will identify a with σ.

Some facts about $\operatorname{Aut} \mathcal{T}$: I

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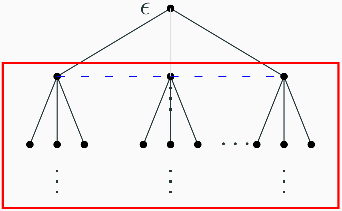

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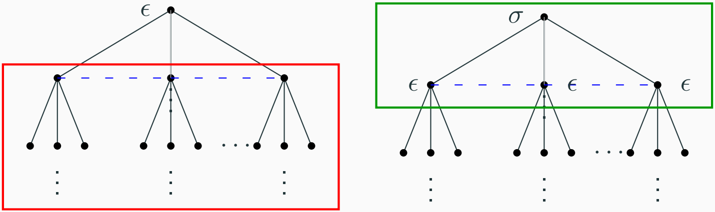

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We define the isomorphism

$$\begin{aligned}\psi : \mathrm{st}(1) &\longrightarrow \mathrm{Aut}\,\mathcal{T} \times \overset{d}{\cdots} \times \mathrm{Aut}\,\mathcal{T} \\ g &\longmapsto (g_1, \dots, g_d)\end{aligned}$$

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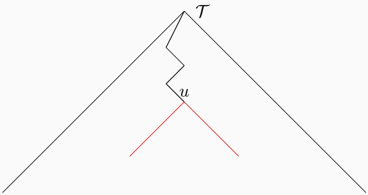

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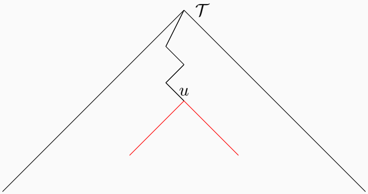

Digression: this implies that $\mathrm{Aut}\,\mathcal{T}$ contains products $\mathrm{Aut}\,\mathcal{T} \times \cdots \times \mathrm{Aut}\,\mathcal{T}$.

I + II = Describing elements of $\operatorname{Aut}\mathcal{T}$

- Any $g \in \operatorname{Aut}\mathcal{T}_d$ can be seen as

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In other words, every $f \in \operatorname{Aut}\mathcal{T}_d$ can be written as

$$f = (f_1, \ldots, f_d)a,$$

where $f_i \in \operatorname{Aut}\mathcal{T}_d$ and a is rooted corresponding to some permutation $\sigma \in \operatorname{Sym}(d)$.

Example

Let $f \in \operatorname{Aut} \mathcal{T}_d$ with $f = (f_1, f_2, \ldots, f_d)a$, where $f_i \in \operatorname{Aut} \mathcal{T}_d$ and a is rooted corresponding to σ. If $f_1 = f_2 = \cdots = f_d = 1$, then f is rooted.

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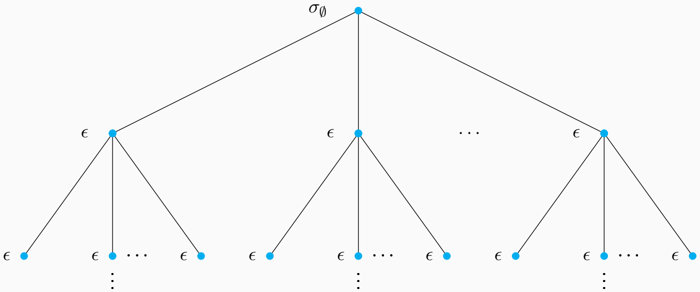

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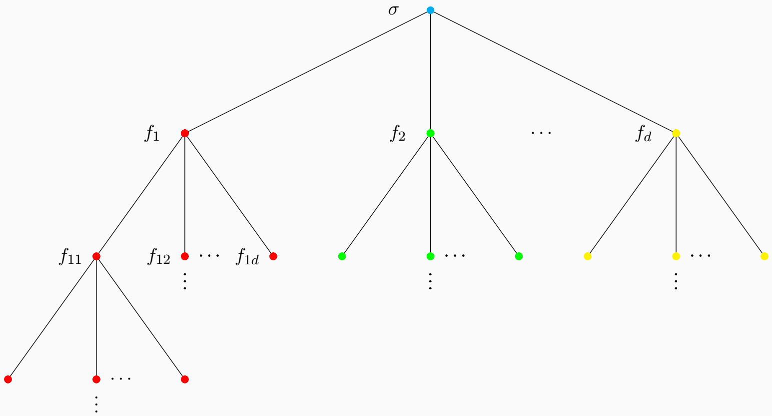

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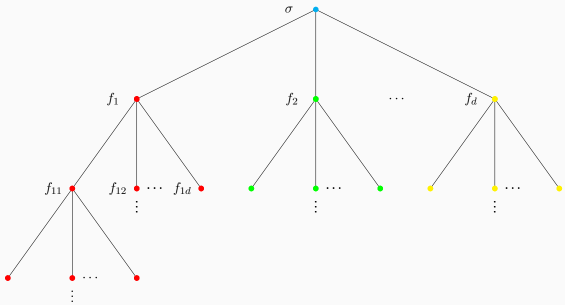

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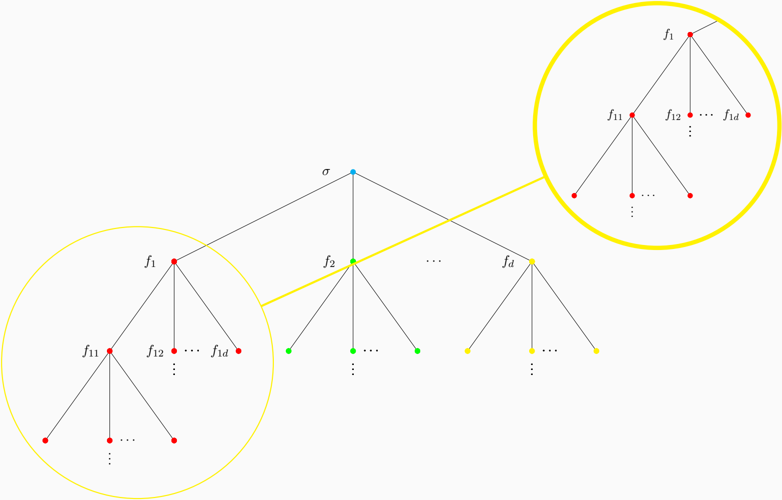

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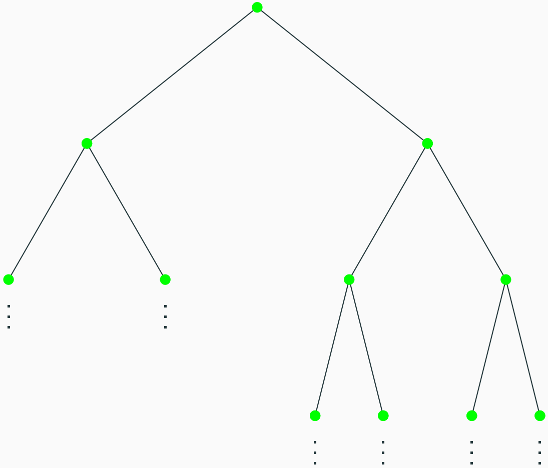

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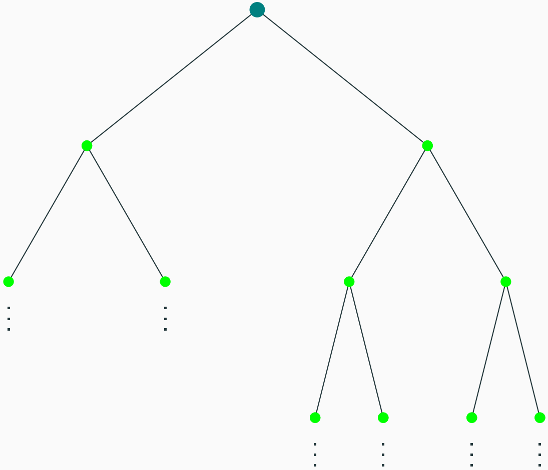

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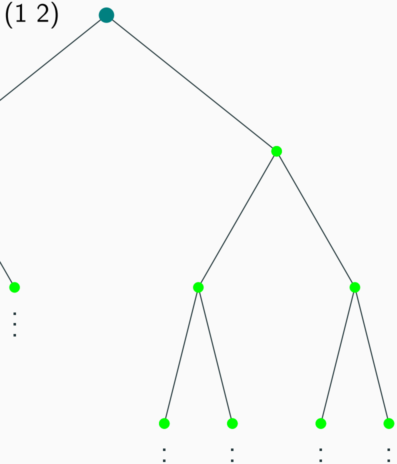

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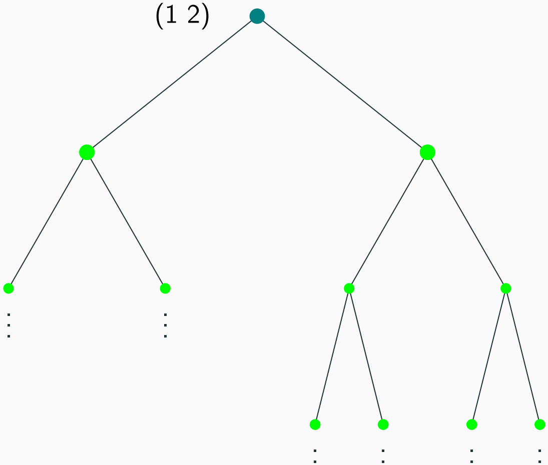

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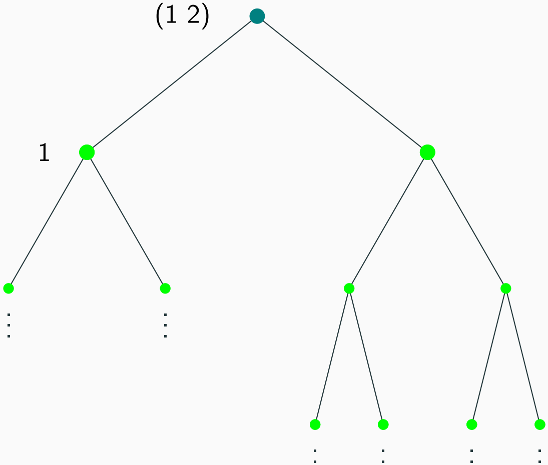

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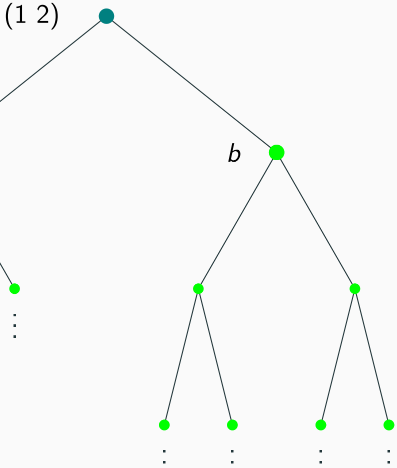

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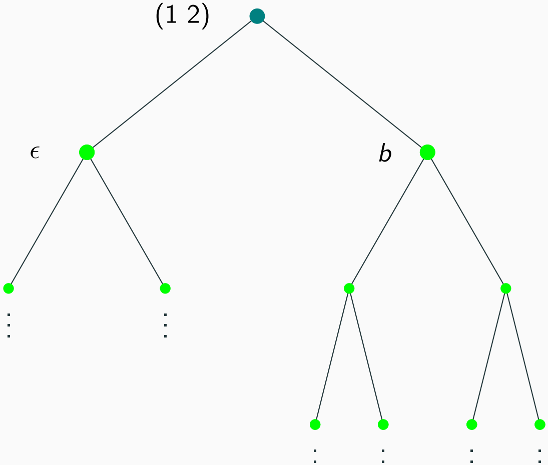

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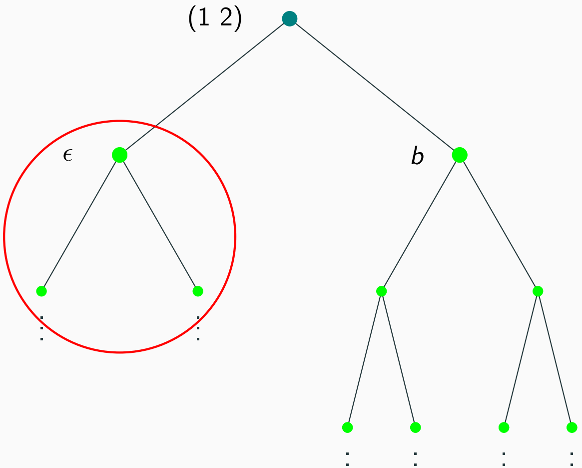

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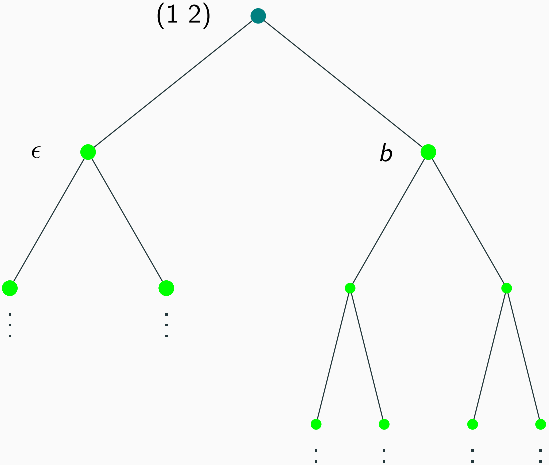

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$$b = (1, b)a.$$

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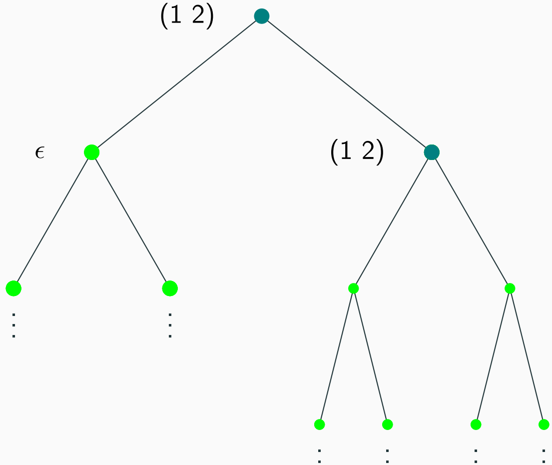

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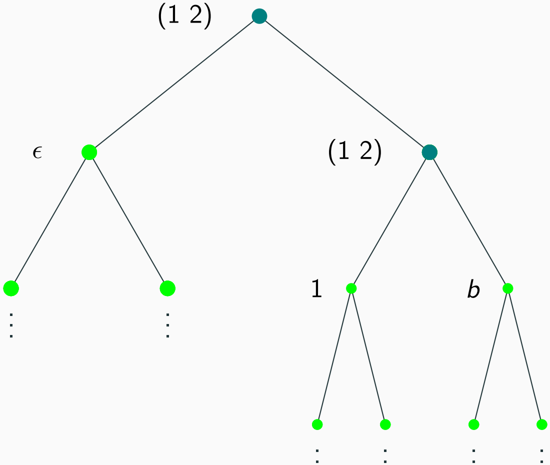

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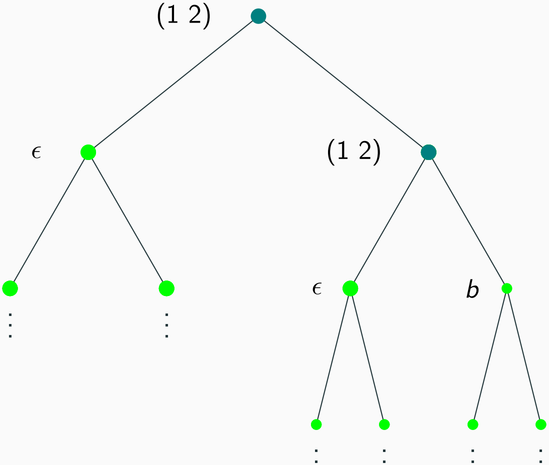

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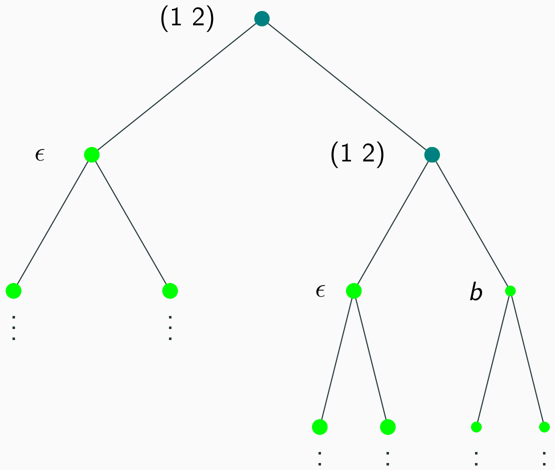

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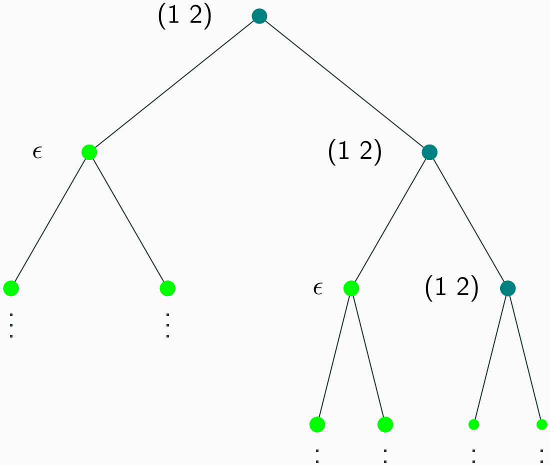

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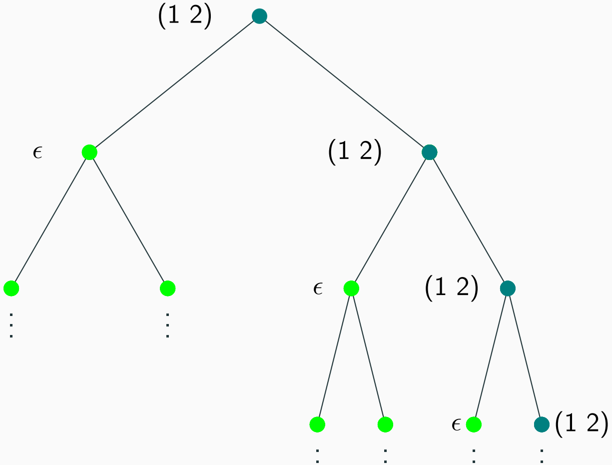

Exercise

If $\mathcal{T}_7$ is the 7-adic tree and a is rooted corresponding to $(1\ 2\ 3\ 4\ 5\ 6\ 7)$, let

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Branch groups

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- Recall that in the full group of automorphisms we have

$$\mathrm{st}(n) \simeq \operatorname{Aut}\mathcal{T} \times \overset{d^n}{\cdots} \times \operatorname{Aut}\mathcal{T},$$

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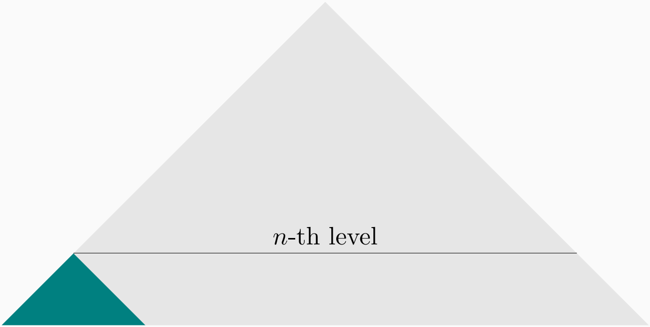
  where $\psi_n(\mathrm{st}_G(n))$ need not be a direct product.
- The question is: given $G \leq \operatorname{Aut}\mathcal{T}$, can we find for every $n \in \mathbb{N}$ a subgroup (eventually of finite index) of $\mathrm{st}_G(n)$ which is a direct product?

Rigid stabilizers

The *rigid stabilizer* of the vertex u is

$$\mathrm{rst}_G(u) = \{g \in G : g \text{ fixes all vertices outside } \mathcal{T}_u\}$$



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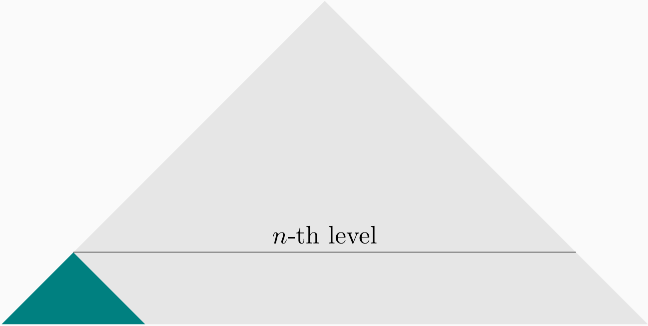

The *rigid stabilizer* of the nth level is $\mathrm{rst}_G(n) = \prod_{u \in X^n} \mathrm{rst}_G(u)$.

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- Good news: in some cases, there exist "nice" rigid stabilizers.
- Informally speaking: the subgroup $\psi_n(\mathrm{rst}_G(n))$ is the largest subgroup of $\psi_n(\mathrm{st}_G(n))$ which is a "geometric" direct product.

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- The first Grigorchuk group is a branch group.

More definitions: Self-similar groups

Let $G \leq \operatorname{Aut} \mathcal{T}$.

- A group G is said to be *self-similar* if taken $g = (g_1, \ldots, g_d)\sigma \in G$ we have $g_i \in G$ for any $i = \{1, \ldots, d\}$.

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- Example: $\operatorname{Aut} \mathcal{T}$ is self-similar, the first Grigorchuk is self-similar.
- Non-example: The group $G = \langle a, b \rangle$, where $a = (b, c)\sigma$ and $c \notin G$, then G is not self-similar.

Regular branch groups

Let G be a self-similar group. We say that G is a *regular branch* if there exists a subgroup K of $\mathrm{st}_G(1)$ of finite index such that

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More precisely we have this situation:

$$\begin{array}{ccc}
G & & G \times \overset{d}{\ldots} \times G \\
\Big| & & \Big| \\
\mathrm{st}_G(1) & \xrightarrow{\psi} & \psi(\mathrm{st}_G(1)) \\
\Big| & & \Big| \\
K & \xrightarrow{\psi} & \psi(K) \\
\Big| & & \Big| \\
L & \xrightarrow{\psi} & K \times \overset{d}{\ldots} \times K
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- Regular branch $\longrightarrow$ branch.

Next lecture

Examples of (weakly) branch groups

Next week we will present the following groups of automorphisms of rooted trees together with their main properties:

- The Grigorchuk groups
- The GGS-groups
- The Basilica group
- The Hanoi Tower group

I am sure I was too quick, so there is still time . . . .

The first Grigorchuk group (finally!)

$$\Gamma = \langle a, b, c, d \rangle$$

$$a = (1, 1)(12) \quad b = (a, c) \quad c = (a, d) \quad d = (1, b)$$

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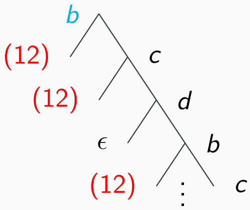

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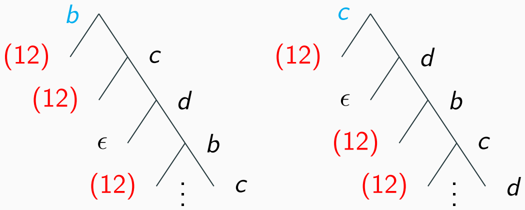

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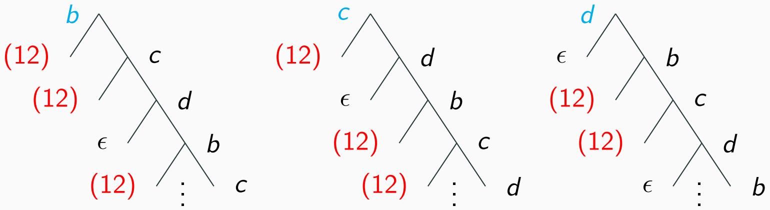

Some properties of Γ

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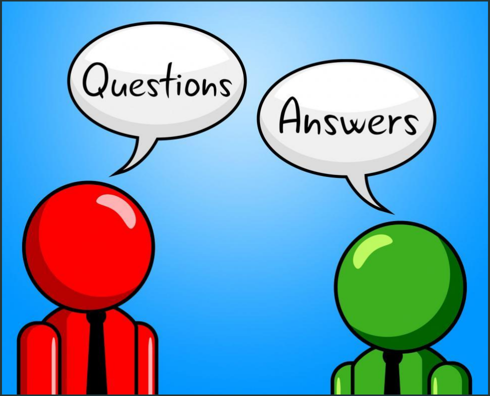
29th of October

$\ddot\smile$


Questions
Answers

Thank you :)