Free groups via graphs, part 2

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Last week.

- Maps of graphs $\leftrightarrow$ subgroups of free groups.
- Number of generators, subgroup membership problem, normality, index, intersection, malnormality…
- Immersions “are” subgroups.
- Finite coverings correspond to finite index subgroups.
- However, coverings are useless for infinite-index subgroups!

This week.

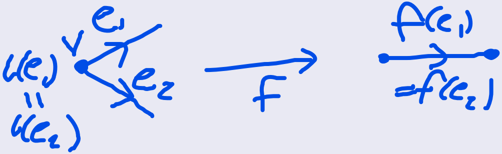
- Foldings take a subgroup and produce an immersion.
- Immersions are brilliant for infinite-index subgroups!

Folding (formal)

$\text{star}(v) = \{e \in E \mid \iota(e) = v\}$

$f : \Gamma \to \Delta$ a graph, $e_1, e_2 \in E\Gamma$ such that:

1. $\iota(e_1) = \iota(e_2)$
2. $f(e_1) = f(e_2)$



then

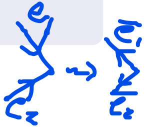
1. f folds e_1 and e_2,
2. f factors through the graph $\Gamma/[e_1 = e_2]$ obtained by identifying $\tau(e_1)$ with $\tau(e_2)$ and e_1 with e_2.

Therefore, if Γ is a finite graph then f factors as

$$\underbrace{\Gamma = \Gamma_0 \to \Gamma_1 \to \cdots \to \Gamma_n}_{\text{folds}} \underbrace{\xrightarrow{\quad f' \quad}}_{\text{immersion}} \Delta$$

The immersion $f' : \Gamma_n \to \Delta$ is unique.

Note. Folding maps are π_1-surjective.



Foldings *correspond* to subgroups

Theorem 4.

Let Δ be a connected graph, $v \in V\Delta$ a vertex, and $H \leq \pi_1(\Delta, v)$ a finitely generated subgroup. Then there exists an immersion $f : \Gamma \to \Delta$ where Γ is connected, $u \in V\Gamma$ a vertex with $f(u) = v$, and $f\pi_1(\Gamma, u) = H$. ← ∃ immersion.

← given subgp

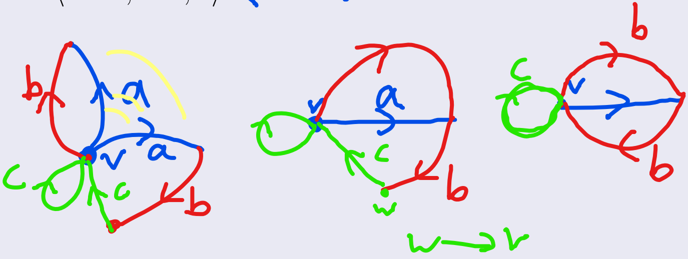
Proof.

Proof is constructive/algorithmic:

1. Input: a finite generating set $\{\alpha_1, \ldots, \alpha_n\}$ for H.
2. Form the map of graphs corresponding to the directed graph with central vertex w and loops $p_1, \ldots, p_n$, where p_i has label α_i.
3. Fold.
4. The resulting map of graphs is an immersion, with image H. (Image is H as folds are π_1-surjective.)

□

Let $H = \langle ba^{-1}, abc, c \rangle$. $\leqslant F(a, b, c)$

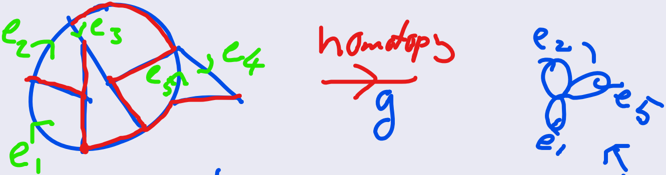


Let Γ be a graph, $v \in V$ a vertex, $T \subset \Gamma$ a spanning tree of Γ. Clearly, for each edge $e_i \in \Gamma \setminus T$ there exist reduced paths $p_i, q_i \subset T$ such that $p_i e_i q_i$ is a loop at v.

Theorem 5.

The set $[p_i e_i q_i]$ forms a basis for $\pi_1(\Gamma, v)$.

Proof.



$\forall\, i = 1, \ldots, 5$, pick a element $b_i \in g^{-1}(e_i)$

$b_i \leftrightarrow p_i e_i q_i$

therefore

$[e_1], \ldots, [e_5]$ form a basis of $\pi_1(\infty)$ □

Application 1: more preliminaries to M. Hall's theorem

As the set $[p_i e_i q_i]$ forms a basis for $\pi_1(\Gamma, v)$, we have:

Corollary 6.

$\pi_1(\Gamma, v)$ has rank $\underline{|E\Gamma| - |ET|} = |E\Gamma| - |V\Gamma| + 1$.

Proof.

obv. because $|ET| = |VT| - 1 = |V\Gamma| - 1$ $\square$

Corollary 7 (Subgraphs "are" free factors).

Let Σ be a subgraph of Γ, and let $v \in V\Sigma \cap V\Gamma$. Then $\pi_1(\Sigma, v)$ is a free factor of $\pi_1(\Gamma, v)$.

Proof.

Extra edges in Γ $\longleftrightarrow$ extra generators

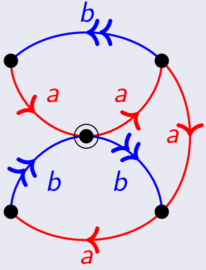
$\square$

Application 1: M. Hall's theorem

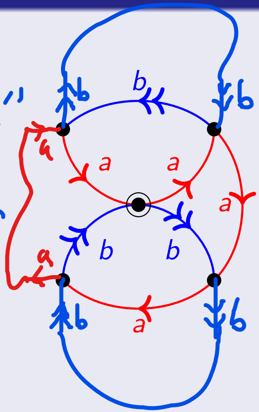
Theorem 8 (M. Hall's theorem).

Let F be a f.g. free group, H a f.g. subgroup. Then there exists $K \leq F$ with finite index such that $K = K' * H$.

Proof.



"Complete" immersion to get a cover



$\square$

Pullbacks of immersions represent intersection

Pullback: $\Gamma_3 \subseteq \Gamma_1 \times \Gamma_2$ with $V\Gamma_3 = \{(u_1, u_2) \mid f_1(u_1) = f_2(u_2)\}$ &
$E\Gamma_3 = \{(e_1, e_2) \mid f_1(e_1) = f_2(e_2)\}$.

Theorem 9.

Let $f_i : \Gamma_i \to \Delta$, $i = 1, 2$, be immersions and let

$$\begin{tikzcd} \Gamma_3 \arrow[r, "g_1"] \arrow[d, "g_2"'] \arrow[dr, "f_3"] & \Gamma_1 \arrow[d, "f_1"] \\ \Gamma_2 \arrow[r, "f_2"'] & \Delta \end{tikzcd}$$

$g_1 : \Gamma_3 \to \Gamma_1$ forgets second component, $g_1(x,y) = x$.

be their pullback diagram. Let $v_1 \in \Gamma_1$, $v_2 \in \Gamma_2$ be such that $f_1(v_1) = w = f_2(v_2)$; let v_3 be the corresponding vertex in Γ_3. Define $f_3 = f_1 g_1 = f_2 g_2 : \Gamma_3 \to \Delta$, and define

$$S_i = f_i \pi_1(\Gamma_i, v_i) \qquad i = 1, 2, 3.$$

$\Gamma_1 \leftrightarrow S_1$
$\Gamma_2 \leftrightarrow S_2$
$\Gamma_3 \leftrightarrow S_3$

(These are subgroups of $\pi_1(\Delta, w)$.)
Then

$$S_3 = S_1 \cap S_2. \quad //$$

Application 2: Howson's theorem

Theorem 10 (Howson's theorem).

Let H, K be f.g. subgroups of a free group F. Then $H \cap K$ is finitely generated (and a free basis of $H \cap K$ can be determines by an easy algorithm).

Proof.

1. Use the folding algorithm to find immersions $f_1 : \Gamma_1 \to \Delta$ and $f_2 : \Gamma_2 \to \Delta$ representing H and K respectively.
2. Consider the pullback map $f_3 : \Gamma_3 \to \Delta$.
3. Then $H \cap K = f_3\pi_1(\Gamma_3, v_3)$ is finitely generated as Γ_3 is a finite graph.
4. Construction of pullback is algorithmic.
5. A basis B for $\pi_1(\Gamma_3, v_3)$ is found via Theorem 5.
6. $f_3(B)$ is a basis for $H \cap K$.

□

Hanna Neumann conjecture/Friedman–Mineyev theorem

$$\mathrm{rank}(H \cap K) - 1 \leq (\mathrm{rank}(H) - 1)(\mathrm{rank}(K) - 1)$$

$\mathrm{rk}(H \cap K) \leq (2-1)(\mathrm{rk}(K)-1)+1 = \mathrm{rk}(K)$

Theorem 11 (Tardos, 1992).

Let $H, K \leq F$ with $\underline{\mathrm{rank}(H) = 2}$. Then:

$$\mathrm{rank}(H \cap K) \leq \mathrm{rank}(K)$$

Proof.

Assume $H, K \leq F_2$.
$\Gamma_1, \Gamma_2 \to \infty$.

The branching number of Γ is:
$b(\Gamma) = \#\{\text{vertices of degree} \geq 3\} + \#\{\text{vertices of degree } 4\}$.
For Γ connected, $\mathrm{rank}(\Gamma) = b(\Gamma)/2 + 1$.
If Γ is not a tree, then $\mathrm{core}(\Gamma)$ is the minimal subgraph of Γ containing every loop in Γ. So prove:
Let $f_i : \Gamma_i \to \Delta$, $i = 1, 2$, immersions of connected graphs, $\mathrm{core}(\Gamma_i) = \Gamma_i$ and $b(\Delta) = b(\Gamma_1) = 2$. Then

$$b(\mathrm{core}(\Gamma_1 \times \Gamma_2)) \leq b(\Gamma_2). \qquad \square$$

Summary

- Maps of graphs $\leftrightarrow$ subgroups of free groups.
- Number of generators, subgroup membership problem, normality, index, intersection, malnormality...
- Immersions ~~"are"~~ correspond to subgroups (via foldings).
- Finite coverings correspond to finite index subgroups.

Easy proofs of interesting theorems:

- M. Hall's theorem: f.g. subgroups are free factors of finite index subgroups.
- Howson's theorem: intersections of f.g. subgroups are f.g.

Applications to free groups.

- Karen Vogtmann Automorphisms of free groups and outer space. *Geom. Dedicata* 94 (2002): 1–31
- Mladen Bestvina & Michael Handel Train tracks and automorphisms of free groups. *Ann. Math. (2)* 135.1 (1992): 1–51
- Igor Mineyev Groups, graphs, and the Hanna Neumann conjecture. *J. Topol. Anal.* 4.1 (2012): 1–12
- Joel Friedman Sheaves on graphs, their homological invariants, and a proof of the Hanna Neumann conjecture *Mem. Amer. Math. Soc.* 233.1100 (2015)
- Laura Ciobanu & Alan D. Logan The Post Correspondence Problem and equalisers for certain free group and monoid morphisms *47th International Colloquium on Automata, Languages and Programming (ICALP2020)* 168 (2020) 120:1-120:16.

Applications to other groups.

- Mark Feighn & Michael Handel Mapping tori of free group automorphisms are coherent. *Ann. Math. (2)* 149.3 (1999): 1061–1077

$F \rtimes \mathbb{Z}$

- Daniel T. Wise The residual finiteness of positive one-relator groups. *Comment. Math. Helv.* 76.2 (2001): 314–338.

Small Cancellation theory.

- Daniel T. Wise The residual finiteness of negatively curved polygons of finite groups. *Invent. Math.* 149.3 (2002): 579–617
- Martin Bridson & Henry Wilton The triviality problem for profinite completions. *Invent. Math.* 202.2 (2015): 839–874.
- Alan D. Logan Every group is the outer automorphism group of an HNN-extension of a fixed triangle group *Adv. Math.* 353 (2019) 116–152.

$\phi(M) \cap M = 1 \ \forall \phi \in \mathrm{Aut} \setminus \mathrm{Inn}$

& M malnormal

Generalisations to other groups.

- Ilya Kapovich & Paul Schupp Genericity, the Arzhantseva–Ol'shanskii method and the isomorphism problem for one-relator groups. *Math. Ann.* 331.1 (2005): 1–19
- Henry Wilton Hall's theorem for limit groups. *Geom. Funct. Anal.* 18.1 (2008): 271–202.
- Benjamin Beeker & Nir Lazarovich Stallings' folds for cube complexes. *Israel J. Math.* 227.1 (2018): 331–363
- Michael Ben-Zvi, Robert Kropholler & Rylee Lyman Folding-like techniques for CAT(0) cube complexes. *arXiv:2011.05374*
- Larsen Louder & Henry Wilton Negative immersions for one-relator groups. *Math. Res. Lett., to appear*
- Olga Kharlampovich & Alexei Miasnikov & Pascal Weil Stallings graphs for quasi-convex subgroups. *J. Algebra* 488 (2017) 442–483

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