Free groups via graphs, part 2

Alan D. Logan

a.logan@hw.ac.uk

Maxwell Institute, Heriot–Watt University,
Edinburgh EH14 4AS, Scotland

Last week.

- Maps of graphs $\leftrightarrow$ subgroups of free groups.
- Number of generators, subgroup membership problem, normality, index, intersection, malnormality...
- Immersions "are" subgroups.
- Finite coverings correspond to finite index subgroups.
- However, coverings are useless for infinite-index subgroups!

This week.

- Foldings take a subgroup and produce an immersion.
- Immersions are brilliant for infinite-index subgroups!

Folding (formal)

$f : \Gamma \to \Delta$ a graph, $e_1, e_2 \in E\Gamma$ such that:

1. $\iota(e_1) = \iota(e_2)$
2. $f(e_1) = f(e_2)$

then

1. f folds e_1 and e_2,
2. f factors through the graph $\Gamma/[e_1 = e_2]$ obtained by identifying $\tau(e_1)$ with $\tau(e_2)$ and e_1 with e_2.

Therefore, if Γ is a finite graph then f factors as

$$\underbrace{\Gamma = \Gamma_0 \to \Gamma_1 \to \cdots \to \Gamma_n}_{\text{folds}} \underbrace{\xrightarrow{\quad f' \quad}}_{\text{immersion}} \Delta$$

The immersion $f' : \Gamma_n \to \Delta$ is unique.

Note. Folding maps are π_1-surjective.

Foldings *correspond* to subgroups

Theorem 4.

Let Δ be a connected graph, $v \in V\Delta$ a vertex, and $H \leq \pi_1(\Delta, v)$ a finitely generated subgroup. Then there exists an immersion $f : \Gamma \to \Delta$ where Γ is connected, $u \in V\Gamma$ a vertex with $f(u) = v$, and $f\pi_1(\Gamma, u) = H$.

Proof.

Proof is constructive/algorithmic:

1. Input: a finite generating set $\{\alpha_1, \ldots, \alpha_n\}$ for H.
2. Form the map of graphs corresponding to the directed graph with central vertex w and loops $p_1, \ldots, p_n$, where p_i has label α_i.
3. Fold.
4. The resulting map of graphs is an immersion, with image H. (Image is H as folds are π_1-surjective.)

□

Example of folding

Let $H = \langle ba^{-1}, abc, c \rangle$.

Let Γ be a graph, $v \in V$ a vertex, $T \subset \Gamma$ a spanning tree of Γ. Clearly, for each edge $e_i \in \Gamma \setminus T$ there exist reduced paths $p_i, q_i \subset T$ such that $p_i e_i q_i$ is a loop at v.

Theorem 5.

The set $[p_i e_i q_i]$ forms a basis for $\pi_1(\Gamma, v)$.

Proof.

□

Application 1: more preliminaries to M. Hall's theorem

As the set $[p_i e_i q_i]$ forms a basis for $\pi_1(\Gamma, v)$, we have:

Corollary 6.

$\pi_1(\Gamma, v)$ has rank $|E\Gamma| - |ET| = |E\Gamma| - |V\Gamma| + 1$.

Proof.

□

Corollary 7 (Subgraphs "are" free factors).

Let Σ be a subgraph of Γ, and let $v \in V\Sigma \cap V\Gamma$. Then $\pi_1(\Sigma, v)$ is a free factor of $\pi_1(\Gamma, v)$.

Proof.

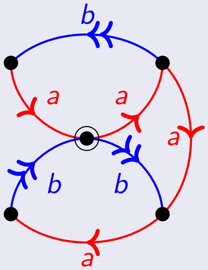
□

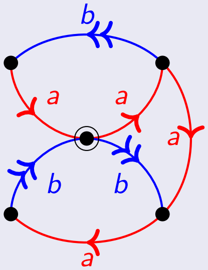
Application 1: M. Hall's theorem

Theorem 8 (M. Hall's theorem).

Let F be a f.g. free group, H a f.g. subgroup. Then there exists $K \leq F$ with finite index such that $K = K' * H$.

Proof.





□

Pullbacks of immersions represent intersection

Pullback: $\Gamma_3 \subseteq \Gamma_1 \times \Gamma_2$ with $V\Gamma_3 = \{(u_1, u_2) \mid f_1(u_1) = f_2(u_2)\}$ &
$E\Gamma_3 = \{(e_1, e_2) \mid f_1(e_1) = f_2(e_2)\}$.

Theorem 9.

Let $f_i : \Gamma_i \to \Delta$, $i = 1, 2$, be immersions and let

$$\begin{array}{ccc} \Gamma_3 & \xrightarrow{g_1} & \Gamma_1 \\ {\scriptstyle g_2}\downarrow & & \downarrow{\scriptstyle f_1} \\ \Gamma_2 & \xrightarrow[f_2]{} & \Delta \end{array}$$

be their pullback diagram. Let $v_1 \in \Gamma_1$, $v_2 \in \Gamma_2$ be such that $f_1(v_1) = w = f_2(v_2)$; let v_3 be the corresponding vertex in Γ_3. Define $f_3 = f_1 g_1 = f_2 g_2 : \Gamma_3 \to \Delta$, and define

$$S_i = f_i \pi_1(\Gamma_i, v_i) \qquad\qquad i = 1, 2, 3.$$

(These are subgroups of $\pi_1(\Delta, w)$.)
Then

$$S_3 = S_1 \cap S_2.$$

Application 2: Howson's theorem

Theorem 10 (Howson's theorem).

Let H, K be f.g. subgroups of a free group F. Then $H \cap K$ is finitely generated (and a free basis of $H \cap K$ can be determines by an easy algorithm).

Proof.

1. Use the folding algorithm to find immersions $f_1 : \Gamma_1 \to \Delta$ and $f_2 : \Gamma_2 \to \Delta$ representing H and K respectively.
2. Consider the pullback map $f_3 : \Gamma_3 \to \Delta$.
3. Then $H \cap K = f_3\pi_1(\Gamma_3, v_3)$ is finitely generated as Γ_3 is a finite graph.
4. Construction of pullback is algorithmic.
5. A basis B for $\pi_1(\Gamma_3, v_3)$ is found via Theorem 5.
6. $f_3(B)$ is a basis for $H \cap K$.

□

Application 3: On the Hanna Neumann Conjecture

Hanna Neumann conjecture/Friedman–Mineyev theorem

$\text{rank}(H \cap K) - 1 \leq (\text{rank}(H) - 1)(\text{rank}(K) - 1)$

Theorem 11 (Tardos, 1992).

Let $H, K \leq F$ *with* $\text{rank}(H) = 2$. *Then:*

$$\text{rank}(H \cap K) \leq \text{rank}(K)$$

Proof.

The branching number of Γ is:
$b(\Gamma) = \#\{\text{vertices of degree} \geq 3\} + \#\{\text{vertices of degree } 4\}$.
For Γ connected, $\text{rank}(\Gamma) = b(\Gamma)/2 + 1$.
If Γ is not a tree, then $\text{core}(\Gamma)$ is the minimal subgraph of Γ containing every loop in Γ. So prove:
Let $f_i : \Gamma_i \to \Delta$, $i = 1, 2$, immersions of connected graphs, $\text{core}(\Gamma_i) = \Gamma_i$ and $b(\Delta) = b(\Gamma_1) = 2$. Then

$$b(\text{core}(\Gamma_1 \times \Gamma_2)) \leq b(\Gamma_2).$$

□

Summary

- Maps of graphs $\leftrightarrow$ subgroups of free groups.
- Number of generators, subgroup membership problem, normality, index, intersection, malnormality...
- Immersions ~~"are"~~ correspond to subgroups (via foldings).
- Finite coverings correspond to finite index subgroups.

Easy proofs of interesting theorems:

- M. Hall's theorem: f.g. subgroups are free factors of finite index subgroups.
- Howson's theorem: intersections of f.g. subgroups are f.g.

Applications to free groups.

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