Free groups via graphs

Alan D. Logan

a.logan@hw.ac.uk

Maxwell Institute, Heriot–Watt University,
Edinburgh EH14 4AS, Scotland

Free groups via graphs.

We shall study subgroups of free groups. Ingredient needed:

1. combinatorics on graphs.
2. a spoonful of category theory.
3. a pinch of topology.

All combined using a simple operation called "folding".

- Stallings, John R. Topology of finite graphs. *Invent. Math.* 71.3 (1983): 551–565.
- Kapovich, Ilya; Myasnikov, Alexei Stallings foldings and subgroups of free groups. *J. Algebra* 248.2 (2002): 608–668
- Clay, Matt Office hour four. *Office hours with a geometric group theorist* (2017): 66–84
- Magnus, Wilhelm, et al. Combinatorial group theory (1966).

Folding (informal)

Γ a graph, $e_1, e_2 \in E$ such that:

1. $\iota(e_1) = \iota(e_2)$ or $\tau(e_1) = \tau(e_2)$
2. $\text{label}(e_1) = \text{label}(e_2)$

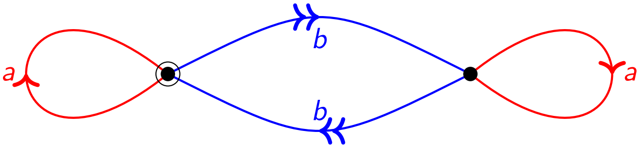
then the *folding* of e_1 and e_2 is the graph $\Gamma/[e_1 = e_2]$ obtained by identifying $\tau(e_1)$ with $\tau(e_2)$, or $\iota(e_1)$ with $\iota(e_2)$, and e_1 with e_2.

Think: make deterministic.

Let $H = \langle a, b^2, bab \rangle$.

Examples: generators, membership problem

So $H = \langle a, b^2, bab \rangle$ has "Stallings' graph":

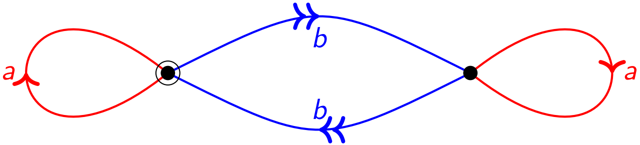


Questions:

- How many generators does H require?
- Is $babab$ in H?
- Is $bab^{-1}a^{-1}$ in H?

Examples: normality, index

So $H = \langle a, b^2, bab \rangle$ has "Stallings' graph":



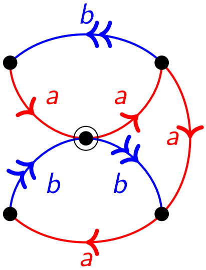
Questions:

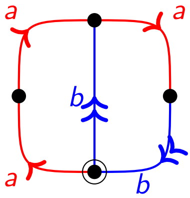
- Is H normal in $F(a, b)$?

- Does H have finite index in $F(a, b)$?

Examples: intersection

Set $K_1 = \langle aba, a^3b, bab\rangle$ and $K_2 = \langle a^2b, bab\rangle$:



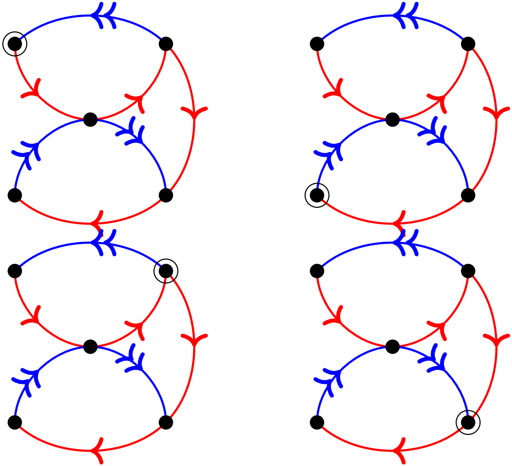


- What is $K_1 \cap K_2$?

A subgroup M of a group G is *malnormal* if for all $g \in G \setminus M$, $M^g \cap M = \{1\}$,

Question:

- Is $K_1 = \langle aba, a^3b, bab \rangle$ malnormal in $F(a, b)$?

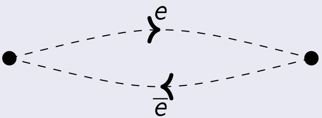


Definition of graphs

Category of graphs

A graph Γ consists of two sets E and V (sometimes "$V\Gamma$"), and two functions $E \to E$ and $E \to V$: For each $e \in E$ there exists $\overline{e} \in E$ and an element $\iota(e) \in V$ such that

- $\overline{\overline{e}} = e$, and
- $\overline{e} \neq e$



Set $\tau(e) = \iota(\overline{e})$.

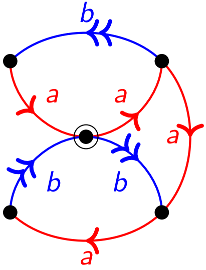
A map of graphs $f : \Gamma \to \Delta$ consists of a pair of functions, edges to edges, vertices to vertices, preserving the structure.

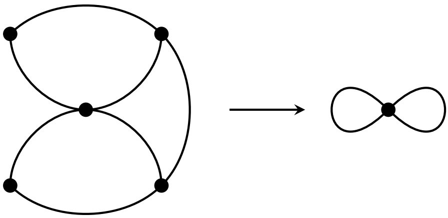
Maps of graphs

Direction

- Each e is a directed edge (or "half edge").
- An orientation of Γ is a choice of exactly one edge in each pair $\{e, \overline{e}\}$.

A Stallings' graph is just a map of graphs:





Fundamental groupoids and fundamental groups

- A round trip is a path of the form $e\bar{e}$.
- If p contains such a subpath, we can delete it to obtain a new subpath p'.
- p and p' are homotopy equivalent.
- $\pi_1(\Gamma)$ is the set of equivalence classes $[p]$ of paths p such that $\iota(p) = \tau(p)$.
- $\pi_1(\Gamma, v)$ is the set of equivalence classes $[p]$ of paths p such that $\iota(p) = v = \tau(p)$.

c.f. Section 1.2 of Magnus, Karrass and Solitar.

$\pi_1(\Gamma, v)$ is a free group of rank $|E| - |V| + 1$.

Immersions and covering

Stars

For $v \in V$, the star of v is the set of edges:

$$\mathrm{st}(v, \Gamma) = \{e \in E \mid \iota(e) = v\}$$

Immersions, covers

A map of graphs $f : \Gamma \to \Delta$ yields for each $v \in V\Gamma$ a function

$$f_v : \mathrm{st}(v, \Gamma) \to \mathrm{st}(f(v), \Delta).$$

- If for all $v \in V\Gamma$ the map f_v is injective, f is an immersion.
- If for all $v \in V\Gamma$ the map f_v is bijective, f is a covering.

Immersions "are" subgroups

Theorem 1.

If $f : \Gamma \to \Delta$ is an immersion, and $u \in V\Gamma$, then the induced map

$$f_* : \pi_1(\Gamma, v) \to \pi_1(\Delta, f(v))$$

is injective.

Sketch Proof.

Let $\alpha \in \pi_1(\Gamma, v)$, $\alpha \neq 1$. So $\alpha = [p]$ with $\iota(p) = v = \tau(p)$, where p is reduced and $|p| \geq 1$.

- As f is an immersion, the circuit $f(p)$ is also reduced.
- As f is an immersion, $|f(p)| = |p| \geq 1$.
- By the theory of free groups (Magnus, Karrass and Solitar, Theorem 1.2), as $f(p)$ is a reduced, non-trivial loop the equivalence class $[f(p)]$ is non-trivial in $\pi_1(\Delta, f(v))$.

□

Coverings *correspond* to subgroups

Theorem 2.

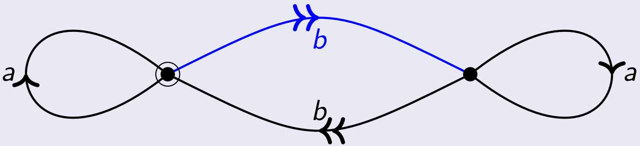
Let Δ *be a connected graph,* $v \in V\Delta$ *a vertex, and* $H \leq \pi_1(\Delta, v)$ *a subgroup. Then there exists a ("unique") covering* $f : \Gamma \to \Delta$ *where* Γ *is connected,* $u \in V\Gamma$ *a vertex with* $f(u) = v$*, and* $f\pi_1(\Gamma, u) = H$*.*

Proposition 3.

If Γ is connected and $f : \Gamma \to \Delta$ is a covering with $f(u) = v$, then the index of $H := f\pi_1(\Gamma, u)$ in $\pi_1(\Delta, v)$ is the cardinality of $f^{-1}(v)$.

Sketch Proof.

- For each $w \in f^{-1}(v) \subseteq V\Gamma$, select a reduced path $q_w := [u, w] \subset \Gamma$.
- Claim. $\mathcal{T} := \{[f(q_w)] : w \in f^{-1}(v)\}$ is a transversal for H.
- As f is a covering, for each reduced circuit $p \subset \Delta$ at v, there exists a reduced path $q \subset \Gamma$ with $f(q) = p$ and $\iota(q) = u$.



□

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- Claim. $\mathcal{T} := \{[f(q_w)] : w \in f^{-1}(v)\}$ is a transversal for H.
- As f is a covering, for each reduced circuit $p \subset \Delta$ at v, there exists a reduced path $q \subset \Gamma$ with $f(q) = p$ and $\iota(q) = u$.
- Therefore, for all $[p] \in \pi_1(\Delta, v)$ there exists q_w such that $[p] = H[f(q_w)]$.
- So $\mathcal{T}$ are coset representatives. In fact...
- $\mathcal{T}$ is a transversal ($[f(q_w)][f(q_{w'})]^{-1} \in H \Rightarrow q_w = q_{w'}$).

The result follows as $|\mathcal{T}| = |f^{-1}(v)|$. □

Summary

- Maps of graphs $\leftrightarrow$ subgroups of free groups.
- Number of generators, subgroup membership problem, normality, index, intersection, malnormality...
- Immersions "are" subgroups.
- Finite coverings correspond to finite index subgroups.
- However, coverings are useless for infinite-index subgroups!

Exercise 1. Fill in the details in the proof of Theorem 1.
Exercise 2. Fill in the details in the proof of Proposition 3.
Exercise 3. Prove that if Γ is connected and $f : \Gamma \to \Delta$ is an immersion but not a cover then $f_*(\pi_1(\Gamma, v))$ has finite index in $\pi_1(\Delta, f(v))$.

Next week

- Immersions are brilliant for infinite-index subgroups!
- Foldings take a subgroup and produce an immersion.

Thank you for your attention!