Introduction to Growth in Groups
Part II: Formal power series

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Last week

Word length

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Definition

The strict growth function $\sigma_{G,S}(n) = \#\{g \in G \mid |g|_S = n\}$,

and the cumulative growth function $\beta_{G,S}(n) = \#\{g \in G \mid |g|_S \leq n\}$.

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Conjugacy classes

Define the length of a conjugacy class κ of G with respect to S to be the length of a shortest word representing κ:

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Definition

The strict conjugacy growth function $s_{G,S}(n) = \#\{\kappa \in \mathcal{C}_G \mid |\kappa|_S = n\}$,

and the cumulative conjugacy growth function $c_{G,S}(n) = \#\{\kappa \in \mathcal{C}_G \mid |\kappa|_S \leq n\}$.

Standard vs Conjugacy growth

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Group invariant	Yes	Yes
Quasi-Isometry Invariant	Yes	No
Polynomial growth	n^d for $d \in \mathbb{N}$	"anything"

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- For any group G, $c_G(n) \preccurlyeq \beta_G(n)$.
- If G is abelian then $c_G(n) \sim \beta_G(n)$ (converse does not hold).

Growth series

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Definition

The (standard/conjugacy/etc.) growth series of G with respect to S is the formal power series

$$\mathbb{S}(z) := \sum_{n=0}^{\infty} \gamma(n) z^n.$$

Algebraic complexity

A formal power series $\mathbb{S}(z)$ is called

- rational if there exist polynomials P, Q with integer coefficients such that $\mathbb{S}(z) = \frac{P(z)}{Q(z)}$;
- algebraic if $\mathbb{S}(z)$ satisfies a polynomial equation with polynomial coefficients;
- holonomic (a.k.a. D-finite) if $\mathbb{S}(z)$ satisfies a finite order differential equation, with polynomial coefficients;
- transcendental if it is not algebraic.

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Question: Into which classes do the various growth functions fall?

Examples

- Standard growth of F_2, with respect to a basis: $\sigma(n) = 4 \cdot 3^{n-1}$ for $n \geq 1$

$$\mathbb{S}(z) = 1 + \sum_{n \geq 1} 4 \cdot 3^{n-1} z^n = 1 + \frac{4}{3} \sum_{n \geq 1} (3z)^n = \frac{1 - 2z}{1 - 3z}$$

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- Standard (and conjugacy) growth of $\mathbb{Z}^2$ with respect to $\{(1,0),(0,1)\}$: $\sigma(n) = 4n$ for $n \geq 1$

$$\mathbb{S}(z) = 1 + \sum_{n \geq 1} (4n) z^n = \frac{(1+z)^2}{(1-z)^2}$$

Rational growth

Rational growth reflects a strong 'regularity' property:

Proposition

A series $\mathbb{S}(z) = \sum \gamma(n) z^n \in \mathbb{Z}[[z]]$ is rational if and only if $\gamma(n)$ satisfies a linear recurrence relation: $\gamma(n) = a_1\gamma(n-1) + \cdots a_k\gamma(n-k)$ for $a_i \in \mathbb{Q}$.

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Proof by example

Let $\gamma(0) = \gamma(1) = 1$ and $\gamma(n) = \gamma(n-1) + \gamma(n-2)$ for $n \geq 2$.

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$$\sum_{n=0}^{\infty} \gamma(n) z^n = 1 + z + \sum_{n=2}^{\infty} (\gamma(n-1) + \gamma(n-2)) z^n$$

$$= 1 + z \sum_{n=0}^{\infty} \gamma(n) z^n + z^2 \sum_{n=0}^{\infty} \gamma(n) z^n$$

$$\sum_{n=0}^{\infty} \gamma(n) z^n = \frac{1}{1 - z - z^2}$$

Fun with generating functions

Product formula: For any functions f, g, we have:

$$\sum_{n=0}^{\infty} f(n)z^n \cdot \sum_{n=0}^{\infty} g(n)z^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} f(k)g(n-k)z^n.$$

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Set $f(n) = 1$ and $g(n) = \sigma(n)$, the strict growth function:

$$\frac{1}{1-z} \sum_{n=0}^{\infty} \sigma(n)z^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sigma(n-k)z^n = \sum_{n=0}^{\infty} \beta(n)z^n.$$

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Proposition

The algebraic complexity of the cumulative (conjugacy) growth series is the same as that of the strict (conjugacy) growth series.

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Corollary

If G has intermediate (conjugacy) growth, it cannot have rational (conjugacy) growth series.

Combination theorems

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Theorem

Direct product:

$$\mathbb{S}_{G \times H, S \cup T}(z) = \mathbb{S}_{G,S}(z) \cdot \mathbb{S}_{H,T}(z)$$

Free product:

$$\frac{1}{\mathbb{S}_{G * H, S \cup T}(z)} = \frac{1}{\mathbb{S}_{G,S}(z)} + \frac{1}{\mathbb{S}_{H,T}(z)} - 1$$

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In particular, if G and H have rational growth series, then so do $G \times H$ and $G * H$.

Decision problems

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If the standard (or conjugacy) growth series is holonomic then the word (or conjugacy) problem has a solution.

Corollary

If G has insoluble word (conjugacy) problem then it has non-holonomic standard (conjugacy) growth.

Growth Series

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Theorem (Stoll, 1996)

The higher Heisenberg groups H_r have rational standard growth series with respect to one choice of generating set and transcendental with respect to another.

$$H_2 = \left\{ \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| a, b, c, d, e \in \mathbb{Z} \right\}$$

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Proof makes use of combination theorems about 'central products'.

What do we know?

In some cases the behaviour is known to be independent of the generators:

	Standard Growth Series	Conjugacy Growth Series
Hyperbolic	Rational (Cannon 1984*)	Transcendental (Antolín-Ciobanu 2017)
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- some automatic groups (Epstein et al 1992),
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Transcendental conjugacy growth for some generators:

- soluble Baumslag-Solitar groups (Ciobanu-E.-Ho 2020),
- some wreath products (Mercier 2016).

A language $L \subset S^*$ is called regular if it is accepted by a finite state automaton (a directed, S-labelled graph with nominated start and accept states).

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$$\sum_{n\geq 0} \left(\mathsf{s}^T M^n \mathsf{a}\right) z^n = \mathsf{s}^T \left(\sum_{n\geq 0} (Mz)^n \right) \mathsf{a} = \mathsf{s}^T (I - Mz)^{-1} \mathsf{a}.$$

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So if we can find a language of geodesic representatives for the elements of G, then the **growth series** is rational.

Example

The language of geodesics in F_2 with respect to a basis is regular.

Hyperbolic groups - standard growth

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This is a consequence of the **geometry**.

Hyperbolic groups - conjugacy growth

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No algebraic series can have these asymptotics (via an analytic combinatorics result of Flajolet).

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Patterns and polyhedral sets

Partition S^* into pieces (aka patterns) that behave like subsets of $\mathbb{N}^r$. Reduce to sets of representatives which are 'polyhedral'. Precise form depends on structure of S^*, but they produce rational growth series in each case.

Definition

For any positive integer r, define the (higher) Heisenberg group H_r as follows:

$$H_r = \left\langle a_1, b_1, a_2, b_2, \ldots, a_r, b_r \;\middle|\; \begin{array}{l} [a_i, a_j] = [a_i, b_j] = [b_i, b_j] = 1 \;\forall i \neq j \\ [a_i, b_i] = [a_j, b_j] \;\forall i \neq j \\ [[a_i, b_i], a_j] = [[a_i, b_i], b_j] = 1 \;\forall i, j \end{array} \right\rangle.$$

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Conjugacy growth

Theorem (Babenko 1989)

The higher Heisenberg groups H_r have conjugacy growth

$$c_{H_r}(n) \sim \begin{cases} n^2 \log n & r = 1 \\ n^{2r} & r \geq 2 \end{cases}.$$

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Corollary

The conjugacy growth series of H_1 is non-holonomic.

Case $r = 1$

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- Normal form $\{a^i b^j c^k \mid i, j, k \in \mathbb{Z}\}$
- Conjugating:

$$aa^i b^j c^k a^{-1} = a^i b^j c^{k+j}, \; ba^i b^j c^k b^{-1} = a^i b^j c^{k-i}$$

and so $[a^i b^j c^k] = a^i b^j c^k \langle c^{\gcd(i,j)} \rangle$.

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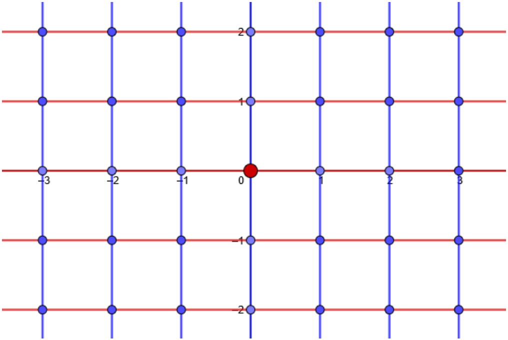
So at each point (i, j), there are exactly $\gcd(i, j)$ many conjugacy classes, all of length $|i| + |j|^*$.

Abelianisation



Conjugacy growth:

$$c(n) \sim \beta_{\mathrm{Ab}(H_1)}(n) \cdot (\text{`expected value' of } \gcd(i,j) \text{ if } |i| + |j| \leq n)$$
$$\sim n^2 \log n$$

Corollary

The conjugacy growth series of H_1 is non-holonomic.

Non-holonomic growth series

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Question: What about the conjugacy growth series of H_r in general?

Open Problems

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- Conjecture: A finitely presented group has rational **conjugacy** growth series if and only if it is virtually abelian.

Further Reading

- M. Clay and D. Margalit (eds.), *Office hours with a geometric group theorist*, Princeton University Press, Princeton, NJ, 2017. MR 3645425
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- M. Stoll, *Rational and transcendental growth series for the higher Heisenberg groups*, Invent. Math. **126** (1996), no. 1, 85–109. MR 1408557