Introduction to Growth in Groups
Part I: Asymptotics

Alex Evetts

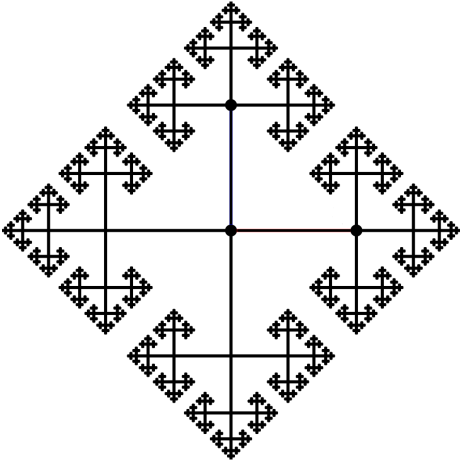
Erwin Schrödinger Institute / University of Vienna

19/11/2020

Back to basics

Given a group G, generated by a finite subset S, define a graph $\Gamma(G, S) = \Gamma$.

- Vertices correspond to elements of G: $V(\Gamma) = \{v_g \mid g \in G\}$
- Directed edges connect vertices v_g to v_h iff $h = gs$ for $s \in S$



Back to basics

Word length

The word length of $g \in G$ with respect to S is the length of a shortest word representing g:

$$|g|_S = \min \{|w| \mid w \in S^*,\ w =_G g\}$$

Back to basics

Word length

The word length of $g \in G$ with respect to S is the length of a shortest word representing g:

$$|g|_S = \min\{|w| \mid w \in S^*,\ w =_G g\}$$

Equivalently, $|g|_S$ is the length of a geodesic path from v_1 to v_g in the Cayley graph.

Back to basics

Word length

The word length of $g \in G$ with respect to S is the length of a shortest word representing g:

$$|g|_S = \min\{|w| \mid w \in S^*,\ w =_G g\}$$

Equivalently, $|g|_S$ is the length of a geodesic path from v_1 to v_g in the Cayley graph.

In fact, Cayley graph becomes a metric space if we define $d(g, h) = |g^{-1}h|_S$.

Counting elements

Definition

The strict growth function $\sigma_{G,S}(n) = \#\{g \in G \mid |g|_S = n\}$,

and the cumulative growth function $\beta_{G,S}(n) = \#\{g \in G \mid |g|_S \leq n\}$.

Counting elements

Definition

The strict growth function $\sigma_{G,S}(n) = \#\{g \in G \mid |g|_S = n\}$,

and the cumulative growth function $\beta_{G,S}(n) = \#\{g \in G \mid |g|_S \leq n\}$.

- counting the number of elements in the metric sphere or ball of radius n in Γ

Counting elements

Definition

The strict growth function $\sigma_{G,S}(n) = \#\{g \in G \mid |g|_S = n\}$,

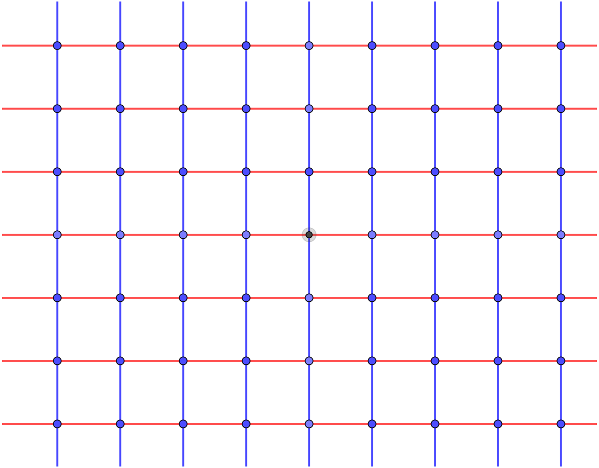
and the cumulative growth function $\beta_{G,S}(n) = \#\{g \in G \mid |g|_S \leq n\}$.

- counting the number of elements in the metric sphere or ball of radius n in Γ

Loose interpretation: groups with 'faster' growth functions are 'larger'.

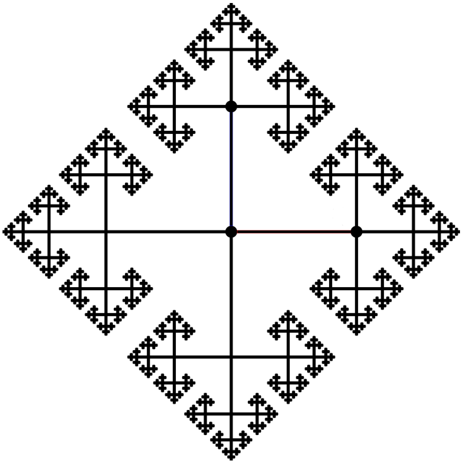
Examples

$\mathbb{Z}^2 = \langle a, b \mid [a, b] \rangle$



Examples

$F_2 = \langle a, b \mid - \rangle$



Invariance

Equivalence of growth functions

- For two functions $f, g\colon \mathbb{N} \to \mathbb{N}$ we write $f \preccurlyeq g$ if there exists $\lambda \geq 1$ s.t.

$$f(n) < \lambda g(\lambda n + \lambda) + \lambda$$

  for all n.

Invariance

Equivalence of growth functions

- For two functions $f, g\colon \mathbb{N} \to \mathbb{N}$ we write $f \preccurlyeq g$ if there exists $\lambda \geq 1$ s.t.

$$f(n) < \lambda g(\lambda n + \lambda) + \lambda$$

  for all n.
- If $f \preccurlyeq g$ and $g \preccurlyeq f$ then we write $f \sim g$ and say that the functions are equivalent.

Invariance

Equivalence of growth functions

- For two functions $f, g \colon \mathbb{N} \to \mathbb{N}$ we write $f \preccurlyeq g$ if there exists $\lambda \geq 1$ s.t.

$$f(n) < \lambda g(\lambda n + \lambda) + \lambda$$

  for all n.
- If $f \preccurlyeq g$ and $g \preccurlyeq f$ then we write $f \sim g$ and say that the functions are equivalent.

Fact: If S and T are two generating sets for a group G, then $\beta_{G,S} \sim \beta_{G,T}$.

Fundamental results

Growth behaves “well”.

Fundamental results

Growth behaves "well".

- If H is a finitely generated subgroup of G then $\beta_H \preccurlyeq \beta_G$.

Fundamental results

Growth behaves "well".

- If H is a finitely generated subgroup of G then $\beta_H \preccurlyeq \beta_G$.
- If $N \lhd G$ then $\beta_{G/N} \preccurlyeq \beta_G$.

Fundamental results

Growth behaves "well".

- If H is a finitely generated subgroup of G then $\beta_H \preccurlyeq \beta_G$.
- If $N \lhd G$ then $\beta_{G/N} \preccurlyeq \beta_G$.
- Commensurability invariant: if H is a finite index subgroup of G then $\beta_H \sim \beta_G$.

Fundamental results

Growth behaves "well".

- If H is a finitely generated subgroup of G then $\beta_H \preccurlyeq \beta_G$.
- If $N \lhd G$ then $\beta_{G/N} \preccurlyeq \beta_G$.
- Commensurability invariant: if H is a finite index subgroup of G then $\beta_H \sim \beta_G$.
- Quasi-Isometry invariant: quasi-isometric groups have equivalent growth functions.

Possible growth

Which functions can occur as growth functions?

Possible growth

Which functions can occur as growth functions?

- Exponential: $\beta(n) \sim a^n$, some $a > 1$, e.g. free groups, hyperbolic groups
  N.B. every group has growth function bounded above by an exponential

Possible growth

Which functions can occur as growth functions?

- Exponential: $\beta(n) \sim a^n$, some $a > 1$, e.g. free groups, hyperbolic groups
  N.B. every group has growth function bounded above by an exponential
- Polynomial: $\beta(n) \preccurlyeq n^d$, some $d > 0$, e.g. abelian groups

Possible growth

Which functions can occur as growth functions?

- Exponential: $\beta(n) \sim a^n$, some $a > 1$, e.g. free groups, hyperbolic groups
  N.B. every group has growth function bounded above by an exponential
- Polynomial: $\beta(n) \preccurlyeq n^d$, some $d > 0$, e.g. abelian groups
- Intermediate: strictly bigger than any polynomial, strictly smaller than any exponential, e.g. $\beta(n) \sim 2^{\sqrt{n}}$ (Grigorchuk's group - see Marialaura's lectures)

Reminder

Subnormal series:

$$G = G_0 \triangleright G_1 \triangleright G_2 \cdots \triangleright G_r = \{1\}$$

Reminder

Subnormal series:

$$G = G_0 \triangleright G_1 \triangleright G_2 \cdots \triangleright G_r = \{1\}$$

G is soluble if it has a subnormal series with each G_i/G_{i+1} is abelian.

Reminder

Subnormal series:

$$G = G_0 \triangleright G_1 \triangleright G_2 \cdots \triangleright G_r = \{1\}$$

G is soluble if it has a subnormal series with each G_i/G_{i+1} is abelian.

G is polycyclic if it has a subnormal series with each G_i/G_{i+1} is cyclic.

Reminder

Subnormal series:

$$G = G_0 \rhd G_1 \rhd G_2 \cdots \rhd G_r = \{1\}$$

G is soluble if it has a subnormal series with each G_i/G_{i+1} is abelian.

G is polycyclic if it has a subnormal series with each G_i/G_{i+1} is cyclic.

G is nilpotent if it has a subnormal series with each $G_{i+1} \lhd G$ and $G_i/G_{i+1} \leq Z(G/G_{i+1})$.

Virtually nilpotent groups

For any group G, define $G^{(i)} = \langle [\cdots[[x_0, x_1], x_2]\cdots x_i] \mid x_k \in G \rangle$

Virtually nilpotent groups

For any group G, define $G^{(i)} = \langle [\cdots[[x_0, x_1], x_2]\cdots x_i] \mid x_k \in G \rangle$

Definition

G is nilpotent if there is some $i \in \mathbb{N}$ with $G^{(i)}$ is trivial. The first such i is the nilpotency class of G.

Virtually nilpotent groups

For any group G, define $G^{(i)} = \langle [\cdots[[x_0, x_1], x_2]\cdots x_i] \mid x_k \in G\rangle$

Definition

G is nilpotent if there is some $i \in \mathbb{N}$ with $G^{(i)}$ is trivial. The first such i is the nilpotency class of G.

Wolf (1968): Every virtually nilpotent group has polynomial growth.

Virtually nilpotent groups

For any group G, define $G^{(i)} = \langle [\cdots[[x_0, x_1], x_2]\cdots x_i] \mid x_k \in G\rangle$

Definition

G is nilpotent if there is some $i \in \mathbb{N}$ with $G^{(i)}$ is trivial. The first such i is the nilpotency class of G.

Wolf (1968): Every virtually nilpotent group has polynomial growth.
Bass (1972), Guivarc'h (1973): The degree of polynomial growth of a virtually nilpotent group G is given by

$$\sum_{i=1}^{c} i \cdot \mathrm{rank}\left(G^{(i)}/G^{(i+1)}\right).$$

Example

The discrete Heisenberg group: $H = \langle a, b \mid [[a, b], a] = [[a, b], b] = 1 \rangle$

Example

The discrete Heisenberg group: $H = \langle a, b \mid [[a, b], a] = [[a, b], b] = 1 \rangle$

- $H^{(1)} = \langle [g_0, g_1] \mid g_i \in H \rangle = \langle [a, b] \rangle \cong \mathbb{Z}$

Example

The discrete Heisenberg group: $H = \langle a, b \mid [[a, b], a] = [[a, b], b] = 1 \rangle$

- $H^{(1)} = \langle [g_0, g_1] \mid g_i \in H \rangle = \langle [a, b] \rangle \cong \mathbb{Z}$
- $H^{(2)} = \langle [[g_0, g_1], g_2] \mid g_i \in H \rangle = \{1\}$

Example

The discrete Heisenberg group: $H = \langle a, b \mid [[a, b], a] = [[a, b], b] = 1 \rangle$

- $H^{(1)} = \langle [g_0, g_1] \mid g_i \in H \rangle = \langle [a, b] \rangle \cong \mathbb{Z}$
- $H^{(2)} = \langle [[g_0, g_1], g_2] \mid g_i \in H \rangle = \{1\}$

Quotients: $H/H^{(1)} \cong \mathbb{Z}^2$, $H^{(1)}/H^{(2)} \cong \mathbb{Z}$.

Example

The discrete Heisenberg group: $H = \langle a, b \mid [[a,b],a] = [[a,b],b] = 1 \rangle$

- $H^{(1)} = \langle [g_0, g_1] \mid g_i \in H \rangle = \langle [a,b] \rangle \cong \mathbb{Z}$
- $H^{(2)} = \langle [[g_0, g_1], g_2] \mid g_i \in H \rangle = \{1\}$

Quotients: $H/H^{(1)} \cong \mathbb{Z}^2$, $H^{(1)}/H^{(2)} \cong \mathbb{Z}$.

Bass-Guivarc'h: $\beta(n) \sim n^{1\times 2 + 2\times 1} = n^4$

Some history

Milnor: A finitely generated soluble group of subexponential growth is polycyclic.

Some history

Milnor: A finitely generated soluble group of subexponential growth is polycyclic.

Wolf: A polycyclic group of subexponential growth is virtually nilpotent.

Some history

Milnor: A finitely generated soluble group of subexponential growth is polycyclic.

Wolf: A polycyclic group of subexponential growth is virtually nilpotent.

So a finitely generated virtually soluble group either has exponential growth, or is virtually nilpotent (and hence has polynomial growth by Bass and Guivarc'h).

Some history

Milnor: A finitely generated soluble group of subexponential growth is polycyclic.

Wolf: A polycyclic group of subexponential growth is virtually nilpotent.

So a finitely generated virtually soluble group either has exponential growth, or is virtually nilpotent (and hence has polynomial growth by Bass and Guivarc'h).

Tits alternative: Every linear group is either virtually soluble or has a non-abelian free subgroup.

Milnor: A finitely generated soluble group of subexponential growth is polycyclic.

Wolf: A polycyclic group of subexponential growth is virtually nilpotent.

So a finitely generated virtually soluble group either has exponential growth, or is virtually nilpotent (and hence has polynomial growth by Bass and Guivarc'h).

Tits alternative: Every linear group is either virtually soluble or has a non-abelian free subgroup.

So every linear group has either exponential growth of is virtually nilpotent (with polynomial growth).

Milnor: A finitely generated soluble group of subexponential growth is polycyclic.

Wolf: A polycyclic group of subexponential growth is virtually nilpotent.

So a finitely generated virtually soluble group either has exponential growth, or is virtually nilpotent (and hence has polynomial growth by Bass and Guivarc'h).

Tits alternative: Every linear group is either virtually soluble or has a non-abelian free subgroup.

So every linear group has either exponential growth of is virtually nilpotent (with polynomial growth).

More generally: G is said to satisfiy a Tits Alternative if every subgroup $H \leq G$ is either virtually soluble or has a non-abelian free subgroup.
This includes Hyperbolic groups, mapping class groups, $\mathrm{Out}(F_n)$.

Converse to Bass-Guivarc'h

Theorem (Gromov 1981)

A finitely generated group has polynomial growth if and only if it is virtually nilpotent.

Converse to Bass-Guivarc'h

Theorem (Gromov 1981)

A finitely generated group has polynomial growth if and only if it is virtually nilpotent.

Consequence: If a group has growth function $\beta(n) \sim n^d$ then $d \in \mathbb{N}$.

Uniform exponential growth

For a group G with generating set S, define:

$$\rho_{G,S} = \lim_{n\to\infty} \left(\beta_{G,S}(n)\right)^{\frac{1}{n}}$$

Uniform exponential growth

For a group G with generating set S, define:

$$\rho_{G,S} = \lim_{n\to\infty} (\beta_{G,S}(n))^{\frac{1}{n}}$$

G has uniform exponential growth if $\inf_S \rho_{G,S} > 1$.

Conjugacy growth

- $\mathcal{C}_G :=$ set of conjugacy classes of G

Conjugacy growth

- $\mathcal{C}_G :=$ set of conjugacy classes of G
- For $\kappa \in \mathcal{C}_G$, define $|\kappa|_S = \min\{|w| \mid w \in S^*,\ \overline{w} \in \kappa\} = \min\{|g|_S \mid g \in \kappa\}$

- $\mathcal{C}_G :=$ set of conjugacy classes of G
- For $\kappa \in \mathcal{C}_G$, define $|\kappa|_S = \min\{|w| \mid w \in S^*,\ \overline{w} \in \kappa\} = \min\{|g|_S \mid g \in \kappa\}$

Definition

The strict and cumulative conjugacy growth functions of G with respect to S are

$$s_{G,S}(n) = \#\{\kappa \in \mathcal{C}_G \mid |\kappa|_S = n\},$$
$$c_{G,S}(n) = \#\{\kappa \in \mathcal{C}_G \mid |\kappa|_S \leq n\}$$

Conjugacy growth

- $\mathcal{C}_G :=$ set of conjugacy classes of G
- For $\kappa \in \mathcal{C}_G$, define $|\kappa|_S = \min\{|w| \mid w \in S^*, \ \overline{w} \in \kappa\} = \min\{|g|_S \mid g \in \kappa\}$

Definition

The strict and cumulative conjugacy growth functions of G with respect to S are

$$s_{G,S}(n) = \#\{\kappa \in \mathcal{C}_G \mid |\kappa|_S = n\},$$
$$c_{G,S}(n) = \#\{\kappa \in \mathcal{C}_G \mid |\kappa|_S \leq n\}$$

- $c_{G,S}(n)$ is the number of distinct conjugacy classes intersecting the ball of radius n in the Cayley graph

Conjugacy growth

- $\mathcal{C}_G :=$ set of conjugacy classes of G
- For $\kappa \in \mathcal{C}_G$, define $|\kappa|_S = \min\{|w| \mid w \in S^*,\ \overline{w} \in \kappa\} = \min\{|g|_S \mid g \in \kappa\}$

Definition

The strict and cumulative conjugacy growth functions of G with respect to S are

$$s_{G,S}(n) = \#\{\kappa \in \mathcal{C}_G \mid |\kappa|_S = n\},$$
$$c_{G,S}(n) = \#\{\kappa \in \mathcal{C}_G \mid |\kappa|_S \leq n\}$$

- $c_{G,S}(n)$ is the number of distinct conjugacy classes intersecting the ball of radius n in the Cayley graph

Facts about conjugacy growth

- Introduced by Babenko in 1989. Previous counting results by Margulis in 1969.

Facts about conjugacy growth

- Introduced by Babenko in 1989. Previous counting results by Margulis in 1969.
- Counts the number of closed geodesics, up to free homotopy, on a Riemannian manifold (with suitable hypothesis).

Facts about conjugacy growth

- Introduced by Babenko in 1989. Previous counting results by Margulis in 1969.
- Counts the number of closed geodesics, up to free homotopy, on a Riemannian manifold (with suitable hypothesis).
- Conjugacy growth is always bounded above by standard growth.

Facts about conjugacy growth

- Introduced by Babenko in 1989. Previous counting results by Margulis in 1969.
- Counts the number of closed geodesics, up to free homotopy, on a Riemannian manifold (with suitable hypothesis).
- Conjugacy growth is always bounded above by standard growth.
- Conjugacy growth and standard growth coincide in abelian groups.

Facts about conjugacy growth

- Introduced by Babenko in 1989. Previous counting results by Margulis in 1969.
- Counts the number of closed geodesics, up to free homotopy, on a Riemannian manifold (with suitable hypothesis).
- Conjugacy growth is always bounded above by standard growth.
- Conjugacy growth and standard growth coincide in abelian groups.

	Standard	Conjugacy
Group invariant	Yes	Yes
Quasi-Isometry Invariant	Yes	No
Polynomial growth	n^d for $d \in \mathbb{N}$	"anything"

Example

$$F_2 = \langle a, b \mid - \rangle$$

Example

$$F_2 = \langle a, b \mid - \rangle$$

- Each element is conjugate to a cyclically reduced element (approximately 3^n such elements).

Example

$$F_2 = \langle a, b \mid - \rangle$$

- Each element is conjugate to a cyclically reduced element (approximately 3^n such elements).
- Cyclically reduced elements are conjugate if and only if they are cyclic permutations of each other.

Example

$$F_2 = \langle a, b \mid - \rangle$$

- Each element is conjugate to a cyclically reduced element (approximately 3^n such elements).
- Cyclically reduced elements are conjugate if and only if they are cyclic permutations of each other.
- A cyclically reduced element has n cyclic permutations (not counting powers which are negligible).

Example

$$F_2 = \langle a, b \mid - \rangle$$

- Each element is conjugate to a cyclically reduced element (approximately 3^n such elements).
- Cyclically reduced elements are conjugate if and only if they are cyclic permutations of each other.
- A cyclically reduced element has n cyclic permutations (not counting powers which are negligible).
- So $c_{F_2}(n) \sim \frac{3^n}{n}$

Example

$$F_2 = \langle a, b \mid - \rangle$$

- Each element is conjugate to a cyclically reduced element (approximately 3^n such elements).
- Cyclically reduced elements are conjugate if and only if they are cyclic permutations of each other.
- A cyclically reduced element has n cyclic permutations (not counting powers which are negligible).
- So $c_{F_2}(n) \sim \frac{3^n}{n}$

This holds for any non-elementary hyperbolic group (Coornaert-Knieper).

Exponential conjugacy growth

Conjecture (Guba-Sapir 2010)

For "ordinary" groups, exponential standard growth should imply exponential conjugacy growth
(so the conjugacy and standard growth functions are equivalent).

Exponential conjugacy growth

Conjecture (Guba-Sapir 2010)

For "ordinary" groups, exponential standard growth should imply exponential conjugacy growth
(so the conjugacy and standard growth functions are equivalent).

Breuillard-Cornulier: This holds for soluble groups.

Exponential conjugacy growth

Conjecture (Guba-Sapir 2010)

For "ordinary" groups, exponential standard growth should imply exponential conjugacy growth
(so the conjugacy and standard growth functions are equivalent).

Breuillard-Cornulier: This holds for soluble groups.

Breuillard-Cornulier-Lubotzky-Meiri: This holds for all linear groups.

Polynomial conjugacy growth

For groups of polynomial standard growth, the standard and conjugacy growth functions can be non-equivalent.

Polynomial conjugacy growth

For groups of polynomial standard growth, the standard and conjugacy growth functions can be non-equivalent.

Example: Heisenberg group: $c(n) \sim n^2 \log n$

Polynomial conjugacy growth

For groups of polynomial standard growth, the standard and conjugacy growth functions can be non-equivalent.

Example: Heisenberg group: $c(n) \sim n^2 \log n$

Open question: What are the asymptotics for the conjugacy growth of virtually nilpotent groups in general?

Can the function be determined from the lower central series, as for standard growth?