# Introduction to Growth in Groups
## Part II: Formal power series

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### Word length

The word length of $g \in G$ with respect to $S$ is the length of a shortest word representing $g$:

$$|g|_S = \min\{|w| \mid w \in S^*,\ w =_G g\}$$

# Last week

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### Definition

The strict growth function $\sigma_{G,S}(n) = \#\{g \in G \mid |g|_S = n\}$,

and the cumulative growth function $\beta_{G,S}(n) = \#\{g \in G \mid |g|_S \leq n\}$.

## Conjugacy classes

Define the length of a conjugacy class $\kappa$ of $G$ with respect to $S$ to be the length of a shortest word representing $\kappa$:

$$|\kappa|_S = \min\{|w| \mid w \in S^*,\ \overline{w} \in \kappa\} = \min\{|g|_S \mid g \in \kappa\}$$

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## Definition

The strict conjugacy growth function $s_{G,S}(n) = \#\{\kappa \in \mathcal{C}_G \mid |\kappa|_S = n\}$,

and the cumulative conjugacy growth function $c_{G,S}(n) = \#\{\kappa \in \mathcal{C}_G \mid |\kappa|_S \leq n\}$.

# Standard vs Conjugacy Growth



| | Standard | Conjugacy |
|---|---|---|
| **Group invariant** | Yes | Yes |
| **Quasi-Isometry Invariant** | Yes | No |
| **Polynomial growth** | $n^d$ for $d \in \mathbb{N}$ | “anything” |

$n^{3/2}$

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- For any group $G$, $c_G(n) \preccurlyeq \beta_G(n)$.
- If $G$ is abelian then $c_G(n) \sim \beta_G(n)$ (converse does not hold).

# Growth series

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### Definition

The (standard/conjugacy/etc.) growth series of $G$ with respect to $S$ is the formal power series

$$\mathbb{S}(z) := \sum_{n=0}^{\infty} \gamma(n) z^n.$$

# Algebraic complexity

$\sum z^n = \frac{1}{1-z}$

A formal power series $\mathbb{S}(z)$ is called

- rational if there exist polynomials $P$, $Q$ with integer coefficients such that $\mathbb{S}(z) = \frac{P(z)}{Q(z)}$;

  $\mathbb{Q}(z)$ $\quad SQ = P$

- algebraic if $\mathbb{S}(z)$ satisfies a polynomial equation with polynomial coefficients;

  $\overline{\mathbb{Q}(z)}$

- holonomic (a.k.a. $D$-finite) if $\mathbb{S}(z)$ satisfies a finite order differential equation, with polynomial coefficients;
- transcendental if it is not algebraic.

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**Question**: Into which classes do the various growth functions fall?

- Standard growth of $F_2$, with respect to a basis: $\sigma(n) = 4 \cdot 3^{n-1}$ for $n \geq 1$

$$\mathbb{S}(z) = 1 + \sum_{n \geq 1} 4 \cdot 3^{n-1} z^n = 1 + \frac{4}{3} \sum_{n \geq 1} (3z)^n = \frac{1-2z}{1-3z}$$

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- Standard (and conjugacy) growth of $\mathbb{Z}^2$ with respect to $\{(1,0),(0,1)\}$: $\sigma(n) = 4n$ for $n \geq 1$

$$\mathbb{S}(z) = 1 + \sum_{n\geq 1} (4n) z^n = \frac{(1+z)^2}{(1-z)^2}$$

Rational growth reflects a strong 'regularity' property:

### Proposition

A series $\mathbb{S}(z) = \sum \gamma(n) z^n \in \mathbb{Z}[[z]]$ is rational if and only if $\gamma(n)$ satisfies a linear recurrence relation: $\gamma(n) = a_1 \gamma(n-1) + \cdots a_k \gamma(n-k)$ for $a_i \in \mathbb{Q}$.

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## Proof by example

Let $\gamma(0) = \gamma(1) = 1$ and $\gamma(n) = \gamma(n-1) + \gamma(n-2)$ for $n \geq 2$.

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$$\sum_{n=0}^{\infty} \gamma(n) z^n = 1 + z + \sum_{n=2}^{\infty} (\gamma(n-1) + \gamma(n-2)) z^n$$

$$= 1 + z \sum_{n=0}^{\infty} \gamma(n) z^n + z^2 \sum_{n=0}^{\infty} \gamma(n) z^n$$

$$\sum_{n=0}^{\infty} \gamma(n) z^n = \frac{1}{1 - z - z^2}$$

$S(z) = \frac{p(z)}{q(z)}$

Product formula: For any functions $f, g$, we have:

$$\sum_{n=0}^{\infty} f(n)z^n \cdot \sum_{n=0}^{\infty} g(n)z^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} f(k)g(n-k)z^n.$$

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Set $f(n) = 1$ and $g(n) = \sigma(n)$, the strict growth function:

$$\frac{1}{1-z} \sum_{n=0}^{\infty} \sigma(n) z^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sigma(n-k) z^n = \sum_{n=0}^{\infty} \beta(n) z^n.$$

Concrete Mathematics
Graham-Knuth-Patashnik

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**Proposition**

The algebraic complexity of the cumulative (conjugacy) growth series is the same as that of the strict (conjugacy) growth series.

## A Restriction on Asymptotics



**Proposition** $\in \mathbb{Z}[[z]]$

If a series $\mathbb{S}(z)$ is rational then the coefficients grow either exponentially or polynomially.

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We can write

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$1 + \alpha_i z + (\alpha_i z)^2 + \cdots$

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If there is a pole inside the unit disc, have some $|\alpha_i| > 1$. This gives exponential growth.
Otherwise, can show the growth is at most polynomial (hint: use the product formula).

Exercise

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## Corollary

If $G$ has intermediate (conjugacy) growth, it cannot have rational (conjugacy) growth series.

# Combination theorems

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### Theorem

(strict, standard)

Direct product:

$$\mathbb{S}_{G \times H, S \cup T}(z) = \mathbb{S}_{G,S}(z) \cdot \mathbb{S}_{H,T}(z)$$

Free product:

$$\frac{1}{\mathbb{S}_{G*H,S\cup T}(z)} = \frac{1}{\mathbb{S}_{G,S}(z)} + \frac{1}{\mathbb{S}_{H,T}(z)} - 1$$

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In particular, if $G$ and $H$ have rational growth series, then so do $G \times H$ and $G * H$.

1911 ?

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$\beta(n)$

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If the standard (or conjugacy) growth series is holonomic then the word (or conjugacy) problem has a solution.

## Corollary

If $G$ has insoluble word (conjugacy) problem then it has non-holonomic standard (conjugacy) growth.

# Growth Series

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**Theorem (Stoll, 1996)**

The higher Heisenberg groups $H_r$ have rational standard growth series with respect to one choice of generating set and transcendental with respect to another.

$$H_2 = \left\{ \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| a, b, c, d, e \in \mathbb{Z} \right\}$$

So the growth series are a useful tool...
but the algebraic complexity is not a group invariant!

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Proof makes use of combination theorems about 'central products'.

# What do we know?

In some cases the behaviour is known to be independent of the generators:

| | Standard Growth Series | Conjugacy Growth Series |
|---|---|---|
| **Hyperbolic** | Rational (Cannon 1984*) | Transcendental non-elem (Antolín-Ciobanu 2017) |
| **Virtually abelian** | Rational (Benson 1983) | Rational (E. 2019) |
| **Heisenberg $H_1$** | Rational (Duchin-Shapiro 2019) 2016 | ~~Transcendental~~ (E. 2020) Non-holonomic |

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Rational standard growth for some generators:

- some automatic groups (Epstein et al 1992),
- soluble Baumslag-Solitar groups $BS(1, k)$ (Collins-Edjvet-Gill 1994),
- and many more. RAAGs, Garside, Coxeter, ... $\mathbb{Z}^n \rtimes \mathbb{Z}$

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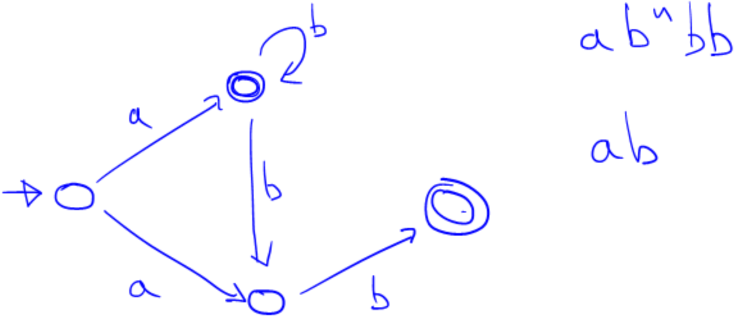
Transcendental conjugacy growth for some generators:

- soluble Baumslag-Solitar groups (Ciobanu-E.-Ho 2020),
- some wreath products (Mercier 2016).

A language $L \subset S^*$ is called regular if it is accepted by a finite state automaton (a directed, $S$-labelled graph with nominated start and accept states).



# Regular languages

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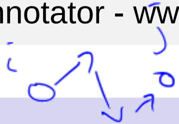

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- The growth series is then

$$\sum_{n\geq 0} (s^T M^n a) z^n = s^T \left( \sum_{n\geq 0} (Mz)^n \right) a = s^T (I - Mz)^{-1} a.$$

$\frac{1}{\det}( \cdot )$

□

Automatic

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□

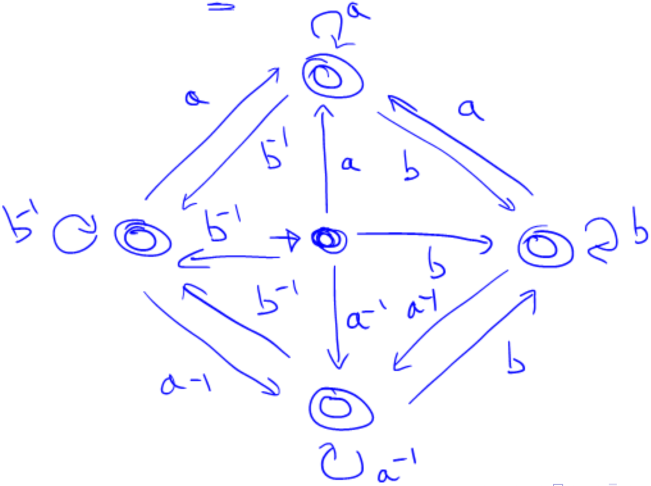
regular

So if we can find a language of geodesic representatives for the elements of $G$, then the **growth series** is rational.

# Example



The language of geodesics in $F_2$ with respect to a basis is regular.



# Hyperbolic groups - standard growth

**Theorem (Cannon)**

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This is a consequence of the **geometry**.

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For any finite generating set, there exist constants $A$, $B$, $\rho$, such that

$3^n / n$

$$A\frac{e^{\rho n}}{n} \leq c_{G,S}(n) \leq B\frac{e^{\rho n}}{n}.$$

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No algebraic series can have these asymptotics (via an analytic combinatorics result of Flajolet).

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## Patterns and polyhedral sets

Partition $S^*$ into pieces (aka patterns) that behave like subsets of $\mathbb{N}^r$. Reduce to sets of representatives which are 'polyhedral'. Precise form depends on structure of $S^*$, but they produce rational growth series in each case.

## Definition

For any positive integer $r$, define the (higher) Heisenberg group $H_r$ as follows:

$$H_r = \left\langle a_1, b_1, a_2, b_2, \ldots, a_r, b_r \;\middle|\; \begin{array}{l} [a_i, a_j] = [a_i, b_j] = [b_i, b_j] = 1 \;\forall i \neq j \\ [a_i, b_i] = [a_j, b_j] \;\forall i \neq j \\ [[a_i, b_i], a_j] = [[a_i, b_i], b_j] = 1 \;\forall i, j \end{array} \right\rangle.$$

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$$H_2 = \left\{ \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, c, d, e \in \mathbb{Z} \right\}$$

$H_r \rhd \langle c \rangle \rhd 1$

### Theorem (Babenko 1989)

The higher Heisenberg groups $H_r$ have conjugacy growth

$$c_{H_r}(n) \sim \begin{cases} n^2 \log n & r = 1 \\ n^{2r} & r \geq 2 \end{cases}.$$

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**Corollary**

The conjugacy growth series of $H_1$ is non-holonomic. $\forall$ gen sets.

# Case $r = 1$

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$aba^{-1}b^{-1} = c$

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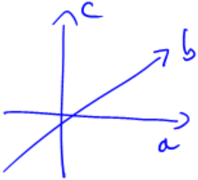

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- Normal form $\{a^i b^j c^k \mid i, j, k \in \mathbb{Z}\}$
- Conjugating:

$$aa^i b^j c^k a^{-1} = a^i b^j c^{k+j}, \;\; ba^i b^j c^k b^{-1} = a^i b^j c^{k-i}$$

and so $[a^i b^j c^k] = a^i b^j c^k \langle c^{\gcd(i,j)} \rangle$.

# Length of a conjugacy class

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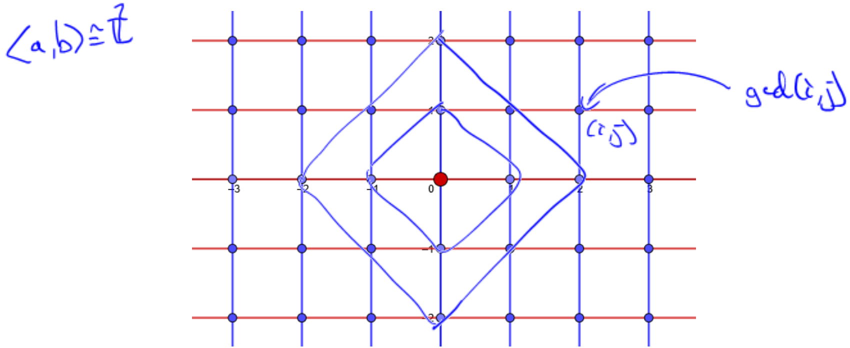
So at each point $(i, j)$, there are exactly $\gcd(i,j)$ many conjugacy classes, all of length $|i| + |j|$.

+2




Conjugacy growth:

$$c(n) \sim \beta_{\mathrm{Ab}(H_1)}(n) \cdot (\text{'expected value' of } \gcd(i,j) \text{ if } |i| + |j| \leq n)$$
$$\sim n^2 \log n$$

$n^{2r}$ gcd(i, j, k)

# Non-holonomic growth series

**Corollary**

The conjugacy growth series of $H_1$ is non-holonomic.

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- If $\gamma(n) \leq n^d$, some $d \in \mathbb{N}$, and $\sum_{n\geq 0} \gamma(n) z^n \in \mathbb{Q}(z)$ then there is some $d' \in \mathbb{N}$ with $\gamma(n) \sim n^{d'}$.

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Question: What about the conjugacy growth series of $H_r$ in general?

$\sim n^{2r}$

# Open Problems

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- Find more examples where the growth series behaviour depends on the choice of generating set.
- Conjecture: A finitely presented group has rational **conjugacy** growth series if and only if it is virtually abelian.

# Further Reading


M. Clay and D. Margalit (eds.), *Office hours with a geometric group theorist*, Princeton University Press, Princeton, NJ, 2017. MR 3645425

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