# Introduction to Growth in Groups
# Part I: Asymptotics

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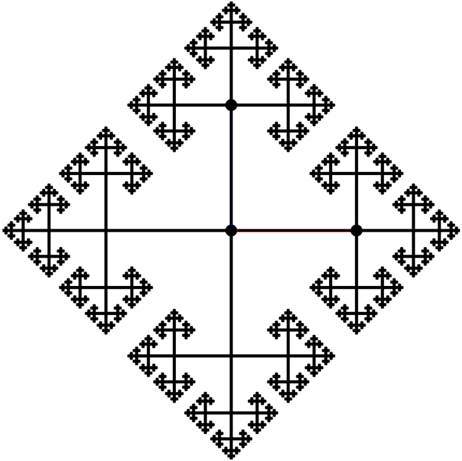
Mann - 'How groups grow'.

Given a group $G$, generated by a finite subset $S$, define a graph $\Gamma(G,S) = \Gamma$.

- Vertices correspond to elements of $G$: $V(\Gamma) = \{v_g \mid g \in G\}$
- Directed edges connect vertices $v_g$ to $v_h$ iff $h = gs$ for $s \in S$



$F_2$

## Word length

The word length of $g \in G$ with respect to $S$ is the length of a shortest word representing $g$:

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In fact, Cayley graph becomes a metric space if we define $d(g,h) = |g^{-1}h|_S$.

$S^{-1} = S$

**Definition**

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and the cumulative growth function $\beta_{G,S}(n) = \#\{g \in G \mid |g|_S \leq n\}$.

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# Counting elements

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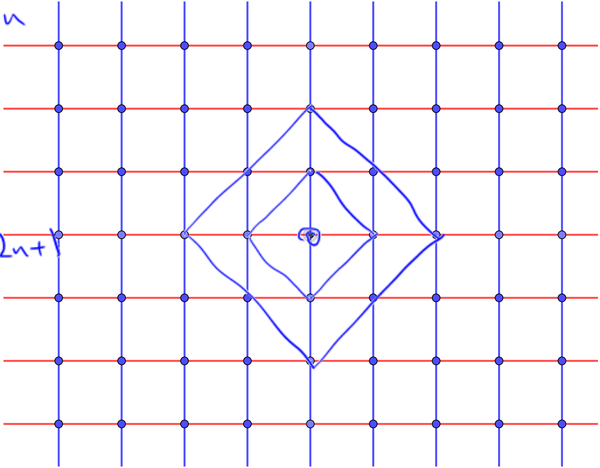
Loose interpretation: groups with 'faster' growth functions are 'larger'.

$\mathbb{Z}^2 = \langle a, b \mid [a, b] \rangle$

$\sigma(n) = 4n$

$\sigma(0) = 1$

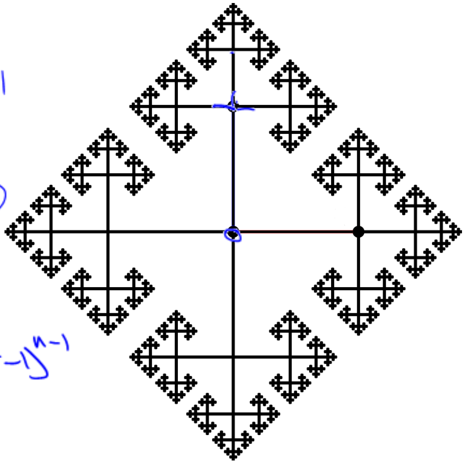
$\beta(n) \simeq 2n^2 + 2n + 1$



$F_2 = \langle a, b \mid - \rangle$

$\sigma(0) = 1$

$\sigma(n) = 4 \cdot 3^{n-1}$




$F_r$

$\sigma(n) = 2r(2r-1)^{n-1}$

## Equivalence of growth functions

- For two functions $f, g \colon \mathbb{N} \to \mathbb{N}$ we write $f \preccurlyeq g$ if there exists $\lambda \geq 1$ s.t.

$$f(n) < \lambda g(\lambda n + \lambda) + \lambda$$

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Fact: If $S$ and $T$ are two generating sets for a group $G$, then $\beta_{G,S} \sim \beta_{G,T}$.

$R_1 = \max_{s \in S} \{|s|_T\}$

$R_2 = \max_{t \in T} \{|t|_S\}$

$|g|_S \leq n$

$g = s_1 s_2 \cdots s_m$

$m \leq n$

$|g|_T \leq R_1 n$

$\Rightarrow \beta_{G,S}(n) \leq \beta_{G,T}(R_1 n)$

$\beta_{G,T}(n) \leq$

# Fundamental results

Growth behaves “well”.

$\langle S \rangle = H \quad \langle T \rangle = G$

$s_1 = t_1 \cdots t_m \in T^*$

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- Commensurability invariant: if $H$ is a finite index subgroup of $G$ then $\beta_H \sim \beta_G$.
- Quasi-Isometry invariant: quasi-isometric groups have equivalent growth functions.

# Possible growth

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$a^n \sim b^n$

$a, b > 1$

$4 \cdot 3^{n-1}$

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  $d \in \mathbb{R}$

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- Polynomial: $\beta(n) \preccurlyeq n^d$, some $d > 0$, e.g. abelian groups $n^d$ $d \in \mathbb{N}$
- Intermediate: strictly bigger than any polynomial, strictly smaller than any exponential, e.g. $\beta(n) \sim 2^{\sqrt{n}}$ (Grigorchuk's group - see Marialaura's lectures)

Subnormal series:

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$G$ is nilpotent if it has a subnormal series with each $G_{i+1} \triangleleft G$ and $G_i/G_{i+1} \leq Z(G/G_{i+1})$.

$x_0 x_1 x_0^{-1} x_1^{-1}$

For any group $G$, define $G^{(i)} = \langle [\cdots[[x_0, x_1], x_2] \cdots x_i] \mid x_k \in G \rangle$

$G \triangleright G^{(1)} \triangleright G^{(2)} \quad \cdots \quad 1$

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Bass (1972), Guivarc'h (1973): The degree of polynomial growth of a virtually nilpotent group $G$ is given by

$$d = \sum_{i=1}^{c} i \cdot \mathrm{rank}\left(G^{(i)}/G^{(i+1)}\right).$$

$\mathbb{Z}^r \times T$

$\beta(n) \sim n^d$

# Example



free nilp of rank 2, class 2

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Quotients: $H/H^{(1)} \cong \mathbb{Z}^2$, $H^{(1)}/H^{(2)} \cong \mathbb{Z}$.

$\langle a, b \rangle$

# Example



$[a^k, b^l] = c^{kl}$   $[a,b] =: c$

$[a^n, b^n] = c^{n^2}$

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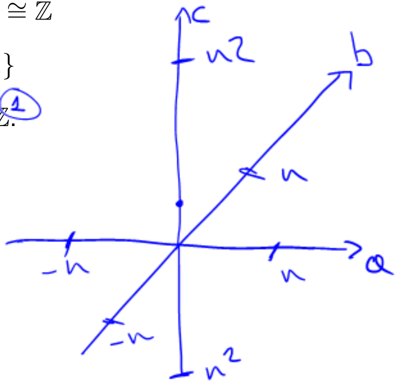
Bass-Guivarc'h: $\beta(n) \sim n^{1\times 2 + 2\times 1} = n^4$

$n^2 \cdot n \cdot n = n^4$



1960s

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$\leq GL_n F$

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More generally: $G$ is said to satisfiy a Tits Alternative if every subgroup $H \leq G$ is either virtually soluble or has a non-abelian free subgroup.
This includes Hyperbolic groups, mapping class groups, $\mathrm{Out}(F_n)$.

1968

# Converse to Bass-Guivarc'h

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Consequence: If a group has growth function $\beta(n) \sim n^d$ then $d \in \mathbb{N}$.

$n^{3/2}$

For a group $G$ with generating set $S$, define:

growth rate $\rho_{G,S} = \lim_{n \to \infty} (\beta_{G,S}(n))^{\frac{1}{n}}$

$\sim a^n$

$\rho = a$

For a group $G$ with generating set $S$, define:

$$\rho_{G,S} = \lim_{n\to\infty} \left(\beta_{G,S}(n)\right)^{\frac{1}{n}}$$

$G$ has uniform exponential growth if $\inf_S \rho_{G,S} > 1$.

Hyp., MCG, ....

John Wilson

$S^*$

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| | Standard | Conjugacy |
|---|---|---|
| **Group invariant** | Yes | Yes |
| **Quasi-Isometry Invariant** | Yes | No |
| **Polynomial growth** | $n^d$ for $d \in \mathbb{N}$ | anything |

# Example

$$F_2 = \langle a, b \mid - \rangle$$

# Example



× $a$ — $a^{-1}$

$4 \cdot 3^{n-1} \approx 3^n$

$$F_2 = \langle a, b \mid - \rangle$$

- Each element is conjugate to a cyclically reduced element (approximately $3^n$ such elements).

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$aba^{-1}b$
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This holds for any non-elementary hyperbolic group (Coornaert-Knieper).

# Exponential conjugacy growth

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- amenable
- diagonal grps ∋ F ✓

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Breuillard-Cornulier: This holds for soluble groups.

Breuillard-Cornulier-Lubotzky-Meiri: This holds for all linear groups.

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Open question: What are the asymptotics for the conjugacy growth of virtually nilpotent groups in general?

Can the function be determined from the lower central series, as for standard growth?