SYMMETRIC QUIVER SETTINGS WITH A POLYNOMIAL RING OF INVARIANTS

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Abstract. In this paper we classify all the symmetric quivers and corresponding dimension vectors having a smooth space of semisimple representation classes. The result we obtain is that such quivers can be decomposed as a connected sum of a few number of basic quivers.

1. Introduction

An interesting problem in invariant theory is the following. Consider a complex vector space V and a reductive algebraic group G with a linear action on V. The ring of polynomial functions over V will be $\mathbb{C}[X_1,\dots,X_n]$ where n is the dimension of V. This ring will have a corresponding action of G on it. One can now look at the subring of invariant polynomial functions,

$$\mathbb{C}[X_1,\dots,X_n]^{\mathsf{G}} := \{f \in \mathbb{C}[X_1,\dots,X_n] | f^g = f\},$$

And ask whether this subring is also a polynomial ring or (which is equivalent, see 2.2) a regular ring. In general this is not an easy problem. In this paper we will look at the special case of symmetric quiver representations.

A *quiver* $Q = (V, A, s, t)$ consists of a set of vertices V, a set of arrows A between those vertices and maps $s, t : A \to V$ which assign to each arrow its starting and terminating vertex. We also denote this as

$$\textcircled{t(a)} \xleftarrow{\;a\;} \textcircled{s(a)}\ .$$

A quiver $Q = (V, A, s, t)$ is *symmetric* if and only if the number of arrows between two vertices is the same in both directions, that is,

$$\forall v, w \in V : \#\{a \in A | \textcircled{v} \xrightarrow{\;a\;} \textcircled{w}\} = \#\{a \in A | \textcircled{v} \xleftarrow{\;a\;} \textcircled{w}\}.$$

A *dimension vector* of a quiver is a map $\alpha : V \to \mathbb{N}$, the size of a dimension vector is defined as $|\alpha| := \sum_{v\in V} \alpha_v$. A couple (Q, α) consisting of a quiver and a dimension vector is called a *quiver setting* and for every vertex $v \in V$, α_v is refered to as the dimension of v. An *α-dimensional complex representation* W of Q assigns to each vertex v a linear space $\mathbb{C}^{\alpha_v}$ and to each arrow a a matrix

$$W_a \in \mathsf{Mat}_{\alpha_{t(a)}\times\alpha_{s(a)}}(\mathbb{C})$$

The space of all α-dimensional representations is denoted by $\mathsf{Rep}_\alpha Q$.

$$\mathsf{Rep}_\alpha Q := \bigoplus_{a\in A} \mathsf{Mat}_{\alpha_{t(a)}\times\alpha_{s(a)}}(\mathbb{C})$$

To the dimension vector α we can also assign a reductive group

$$\mathsf{GL}_\alpha := \bigoplus_{v\in V} \mathsf{GL}_{\alpha_\mathsf{v}}(\mathbb{C}).$$

An element of this group, g, has a natural action on $\mathsf{Rep}_\alpha Q$:

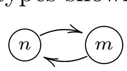
$$W := (W_a)_{a\in A},\ W^g := (g_{t(a)}W_a g_{s(a)}^{-1})_{a\in A}$$

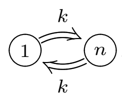
With these definitions the special vector space we will look at is $\mathsf{Rep}_\alpha Q$ and the reductive group is GL_α. We also suppose that α doesn't contain any vertex with zero dimension because this problem can be reduced to the problem of a new quiver obtained by deleting all vertices that have zero dimension. Quiver settings having this propety are called *genuine*.

The main theorem we prove here is a classification of all symmetric quiver settings for which the corresponding ring of invariants is a regular ring. Such quiver settings are called *coregular*.

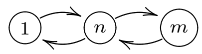
Theorem. A symmetric quiver setting without loops (Q, α) is coregular if and only if the following conditions are satisfied:

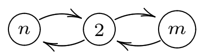
- Q is treelike, by which we mean that the underlying graph, having the same vertices as Q and 1 edge between two vertices whenever there is at least one arrow between them in Q, is a tree.
- The branching vertices (i.e. vertices having more than two incoming arrows) have dimension 1.
- The quiver setting is constructed by sticking together subquiver settings of the types shown below identifying only vertices with dimension 1.

I 

II 

, $k \leq n$

III 

IV 

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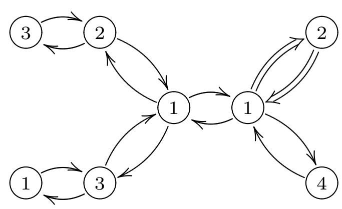
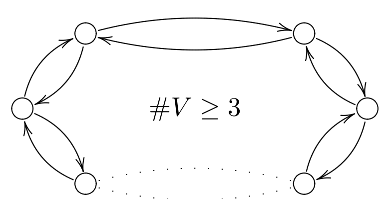
As an example illustrating this theorem we show a coregular quiver setting made by sticking together 2 settings of type I, and 1 of type II, III, and IV.



The proof of the theorem uses mainly two observations.

(1) If a quiver setting is coregular then all its subquiver settings and all its possible local quiver settings (see section 3) are coregular.
(2) If one sticks together two quiver settings by identifying a vertex with dimension 1, the ring of invariants of this new setting will be the tensor product of the rings of invariants of the two original quiver settings.

First one proves that a coregular quiver setting must be treelike because of observation 1 and the fact that a quiver of the form



is not coregular for any dimension vector. A similar argument is used to prove that the branching vertices must have dimension 1.

Observation 2 allows us now to cut the tree into pieces, (or in the algebraic way decomposing the ring of invariants into a tensor product) and to look at the pieces separately. Finally one concludes the proof by a classification of all coregular pieces that can't be cut into smaller ones.

2. The quotient space and the ring of invariants

As we stated in the introduction we want to study the ring of invariants $\mathbb{C}[\mathsf{Rep}_\alpha]^{\mathsf{GL}_\alpha}$. If we look at the problem in a geometric way this ring of invariants corresponds to a new affine variety that classifies the closed orbits of the GL_α-action on $\mathsf{Rep}_\alpha Q$.

Definition 2.1. If we divide out the action of GL_α on $\mathsf{Rep}_\alpha Q$, by taking the affine quotient we obtain a new space $\mathsf{iss}_\alpha Q$. The points of the space $\mathsf{iss}_\alpha Q$ are the closed GL_α-orbits in $\mathsf{Rep}_\alpha Q$. The coordinate ring of this variety is the ring of GL_α-invariant polynomial functions on $\mathsf{Rep}_\alpha Q$.

$$\mathbb{C}[\mathsf{iss}_\alpha Q] := \mathbb{C}[\mathsf{Rep}_\alpha]^{\mathsf{GL}_\alpha}$$

For more details of this construction see [1]

The question whether the ring of invariants is regular or polynomial is the same as asking whether $\mathsf{iss}_\alpha Q$ is a smooth variety or an affine space.

Another way of looking at this problem comes from the representation theoretic point of view. Two representations in $\mathsf{Rep}_\alpha Q$ are called equivalent, if they belong to the same orbit under the action of GL_α

A representation W is called *simple* if the only collections of subspaces $(\mathsf{V}_v)_{v\in V}$, $\mathsf{V}_v \subseteq \mathbb{C}^{\alpha_v}$ having the property

$$\forall a \in A : W_a \mathsf{V}_{s(a)} \subset \mathsf{V}_{t(a)}$$

are the trivial ones (i.e. the collection of zero-dimensional subspaces and $(\mathbb{C}^{\alpha_v})_{v\in V}$).

The direct sum $W \oplus W'$ of two representations W, W' has as dimension vector the sum of the two dimension vectors and as matrices $(W \oplus W')_a := W_a \oplus W'_a$. A representation equivalent to a direct sum of simple representations is called *semisimple*.

In [1] it is proven that an orbit of a representation is closed if and only if this representation is semisimple. So one can also consider $\mathsf{iss}_\alpha Q$ as the space classifying all semisimple α-dimensional representation classes.

In order to study $\mathsf{iss}_\alpha Q$ more closely, we recall some of the results of the article [2] by Le Bruyn and Procesi, which studies the local structure of the invariant ring $\mathbb{C}[\mathsf{iss}_\alpha Q]$.

A sequence of arrows $a_1 \dots a_p$ in a quiver Q is called a *path of length* p if $s(a_i) = t(a_{i+1})$, this path is called a cycle if $s(a_p) = t(a_1)$. To a cycle we can associate a polynomial function

$$f_c : \mathsf{Rep}_\alpha Q \to \mathbb{C} : W \mapsto \mathsf{Tr}(W_{a_1} \cdots W_{a_p})$$

which is definitely GL_α-invariant. Two cycles that are a cyclic permutation of each other give the same polynomial invariant, because of the basic properties of the trace map. Two such cycles are called equivalent.

A cycle $a_1 \dots a_p$ is called *primitive* if every arrow has a different starting vertex. This means that the cycle runs through each vertex at most 1 time. It is easy to see that every cycle has a decomposition in primitive cycles. It is however not true that the corresponding polynomial invariant decomposes to a product of the polynomial functions of the primitive cycles.

We will call a cycle *quasi-primitive* for a dimension vector α if the vertices that are ran through more than once have dimension bigger than 1. By cyclicly permuting a cycle and splitting the trace of a product of two 1×1 matrices into a product of traces, we can always decompose an f_c into a product of traces of quasi-primitive cycles. We now have the following result

Theorem 2.1 (Le Bruyn-Procesi)**.** $\mathbb{C}[\mathsf{iss}_\alpha Q]$ is generated by all f_c where c is a quasi-primitive cycle of degree smaller than $|\alpha|^2 + 1$. We can turn $\mathbb{C}[\mathsf{iss}_\alpha Q]$ into a graded ring by giving f_c the length of its cycle as degree.

Because $\mathsf{iss}_\alpha Q$ is a quotient of the linear representation $\mathsf{Rep}_\alpha Q$, the only smooth varieties that can occur are affine spaces. This is a direct consequence of [1, II.4.3. lemma 1 p139]

Theorem 2.2. Suppose G is a reductive group and V a linear G-representation. If the affine quotient V/G is smooth in the zero orbit then $V/\mathsf{G} \cong \mathbb{C}^t$ for some $t \in \mathbb{N}$.

Definition 2.2. Define a partial ordering on the set of quivers as follows A quiver $Q' = (V', A', s', t')$ is smaller than $Q = (V, A, s, t)$ if (up to isomorphism)

$$V' \subseteq V,\ A' \subseteq A,\ s' = s|_{A'} \text{ and } t' = t|_{A'},$$

Q' is called a *subquiver* of Q.

Lemma 2.3. If $\mathsf{iss}_\alpha Q$ is smooth and $Q' \leq Q$ then $\mathsf{iss}_{\alpha'} Q'$ is also smooth, where $\alpha' := \alpha|_{V'}$

Proof. We have an embedding

$$\mathsf{Rep}_{\alpha'} Q' \hookrightarrow \mathsf{Rep}_\alpha Q$$

by assigning to the additional arrows in Q zero matrices. So

$$\mathbb{C}[\mathsf{Rep}_\alpha Q] \twoheadrightarrow \mathbb{C}[\mathsf{Rep}_{\alpha'} Q'] \;\Rightarrow\; \mathbb{C}[\mathsf{Rep}_\alpha Q]^{\mathsf{GL}_\alpha} \twoheadrightarrow \mathbb{C}[\mathsf{Rep}_{\alpha'} Q']^{\mathsf{GL}_\alpha} \;.$$

Because the action of GL_α on $\mathsf{Rep}_{\alpha'} Q'$ reduces to that of $\mathsf{GL}_{\alpha'}$, $\mathbb{C}[\mathsf{iss}_{\alpha'} Q']$ is a quotient ring of $\mathbb{C}[\mathsf{iss}_\alpha Q] = \mathbb{C}[X_1, \ldots, X_n]$. The only relations that we have to divide out are the X_i that correspond to a cycle containing one of the additional arrows we put zero, so $\mathbb{C}[\mathsf{iss}_{\alpha'} Q']$ is just a polynomial ring with less variables. □

Two vertices v and w are said to be *strongly connected* if there is a path from v to w and vice versa. It is easy to check that this relation is an equivalence so we can divide the set of vertices into equivalence classes V_i. The subquiver Q_i having V_i as set of vertices, and as arrows all arrows between vertices of V_i is called a *strongly connected component* of Q.

Lemma 2.4.

(1) If (Q, α) is a quiver setting then

$$\mathbb{C}[\mathsf{iss}_\alpha Q] := \bigotimes_i \mathbb{C}[\mathsf{iss}_{\alpha_i} Q_i]$$

where $Q_i = (V_i, A_i, s_i, t_i)$ are the strongly connected components of Q and $\alpha_i := \alpha|_{V_i}$.

(2) $\mathsf{iss}_\alpha Q$ is smooth if and only if the $\mathsf{iss}_\alpha Q_i$ of all its strongly connected components are smooth.

Proof. (1) By theorem 2.1 $\mathbb{C}[\mathsf{iss}_\alpha Q]$ is generated by the traces of cycles. Every cycle belongs to a certain connected component of Q. Between f_c's coming from cycles of different components there cannot be any relations, so we can consider the ring of invariants as a tensor-products of the rings of invariants different strongly connected components.

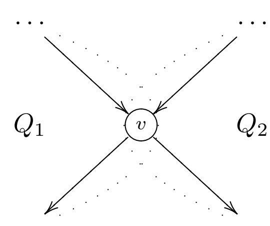
(2) If all the strongly connected components are coregular the ring of invariants of the total quiver setting will be the tensor product of polynomial rings

and hence a polynomial ring. The inverse implication follows directly from lemma 2.3.

□

Definition 2.3. A quiver $Q = (V, A, s, t)$ is said to be the *connected sum* of 2 subquivers $Q_1 = (V_1, A_1, s_1, t_1)$ and $Q_1 = (V_2, A_2, s_2, t_2)$ at the vertex v, if the two subquivers make up the whole quiver and only intersect in the vertex v. So in symbols $V = V_1 \cup V_2$, $A = A_1 \cup A_2$, $V_1 \cap V_2 = \{v\}$ and $A_1 \cap A_2 = \emptyset$.

$Q_1 \underset{v}{\#} Q_2 :=$



If we connect three or more components we write $Q_1 \underset{v}{\#} Q_2 \underset{w}{\#} Q_3$ instead of $(Q_1 \underset{v}{\#} Q_2) \underset{w}{\#} Q_3$ for sake of simplicity.

Lemma 2.5. Suppose $Q = Q_1 \underset{v}{\#} Q_2$ and $\alpha_v = 1$ then

$$\mathbb{C}[\mathsf{iss}_\alpha Q] := \mathbb{C}[\mathsf{iss}_{\alpha_1} Q_1] \otimes \mathbb{C}[\mathsf{iss}_{\alpha_2} Q_2]$$

where $\alpha_i := \alpha|_{Q_i}$.

Proof. By theorem 2.1 $\mathbb{C}[\mathsf{iss}_\alpha Q]$ is generated by the traces of quasi-primitive cycles. Because the dimension of v is one every quasi-primitive cycle is either in subquiver Q_1 or Q_2 and there cannot be any relations between invariants coming from cycles in different subquivers. This implies that the ring of invariants of (Q, α) is the tensorproduct of the rings of invariants of the two subquiver settings. □

Finally we can restrict to quivers without loops. This is a consequence of the first fundamental theorem of GL_n [1, II.4.1 p116].

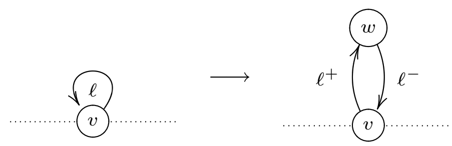
Theorem 2.6. For every $l, n, m \in \mathbb{N}$, the affine quotient

$$\mathsf{Mat}_{l \times n}(\mathbb{C}) \oplus \mathsf{Mat}_{n \times m}(\mathbb{C}) / \mathsf{GL}_n(\mathbb{C})$$

is isomorphic to the space of all $l \times m$-matrices of rank smaller than n. The identification is obtained via the projection map

$$\pi : \mathsf{Mat}_{l \times n}(\mathbb{C}) \oplus \mathsf{Mat}_{n \times m}(\mathbb{C}) \to \mathsf{Mat}_{l \times m}(\mathbb{C}) : (A, B) \mapsto AB$$

If we have a quiver setting with loops than we can construct a new quiver setting $(Q^\times, \alpha^\times)$ such that $\mathsf{iss}_{\alpha^\times} Q^\times$ is isomorphic to the original $\mathsf{iss}_\alpha Q$. We alter every loop in the original quiver into a vertex and two arrows as in the picture.



The dimension at w is bigger or equal than on the vertex v ($\alpha^\times_w \geq \alpha_v$). If we divide out the base change action of on the vertex w, the quotient space $\mathsf{Rep}_{\alpha^\times} Q^\times / \mathsf{GL}_{\alpha^\times_w}(\mathbb{C})$ is isomorphic to $\mathsf{Rep}_\alpha Q$ by the fundamental theorem.

Lemma 2.7.

$$\mathsf{iss}_{\alpha^\times} Q^\times \cong (\mathsf{Rep}_{\alpha^\times} Q^\times / \mathsf{GL}_{\alpha^\times_w}(\mathbb{C}))/\mathsf{GL}_\alpha \cong \mathsf{iss}_\alpha Q$$

Lemmas 2.4 and 2.7 allow us to consider only strongly connected quivers without any loops.

3. The Luna-slice machinery

In this section we review briefly the Luna-slice theorem and indicate in what way we will use it to obtain our classification. Most of the results in this section are taken from [3] or [5].

If we want to prove that a certain $\mathsf{iss}_\alpha Q$ is a smooth space, we have to check that it is smooth in every point. Take a point $p \in \mathsf{iss}_\alpha Q$, this point will correspond to the the isomorphism class of a semisimple representation $V \in \mathsf{Rep}_\alpha Q$ which can be decomposed as a direct sum of simple representations.

$$V = S_1^{\oplus a_1} \oplus \cdots \oplus S_k^{\oplus a_k},$$

The Luna-slice theorem connects the structure of $\mathsf{Rep}_\alpha Q$ around the closed orbit of V under the action of GL_α to the structure of $\mathsf{iss}_\alpha Q$ around p.

The orbit of V, $\mathcal{O}_V$, has a tangent space in V: $\mathrm{T}_V \mathcal{O}_V$. This tangent space forms a subspace of the complete tangent space $\mathrm{T}_V \mathsf{Rep}_\alpha Q$ and has a quotient space denoted by $\mathsf{N}_V := \mathrm{T}_V \mathsf{Rep}_\alpha Q / \mathrm{T}_V \mathcal{O}_V$. Every $g \in \mathsf{GL}_\alpha$ defines a natural linear map g^* on the tangent spaces.

$$g^* : \mathrm{T}_V \mathsf{Rep}_\alpha Q \to \mathrm{T}_{V^g} \mathsf{Rep}_\alpha Q.$$

Hence the stabilizer of V in GL_α, Stab_V, acts on $\mathrm{T}_V\mathsf{Rep}_\alpha Q$. Moreover is $\mathrm{T}_V\mathcal{O}_V$ mapped onto itself and therefore one can factor out $\mathrm{T}_V\mathcal{O}_V$ to obtain an Stab_V-action on N_V. In [3] the following result is obtained:

Theorem 3.1 (Luna 1973)**.** There exists an étale isomorphism φ between an open neighborhood of the point $0 \in \mathsf{N}_V/\mathsf{Stab}_V$ and an open neighborhood of $p \in \mathsf{iss}_\alpha Q$ mapping 0 to V. onto $\mathcal{O}_V$. So locally we have the following diagram:

$$\begin{array}{ccc} \mathsf{Rep}_\alpha Q & \xrightarrow{/\mathsf{GL}_\alpha} & \mathsf{iss}_\alpha Q \\ & & \uparrow \varphi \\ \mathsf{N}_V & \xrightarrow{/\mathsf{Stab}_V} & \mathsf{N}_V/\mathsf{Stab}_V \end{array}$$

Near the point p $\mathsf{iss}_\alpha Q$ is analytically isomorphic to the quotient $\mathsf{N}_V/\mathsf{Stab}_V$.

Because we are only interested in smoothness which is an analytic property we can simplify the problem of studying $\mathsf{iss}_\alpha Q$ p to the study of the simpler quotient $\mathsf{N}_V/\mathsf{Stab}V$ around the zero. At this point the technique of local quivers comes in action ([2] section 6).

The stabilizer of a simple representation is isomorphic to the group of scalar matrices. The stabilizer of the direct sum of k copies of a simple representation is isomorphic to $\mathsf{GL}_k(\mathbb{C})$. Keeping this in mind and looking at the decomposition of V into simple representations, we obtain that the stabilizer of

$$V := S_1^{\oplus a_1} \oplus \cdots \oplus S_k^{\oplus a_k}$$

must be equal to the group

$$\mathsf{Stab}_V \cong \mathsf{GL}_{a_1}(\mathbb{C}) \times \cdots \times \mathsf{GL}_{a_k}(\mathbb{C})$$

The tangent space in V will be identified with $\mathsf{Rep}_\alpha Q$. Due to the action of GL_α we can map the Lie-algebra

$$\mathfrak{gl}_\alpha := \mathrm{T}_e\mathsf{GL}_\alpha = \bigoplus_{v \in V} \mathfrak{gl}_{\alpha_v}(\mathbb{C})$$

surjectively onto the tangent space $\mathrm{T}_V\mathcal{O}_V$. A little calculation shows us that we can identify $\mathrm{T}_V\mathcal{O}_V$ with the following subset of $\mathsf{Rep}_\alpha Q$.

$$\{[m, V] | m \in \mathfrak{gl}_\alpha\}$$

Calculating the action of Stab_V on the space above and on $\mathsf{Rep}_\alpha Q$ leads to the following theorem (see [2])

Theorem 3.2 (Le Bruyn-Procesi)**.** For a point $p \in \mathsf{iss}_\alpha Q$ corresponding to a semisimple representation $V = S_1^{\oplus a_1} \oplus \cdots \oplus S_k^{\oplus a_k}$, we can identify N_V canonically with $\mathsf{Rep}_{\alpha_p} Q_p$ where Q_p is the *local quiver* of p. Q_p has k vertices corresponding to the set $\{S_i\}$ of simple factors of V and between S_i and S_j the number of arrows equals

$$\delta_{ij} - \chi_Q(\alpha_i, \alpha_j)$$

where α_i is the dimension vector of the simple component S_i and χ_Q is the Euler form of the quiver Q. The *Euler form* of Q is the bilinear form $\chi_Q : \mathbb{Z}^{\# V} \times \mathbb{Z}^{\# V} \to \mathbb{Z}$ defined by the matrix

$$m_{ij} = \delta_{ij} - \#\{a | \, \textcircled{\scriptsize i} \xleftarrow{\;a\;} \textcircled{\scriptsize j}, \}$$

where δ is the Kronecker delta.

The dimension vector α_p is defined to be $(a_1, \ldots, a_k)$, where the a_i are the multiplicities of the simple components in V.

The action of Stab_V on N_V corresponds to the normal action of GL_{α_p} on $\mathsf{Rep}_{\alpha_p} Q_p$.

Putting all these results together we get:

Theorem 3.3 (Le Bruyn-Procesi)**.** For every point $p \in \mathsf{iss}_\alpha Q$ we have an étale isomophism between an open neighborhood of the zero representation in $\mathsf{iss}_{\alpha_p} Q_p$ and an open neighborhood of p.

How are we going to apply this theorem? If we want to compute whether a certain space $\mathsf{iss}_\alpha Q$ is smooth, then we can choose a certain point p and look at this locally. Because of the local étale isomorphism, the corresponding local quiver Q_p must have a quotient space $\mathsf{iss}_{\alpha_p} Q_p$ that is smooth in the zero-representation. Therefore by 2.2, $\mathbb{C}[\mathsf{iss}_{\alpha_p} Q_p]$ must be a polynomial ring and hence (Q_p, α_p) is coregular. To find out whether (Q, α) is coregular we have to check all possible points p.

Theorem 3.4. (Q, α) is coregular if and only if for every possible semisimple α-dimensional representation V, the corresponding local quiver setting is coregular.

One of the local quivers is equal to the original quiver, namely the one corresponding to the representation

$$\bigoplus_{v \in V} S_v^{\oplus \alpha_v},$$

Where S_v corresponds to the simple representation with dimension vector

$$\epsilon_v : V \to \mathbb{N} : w \mapsto \delta_{vw},$$

assigning to every arrow the zero matrix. This implies that we can only use this result to rule out quiver settings that are not coregular.

Because the structure of the local quiver setting only depends on the dimension vectors of the simple components. Therefore one can restrict to looking at decompositions of α into dimension vectors f.i.

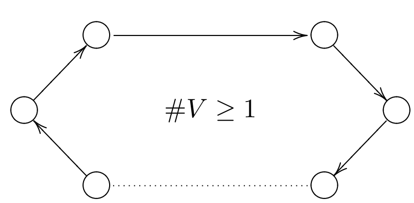
$$\alpha = a_1\beta_1 + \cdots + a_k\beta_k \text{ (the } \beta_i \text{ need not to be different).}$$

One can now ask whether there is a semisimple representation corresponding to such a decomposition. The answer to this question will be positive whenever for all the β_i there exist simple representations of that dimension vector and whenever there are two or more β_i equal there are at least as much different simple representation classes with dimesion vector β_i (otherwise you can't make a direct sum with different simple representations having the same dimension vector).

To check the above conditions we must also have a characterization of the dimension vectors for which a quiver has simple representations. We recall a result from Le Bruyn and Procesi [2].

Theorem 3.5. Let (Q, α) be a genuine quiver setting. There exist simple representations of dimension vector α if and only if

- If Q is of the form



and $\alpha = \mathbf{1}$ (this is the constant map from the vertices to 1).

- Q is not of the form above, but strongly connected and

$$\forall v \in V : \chi_Q(\alpha, \epsilon_v) \le 0 \text{ and } \chi_Q(\epsilon_v, \alpha) \le 0$$

In both cases the dimension of $\mathsf{iss}_\alpha Q$ is given by $1 - \chi_Q(\alpha, \alpha)$. If this dimension is zero then there is only one simple representation class with dimension vector α.

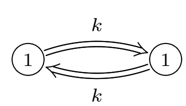
If (Q, α) is not genuine, there exist simple representations if and only if there exist for the genuine quiver setting obtained by deleting all vertices with dimension zero.

4. Necessary conditions

In this section we determine some neccesary conditions for a quiver setting to be coregular in the next section we will use these conditions to generate all coregular quiver settings.

We first look at a simple case and then we rule out more complex quiver settings by looking the local quiver of some decompositions. In the following pictures of quiver settings, we wil write down the dimensions inside the corresponding vertices.

Lemma 4.1. The following quiver setting is not coregular if $k > 1$.



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Proof. The representation space is spanned by all the cycles

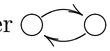
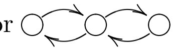
$$X_{ij} = f_{a_i b_j}$$

Where a_i stands for one of the arrows to the right and b_j one to the left. All these cycles are neccesary to generate the algebra, because the representation for which all the arrows are zero except a_i and b_j is not equivalent to the zero and has as values in the cycles all zero's except for X_{ij}. The relations between the cycles are of the form

$$X_{ij} X_{kl} = X_{il} X_{kj}$$

These relations prevent $\mathsf{iss}_1 Q$ from being an affine space. The only way to make $\mathsf{iss}_1 Q$ into an affine space is that there is only 1 such cycle. □

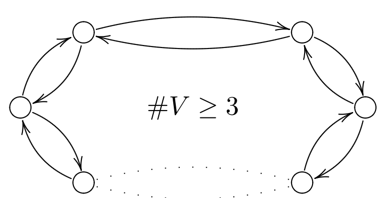
We will use this lemma in the following way. Suppose we have a quiver which has a decomposition that contains two simple factors that occur with multiplicity 1. The corresponding local quiver will contain a subsetting of the form $(*)$ with $k = -\chi_Q(\beta, \gamma)$ where β and γ are the dimension vectors of the simple factors.

Lemma 4.2. Suppose that (Q, α) is a coregular symmetric strongly connected quiver without loops and $\forall v \in V : \alpha(v) > 1$ then Q is either  or .

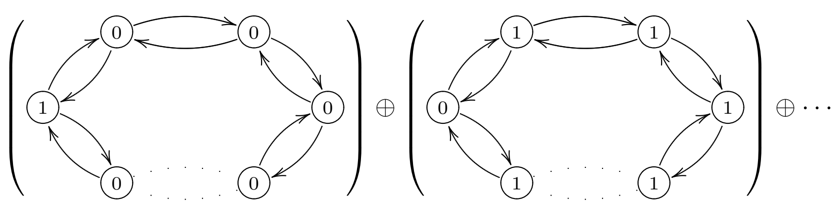
Proof. We can make a decomposition containing 2 different simple components with dimension vector $\mathbf{1}$ and all the other componenents with dimension vectors of the form ϵ_v.

The number of arrows in the local quiver between the first two components is $\chi(\mathbf{1}, \mathbf{1}) = \#A - \#V$. In order to be coregular this number must be at most 1. Because the quiver is symmetric and strongly connected $\#A \geq 2(\#V - 1)$ so $1 \geq \chi(\mathbf{1}, \mathbf{1}) \geq \#V - 2$. The only quivers satisfying $\#V \leq 3$ and $\#A \leq \#V + 1$ are the one listed above. $\square$

Lemma 4.3. The following quiver is not coregular for any dimension vector



Proof. By the previous lemma we can suppose that the dimension of the left vertex is 1. Make the folowing decomposition:



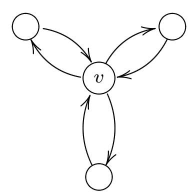
Where the dots stand for components with dimension vectors of the form ϵ_v. The number of arrows in the local quiver between the first two components is 2 so (Q, α) is not coregular. $\square$

The lemma above, in combination with lemma 2.3 shows us that if we look at the underlying graph of a coregular quiver setting (Q, α) (having the same vertices as

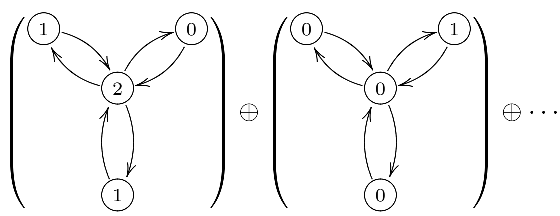
Q, and 1 edge between two vertices if there is an arrow between them), this graph must have the form of a tree. So we have restrictions on the form of the quiver.

There are also restrictions on the possible dimension vectors. We will determine at which vertices the dimension vector has to be 1.

Lemma 4.4. The following quiver setting is not coregular if the dimension in the centre v is bigger than 1.



Proof. If the dimension vector of the centre is bigger than 1 then by 4.2 we can suppose that at least 1 of the dimensions of the other vertices is 1 (take this to be the upper right one). We can find a decomposition of the form

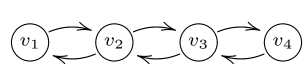


The number of arrows between the first and the second simple component equals

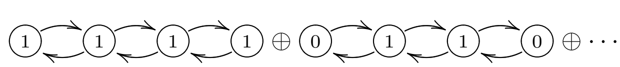
$$-\begin{pmatrix}2 & 1 & 1 & 0\end{pmatrix}\begin{pmatrix}1 & -1 & -1 & -1\\ -1 & 1 & 0 & 0\\ -1 & 0 & 1 & 0\\ -1 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = 2$$

□

Lemma 4.5. The following quiver setting is not coregular if $v_2, v_3 \geq 2$



Proof. We have a decomposition



The number of arrows between the first and the second simple component is again given by

$$-\begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & -1 & 0 & 0\\ -1 & 1 & -1 & 0\\ 0 & -1 & 1 & -1\\ 0 & 0 & -1 & 1\end{pmatrix}\begin{pmatrix}0\\1\\1\\0\end{pmatrix} = 2$$

□

Theorem 4.6. If a quiver setting (Q, α) (Q is symmetric and connected without loops) is coregular then Q is a connected sum

$$Q := Q_1 \underset{v_1}{\#} Q_2 \underset{v_2}{\#} \cdots \underset{v_{k-1}}{\#} Q_k,$$

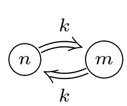
where the Q_i have at most 3 vertices. and $\alpha_{v_j} = 1,\ j = 1, \ldots, k-1$.

Proof. If a symmetric connected quiver setting is coregular and it has more than 3 vertices then it is treelike. Cutting at the vertices with dimension 1, we can consider Q as a connected sum of smaller components. Because branching vertices have dimension 1 and there are no 2 consecutive vertices with dimension bigger than 1 unless they are at the end of a branch, the components of this connected sum have at most 3 vertices. □

5. The characterisation result

In this section we determine all coregular quiver settings with 2 or 3 vertices. After that we combine them to bigger quivers in order to construct all symmetric quiver settings that are coregular.

Lemma 5.1. The quiver setting

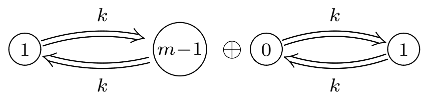


$,\ n \leq m$

is coregular if and only if $k = 1$ or $1 = n \leq k \leq m$.

Proof. if $k = 1$ then lemma 2.7 shows that the space is equal to that of the quiver with one vertex and one loop, such that the dimension vector is n. This problem is the same as the conjugacy problem of matrices, which is known to have a smooth space (see classical invariant theory H.P. Kraft [1]).

If $k > 1$ then by lemma 4.2 at least n must be 1. If $k > m$ then we can make the following decomposition in simples:



Computing the number of arrows in the local quiver gives us

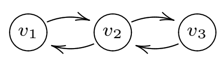
$$-\begin{pmatrix}1 & a-1\end{pmatrix}\begin{pmatrix}1 & -k\\ -k & 1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = k - a + 1 > 1.$$

If $k = m$ then there are exactly k^2 quasi-primitive cycles. Moreover (Q, α) is simple and the dimension of $\mathsf{iss}_\alpha Q$ is $\chi_Q(\alpha, \alpha) = k^2$. Hence there can be no relations between the generators of $\mathbb{C}[\mathsf{iss}_\alpha Q]$. By lemma 2.3 is the case $k < m$ also coregular.

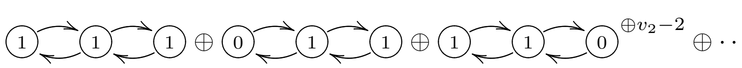
□

For quivers with 3 vertices we only have to look at the settings where the dimension of the middle vertex is bigger than 1 because otherwise we can consider it as a connected sum of two quiver settings with 2 vertices.

Lemma 5.2. The following quiver setting is not coregular if $v_2 \geq 3$ and $v_1, v_3 \geq 2$



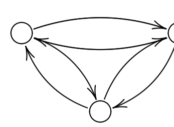
Proof. We have a decomposition



computing the arrows:

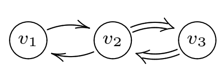
$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & -1 & 0\\ -1 & 1 & -1\\ 0 & -1 & 1\end{pmatrix}\begin{pmatrix}0\\ 1\\ 1\end{pmatrix} = \begin{pmatrix}1 & 1 & 0\end{pmatrix}\begin{pmatrix}1 & -1 & 0\\ -1 & 1 & -1\\ 0 & -1 & 1\end{pmatrix}\begin{pmatrix}0\\ 1\\ 1\end{pmatrix} = \begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & -1 & 0\\ -1 & 1 & -1\\ 0 & -1 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\\ 0\end{pmatrix} = -1$$

shows that the associated local quiver has a subquiver of the form

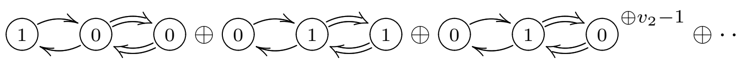

(∗∗)

which is not coregular according to lemma 4.3. □

Lemma 5.3. The following quiver setting is not coregular if $v_2 \geq 2$

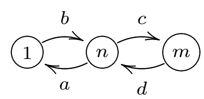


Proof. We have a decomposition



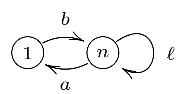
computing the arrows as in the previous lemma shows that the associated local quiver has also a subquiver of the form (**). □

Lemma 5.4. The quiver setting



is coregular.

Proof. If $m > n$ the situation is the same as



this quiver setting has simple representations by 3.5 and the dimension of quotient space $\mathsf{iss}_\alpha Q$ is

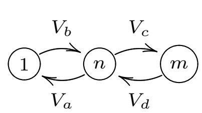
$$1 - \begin{pmatrix} 1 & n \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ n \end{pmatrix} = 2n$$

The only quasi-primitive cycles are of the form

$$\ell^i \text{ and } a\ell^i b$$

By the Cayley-Hamilton identity for V_ℓ, we only need the $2n$ quasi-primitive cycles $\ell^i, 1 \leq i \leq n$ and $a\ell^i b, 0 \leq i \leq n-1$ to generate the ring of invariants. Because the dimension of $\mathsf{iss}_\alpha Q$ is $2n$, $\mathbb{C}[\mathsf{iss}_\alpha Q]$ must be polynomial.

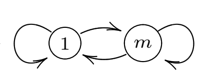
If $m < n$ a general representation consist of 2 big matrices as shown in the picture



$$\Rightarrow \qquad \mathbb{C}^1 \oplus \mathbb{C}^m \xrightarrow{V_b \oplus V_d} \mathbb{C}^n \xrightarrow{V_a \oplus V_c} \mathbb{C}^1 \oplus \mathbb{C}^m$$

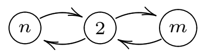
If we divide out the action of base change on $\mathbb{C}^n$ using the fundamental theorem 2.6, the quotient gives us one big composite map which corresponds to a new quiver situation

$$\mathbb{C}^1 \oplus \mathbb{C}^m \xrightarrow{(V_a \oplus V_c)(V_b \oplus V_d)} \mathbb{C}^1 \oplus \mathbb{C}^m \qquad \Rightarrow$$



This new quiver setting is coregular because it is the connected sum of two coregular quiver settings: one vertex of dimension 1 with one loop and the quiver setting we encountered in the first part of the proof (4.6). □

Lemma 5.5. The quiver setting



is coregular.

Proof. If both n and m are bigger then 1, this problem is the same as a vertex with two loops, or two simultaneously conjugated 2×2-matrices. As we know from [4], this problem is coregular and its ring of invariants is generated by

$$\mathsf{Tr}A, \mathsf{Tr}B, \mathsf{Tr}A^2, \mathsf{Tr}B^2 \text{ and } \mathsf{Tr}AB,$$

where A and B are the matrices corresponding to the two loops. If $n = 1$, we can take the same invariants as above, taking for A only rank 1 matrices, therefore $\mathsf{Tr}A^2 = (\mathsf{Tr}A)^2$ is the only extra relation. This means that the ring of invariants is indeed a polynomial ring generated by

$$\mathsf{Tr}A, \mathsf{Tr}B, \mathsf{Tr}B^2 \text{ and } \mathsf{Tr}AB.$$

If both n and m are 1 we are in the situation of the previous lemma. □

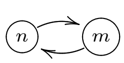
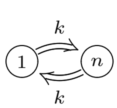
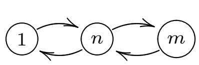
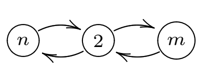
Keeping 5.2 and 5.3 in mind, the last two lemmas give us all coregular quiver settings with 3 vertices that are not the connected sum of smaller ones.

Combining all the results we get a characterisation:

Theorem 5.6. A quiver setting (Q, α) (Q is symmetric and connected without loops) is coregular if and only if Q is a connected sum

$$Q := Q_1 \underset{v_1}{\#} Q_2 \underset{v_2}{\#} \cdots \underset{v_{k-1}}{\#} Q_k,$$

where the (Q_i, α_i) are of the form

- 
-  , $k \leq n$
- 
- 

and $\alpha_{v_j} = 1,\ j = 1, \ldots, k-1$

Proof. The proof follows from theorem 4.6 and the fact that the above list characterizes all coregular quiver settings with 3 or less vertices that cannot be written as a connected sum of smaller ones. □

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