CALABI-YAU ALGEBRAS AND WEIGHTED QUIVER POLYHEDRA

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ABSTRACT. Dimer models have been used in string theory to construct path algebras with relations that are 3-dimensional Calabi-Yau Algebras. These constructions result in algebras that share some specific properties: they are finitely generated modules over their centers and their representation spaces are toric varieties. In order to describe these algebras we introduce the notion of a toric order and show that all toric orders which are 3-dimensional Calabi-Yau algebras can be constructed from dimer models on a torus.

Toric orders are examples of a much broader class of algebras: positively graded cancellation algebras. For these algebras the CY-3 condition implies the existence of a weighted quiver polyhedron, which is an extension of dimer models obtained by replacing the torus with any two-dimensional compact orientable orbifold.

1. INTRODUCTION

Calabi-Yau algebras play an important role in theoretical physics because their derived categories can be used to describe brane configurations in the B-model of topological string theory. There are several ways to construct examples of this kind of algebras such as McKay correspondence [15, 9] or exceptional sequences [2]. Another important construction method are dimer models [17, 13, 16]. A dimer model $\mathcal{D}$ consists of a bipartite graph (with black and white vertices) that is embedded in a compact surface. The corresponding algebra $A_{\mathcal{D}}$ is the path algebra with relations of the dual graph oriented such that a cycle around a black (white) vertex has a (anti-)clockwise orientation. The relations come from the partial derivatives of a superpotential which is the sum of all clockwise cycles minus the sum of all anti-clockwise cycles.

It was shown by Nathan Broomhead in [6], by Sergey Mozgovoy and Markus Reineke in [26] and by Ben Davison in [11] that if the dimer model satisfies certain consistency conditions, the algebra $A_{\mathcal{D}}$ is a 3-dimensional Calabi-Yau Algebra.

In this paper we will show why dimer models appear in this setting and to which extent they arise from the Calabi-Yau property.

The Calabi-Yau algebras that one obtains from dimer models on a torus share quite specific properties. They are meant to be noncommutative toric resolutions of a toric variety and therefore these algebras are prime and finitely generated modules over their centers, which are the coordinate rings of the affine toric varieties one wishes to resolve. The fact that the resolution is supposed to be toric implies that the algebra is a positively graded subalgebra of $\mathsf{Mat}_n(T)$ where $T = \mathbb{C}[\mathbb{Z}^k]$ is the coordinate ring of the torus inside the toric variety. We will call any such algebra a toric order and discuss how they fit in the notion of a noncommutative crepant resolution as introduced by Van den Bergh.

In this paper we will prove that if a positively graded toric order is CY-3 then it comes from a dimer model on a torus. The way we prove this result is by generalizing both sides and proving a similar theorem in this generalized context. On the one hand, we relax the definition of a toric order to cancellation algebras (Definition 8) and on the other hand we introduce the notion of a weighted quiver polyhedron which corresponds roughly to a dimer model two-dimensional orientable orbifold (Examples can be found in section 9). Our main theorem then states

Theorem 1.1 (=Theorem 7.1)**.** *Every positively graded cancellation algebra that is CY-3 comes from a weighted quiver polyhedron.*

In the specific situation of toric orders, theorem 8.7 then shows that this quiver polyhedron must in fact come from a dimer model.

The paper is structured as follows. After the preliminary sections on path algebras, Calabi-Yau algebras and noncommutative resolutions, we introduce toric orders in section 4 and discuss how they fit in the theory of noncommutative resolutions. In section 5 we generalize toric orders to cancellation algebras and discuss bimodule resolutions in this setup. In section 6 we define the combinatorial notion of a quiver polyhedron, relate it to dimer models and work out a theory of Galois covers for them. In section 7 we prove the main theorem. Section 8 contains a short discussion on the cancellation property for quiver polyhedra and uses this to prove that toric CY-3 orders come from dimer models. We end with some examples of quiver polyhedra and their corresponding Jacobi Algebras.

2. Acknowledgements

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3. Preliminaries

3.1. **Path algebras with relations.** As usual a *quiver* Q is an oriented graph. We denote the set of vertices by Q_0, the set of arrows by Q_1 and the maps h, t assign to each arrow its head and tail. A *nontrivial path* p is a sequence of arrows $a_1 \cdots a_k$ such that $t(a_i) = h(a_{i+1})$, whereas a *trivial path* is just a vertex. We will denote the length of a path by $|p| := k$ and the head and tail by $h(p) = h(a_1),\ t(p) = t(a_k)$. A path is called cyclic if $h(p) = t(p)$. Later on we will denote by $p[i]$ the $n - i^{th}$ arrow of p and by $p[i \dots j]$ the subpath $p[i] \dots p[j]$.

$$\circ \xleftarrow{p[n-1]} \circ \xleftarrow{p[n-2]} \circ \xleftarrow{p[1]} \cdots \circ \xleftarrow{p[0]} \circ \text{ and } p = p[n-1]p[n-2]\dots p[1]p[0].$$

A quiver is called *connected* if it is not the disjoint union of two subquivers and it is *strongly connected* if there is a cyclic path through each pair of vertices.

The *path algebra* $\mathbb{C}Q$ is the complex vector space with as basis the paths in Q and the multiplication of two paths p, q is their concatenation pq if $t(p) = h(q)$ or else 0. The span of all paths of nonzero length form an ideal which we denote by $\mathcal{J}$. A *path algebra with relations* $A = \mathbb{C}Q/\mathcal{I}$ is the quotient of a path algebra by a finitely generated ideal $\mathcal{I} \subset \mathcal{J}^2$. A path algebra is connected or strongly connected if and only if its underlying quiver is.

We will call a path algebra with relations $\mathbb{C}Q/\mathcal{I}$ *positively graded* if there exists a grading $\mathtt{R} : Q_1 \to \mathbb{R}_{>0}$ such that $\mathcal{I}$ is generated by homogeneous relations. Borrowing terminology from physics, we will sometimes call this map the R-charge.

A special type of path algebras with relations are Jacobi algebras. To define these we need to introduce some notation. The vector space $\mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$ has as basis the set of cyclic paths up to cyclic permutation of the arrows. We can embed this space into $\mathbb{C}Q$ by mapping a cyclic path onto the sum of all its possible cyclic permutations:

$$\circlearrowright : \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q] \to \mathbb{C}Q : a_1 \cdots a_n \mapsto \sum_i a_i \cdots a_n a_1 \cdots a_{i-1}.$$

An element of the form $p + [\mathbb{C}Q, \mathbb{C}Q]$ where p is a cyclic path will be called a *cycle*. Usually we will drop the $+[\mathbb{C}Q, \mathbb{C}Q]$ from the notation and represent the cycle by one of its cyclic paths.

Another convention we will use is the deletion of arrows: if $p := a_1 \cdots a_n$ is a path and b an arrow, then $p\not{b} = a_1 \cdots a_{n-1}$ if $b = a_n$ and zero otherwise. Similarly one can define $\not{b}p$. These new defined maps can be combined to obtain a 'derivation'

$$\partial_a : \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q] \to \mathbb{C}Q : p \mapsto \circlearrowright(p)\not{a} = \not{a} \circlearrowright(p).$$

An element $W \in \mathcal{J}^3/[\mathbb{C}Q, \mathbb{C}Q] \subset \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$ is called a *superpotential*. This element does not need to be homogeneous. If we quotient out the partial derivatives of a superpotential we get an algebra which is called the *Jacobi algebra*:

$$A_W := \mathbb{C}Q/\langle \partial_a W : a \in Q_1 \rangle.$$

Note that if W is homogeneous for some R-charge $\mathtt{R}$, then the corresponding Jacobi Algebra is a positively graded algebra. The converse does not need to be true.

3.2. **Calabi-Yau Algebras.**

Definition 3.1. A path algebra with relations A is *n-dimensional Calabi-Yau* (CY$-n$) if A has a projective bimodule resolution $\mathcal{P}^\bullet$ that is dual to its n^{th} shift

$$\mathrm{Hom}_{A-A}(\mathcal{P}^\bullet, A \otimes A)[n] \cong \mathcal{P}^\bullet$$

For further details about this property we refer to [3] and [15]. In this paper we will only need the following results:

Property 3.2. *If A is CY-n then*

- C1 *The global dimension of A is n*
- C2 *If $X, Y \in \mathsf{Mod}A$ then*

$$\mathsf{Ext}^k_A(X, Y) \cong \mathsf{Ext}^{n-k}_A(Y, X)^*.$$

- C3 *The identifications above give us a pairings $\langle , \rangle^k_{XY} : \mathsf{Ext}^k_A(X, Y) \times \mathsf{Ext}^{n-k}_A(Y, X) \to \mathbb{C}$ which satisfy*

$$\langle f, g\rangle^k_{XY} = \langle 1_X, g * f\rangle^0_{XX} = (-1)^{k(n-k)}\langle 1_Y, f * g\rangle^0_{YY},$$

where $*$ denotes the standard composition of extensions.

Proofs can be found in [3].

3.3. **Noncommutative resolutions and orders.** Suppose $\mathbb{V}$ is a normal variety with coordinate ring R and function field K. A resolution of $\mathbb{V}$ is a proper birational surjective map $\pi : \tilde{\mathbb{V}} \to \mathbb{V}$ such that $\tilde{\mathbb{V}}$ is smooth. The birationality of π implies that it gives an isomorphism on the level of the function fields: $K(\tilde{\mathbb{V}}) = K$.

A nice method to try to construct a resolution is by using orders. An *R-order* in $\mathsf{Mat}_n(K)$ is an R-algebra $A \subset \mathsf{Mat}_n(K)$ that is a finitely generated R-module and

$$A \cdot K = A \otimes_R K = \mathsf{Mat}_n(K).$$

The embedding $R \subset A$ can be seen as a noncommutative generalization of the resolution because birationally (i.e. tensoring with K) it gives a Morita equivalence instead of an isomorphism.

Given an order A, we have a notion of a trace $\mathrm{Tr} : A \to R$, which is the restriction of the standard trace function in $\mathsf{Mat}_n(K)$. Traces of elements in A sit in R because R is a normal domain. This trace allows us to consider the n-dimensional trace preserving representations of A:

$$\mathsf{trep}A := \{\rho : A \to \mathsf{Mat}_n(\mathbb{C}) | \mathrm{Tr}\rho(a) = \rho(\mathrm{Tr}a)\}$$

This object can be given the structure of an affine scheme (take care, it can consist of several components). It has an action of $\mathsf{GL}_n(\mathbb{C})$ by conjugation and using this action we can reconstruct A as the ring of equivariant maps and R as the ring of invariant maps (see [28]):

$$A = \mathsf{Eqv}_{\mathsf{GL}_n}(\mathsf{trep}A, \mathsf{Mat}_n(\mathbb{C})) := \{f : \mathsf{trep}A \to \mathsf{Mat}_n(\mathbb{C}) | \forall g \in \mathsf{GL}_n : f(\rho^g) = f(\rho)^g\},$$
$$R = \mathsf{Inv}_{\mathsf{GL}_n}(\mathsf{trep}A, \mathbb{C}) := \{f : \mathsf{trep}A \to \mathbb{C} | \forall g \in \mathsf{GL}_n : f(\rho^g) = f(\rho)\}.$$

Geometrically this means that R is the coordinate ring of the categorical quotient $\mathsf{trep}A /\!\!/ \mathsf{GL}_n$ and this quotient parameterizes the *isomorphism classes of semisimple trace preserving representations of A*. In general the space $\mathsf{trep}A$ consists of more than one component but there is only one component that maps surjectively onto the quotient. This is the component that contains the generic simples and we denote it by $\mathsf{srep}A$.

To construct a resolution of $\mathbb{V} = \mathsf{srep}A/\!\!/\mathsf{GL}_n$, we can try to take a Mumford quotient instead of the categorical quotient. To do this, one must specify a stability condition, which in the case of path algebras with relations mounts to choosing a $\theta \in \mathbb{Z}^{Q_0}$ (see [23]). The new quotient $\mathbb{V}_\theta = \mathsf{srep}A/\!\!/_\theta\mathsf{GL}_n$ parameterizes the *isomorphism classes of (direct sums of) θ-stable trace preserving representations of A.* If one is lucky the new quotient is smooth and then it provides a resolution of $\mathbb{V}$.

The idea of using orders for the construction of resolutions motivates the notion of a noncommutative resolution. There are many possible definitions but they all share the following properties:

- A is an R-order in $\mathsf{Mat}_n(K)$
- A has some smoothness property: finite homological dimension/homologically homogeneous/Calabi-Yau.

In this paper the focus is on the Calabi-Yau property, so for us noncommutative resolutions are Calabi-Yau orders. Although at first sight this seems to be slightly different from the notion of the noncommutative crepant resolutions introduced by Van den Bergh in [33], in the 3-dimensional toric setting these notions will coincide (see [4]).

4. Toric orders

If $\mathbb{V}$ is a toric variety, then it has a faithful action of a torus $\mathbb{T}^k = \mathbb{C}^{*k}$ with a dense open orbit. Ring-theoretically this means that R is $\mathbb{Z}^k$-graded and we can embed it as a graded subring of $T := \mathbb{C}[\mathbb{T}^k] = \mathbb{C}[X_1, X_1^{-1}, \dots, X_k, X_k^{-1}]$.

To resolve the singularities of $\mathbb{V}$, we want to keep the toric structure of $\mathbb{V}$ so we need to construct a toric resolution. By this we mean that the map $\pi : \tilde{\mathbb{V}} \to \mathbb{V}$ is a $\mathbb{C}^{*k}$-equivariant map that is one to one on the torus $\mathbb{T}^k$. From the point of view of rings, the coordinate ring of the torus now substitutes for the function field K and everything gets $\mathbb{Z}^k$-graded.

This enables us to define *toric orders*.

Definition 4.1. Let $R \subset T = \mathbb{C}[X_1, X_1^{-1}, \dots, X_k, X_k^{-1}]$ be the coordinate ring of a toric variety. A toric R-order A is a positively $\mathbb{Z}^k$-graded R-subalgebra of $\mathsf{Mat}_n(T)$ that is a finitely generated R-module and

TO1 $A \cdot T = \mathsf{Mat}_n(T)$
TO2 $R^{\oplus n} \subset A$

Remark 4.2. By positively $\mathbb{Z}^k$-graded, we mean there is a vector $u \in \mathbb{Z}^k$ such that if there are nonzero homogeneous elements with degree $v \in \mathbb{Z}^k \setminus \{0\}$, then $u \cdot v > 0$.

Toric orders are special orders, so we can also reconstruct R and A from the invariant and equivariant maps on $\mathsf{trep}A$. If we do this we leave the toric context because GL_n is not toric. However, with a slight modification we can make everything we said in the previous section work in the toric context.

We can get rid of GL_n by looking at α-dimensional representations with $\alpha = (1, \dots, 1)$. Because of condition TO2, the standard idempotents $e_i \subset \mathsf{Mat}_n(\mathbb{C}) \subset \mathsf{Mat}_n(T)$ must sit in A. We now define

$$\mathsf{trep}_\alpha A := \{\rho \in \mathsf{trep}A | \rho(e_i) = e_i\}$$

This is a closed subscheme of $\mathsf{trep}A$ that meets every orbit. The action of GL_n on $\mathsf{trep}A$ restricts to an action of $\mathsf{GL}_\alpha = \mathbb{C}^{*n} \subset \mathsf{GL}_n$ on $\mathsf{trep}_\alpha A$ and

$$\mathsf{trep}A = \mathsf{trep}_\alpha A \times_{\mathsf{GL}_\alpha} \mathsf{GL}_n.$$

Furthermore we have again that

$$A = \mathsf{Eqv}_{\mathsf{GL}_\alpha}(\mathsf{trep}_\alpha A, \mathsf{Mat}_n(\mathbb{C})) \text{ and } R = \mathsf{Inv}_{\mathsf{GL}_\alpha}(\mathsf{trep}_\alpha A, \mathbb{C}).$$

Just as in before we single out one component $\mathsf{srep}_\alpha A = \mathsf{srep}A \cap \mathsf{trep}_\alpha A$. This component contains a $n-1+k$-dimensional torus coming from the pullback of the representations of $\mathsf{Mat}_n(T)$ and there is a combined action of $\mathbb{C}^{*n-1}$ from $\mathsf{GL}_\alpha/\mathbb{C}^*$ and $\mathbb{C}^{*k}$ by scaling of the variables. Therefore $\mathsf{srep}_\alpha A$ can be seen as a toric variety but it is not necessarily normal.

Unlike in the general case of orders, toric orders have the advantage that one only needs $\mathsf{srep}_\alpha A$ to reconstruct the order and not the whole space $\mathsf{trep}_\alpha A$.

Theorem 4.3. *If A is a toric R-order in $\mathsf{Mat}_n(K)$ then*

$$A = \mathsf{Eqv}_{\mathsf{GL}_\alpha}(\mathsf{srep}_\alpha A, \mathsf{Mat}_n(\mathbb{C})) \text{ and } R = \mathsf{Inv}_{\mathsf{GL}_\alpha}(\mathsf{srep}_\alpha A, \mathbb{C}).$$

Proof. We have a map $\mathsf{Eqv}_{\mathsf{GL}_\alpha}(\mathsf{trep}_\alpha, \mathsf{Mat}_n(\mathbb{C})) \to \mathsf{Eqv}_{\mathsf{GL}_\alpha}(\mathsf{srep}_\alpha A, \mathsf{Mat}_n(\mathbb{C}))$ by restriction. This map decomposes as a direct sum of maps according to the matrix entries

$$\mathsf{Eqv}_{\mathsf{GL}_\alpha}(\mathsf{trep}_\alpha, \mathsf{Mat}_n(\mathbb{C}))_{ij} \to \mathsf{Eqv}_{\mathsf{GL}_\alpha}(\mathsf{srep}_\alpha, \mathsf{Mat}_n(\mathbb{C}))_{ij}$$

But $\mathsf{Eqv}_{\mathsf{GL}_\alpha}(\mathsf{trep}_\alpha A, \mathsf{Mat}_n(\mathbb{C}))_{ij}$ is the subspace $\mathbb{C}[\mathsf{trep}_\alpha A]_{ij} \subset \mathbb{C}[\mathsf{trep}_\alpha A]$ of weight $i-j$ for the GL_α-action. The same holds for $\mathsf{Eqv}_{\mathsf{GL}_\alpha}(\mathsf{srep} A, \mathsf{Mat}_n(\mathbb{C}))_{ij}$.

The map $\mathbb{C}[\mathsf{trep}_\alpha A] \to \mathbb{C}[\mathsf{srep}_\alpha A]$ is a surjection that is compatible with the $\mathbb{C}^{*n}$-action. This means that $\mathbb{C}[\mathsf{trep}_\alpha A]_{ij} \to \mathbb{C}[\mathsf{srep}_\alpha A]_{ij}$ is surjective. $\mathbb{C}[\mathsf{trep}_\alpha A] \to \mathbb{C}[\mathsf{srep}_\alpha A]$ is not an injection but it becomes an injection if we tensor it over R with the torus ring T (note that R sits both in $\mathbb{C}[\mathsf{trep}_\alpha A]$ and $\mathbb{C}[\mathsf{srep}_\alpha A]$ as a subring because $\mathsf{srep}_\alpha A$ is the component that maps surjectively to $\mathbb{V}$). Therefore if $a \in \mathbb{C}[\mathsf{trep}_\alpha A]_{ij}$ sits in the kernel then we can lift a to an element in A such that $a \otimes_R 1_T = 0$ but this is impossible because $A \subset \mathsf{Mat}_n(T)$.

The second statement follows directly from the first. □

Remark 4.4. This property corresponds to the notion of algebraic consistency, introduced by Broomhead. More specific $\mathsf{Eqv}_{\mathsf{GL}_\alpha}(\mathsf{srep}_\alpha A$ can be identified with the algebra B in [6].

5. Category algebras and cancellation algebras

In this section we will extend the notion of toric orders to a non-Noetherian setting. This generalization gives rise to the notion of a cancellation algebra.

5.1. **Motivation and definition.** From any toric algebra $A \subset \mathsf{Mat}_n(T)$ we can construct a category $\mathcal{C}_A$. The objects of this category are the elementary idempotents $e_i \subset A$ and the morphisms between e_i and e_j are the monomials of T which occur on the i,jth entries of elements in A. In other words:

$$\mathrm{Hom}_{\mathcal{C}_A}(i,j) = \{\text{monomials in } iAj\}.$$

We can reconstruct A as the *category algebra* of $\mathcal{C}_A$. This algebra is the vector space with as basis the set of all morphisms of $\mathcal{C}_A$ and as multiplication the composition of morphisms if possible and zero otherwise.

The category $\mathcal{C}_A$ has a special property: it is a cancellation category.

Definition 5.1. A category $\mathcal{C}$ is called a cancellation category if every morphism is epic and monic:

$$\forall a,b,c : ab = ac \implies b = c \text{ and } ac = bc \implies a = b \text{ (when defined).}$$

The category algebra of a cancellation category is called a cancellation algebra.

Remark 5.2. We will assume implicitly that a cancellation category only has a finite number of objects, and can be generated by a finite number of arrows. This is to make sure that the cancellation algebras we will consider are finitely generated as algebras. Later we will also investigate cancellation categories with a countable number of objects. In this case the category algebra will not be unital.

Remark 5.3. A category algebra can equivalently be defined as a path algebra of a quiver with relations $\mathbb{C}Q/\mathcal{I}$ where $\mathcal{I}$ is generated by elements of the form $p-q$ with p,q paths. Therefore it makes sense to use the notation $h(p)$ and $t(p)$ for morphisms in the category. In general it is not easy to check from a set of relations of the required form whether the corresponding category algebra is a cancellation algebra or not.

Just as for quivers we can speak of positively graded categories, category algebras and in specific positively graded toric orders. In this last case all monomials in a toric order are homogeneous for the grading $\mathtt{R}$.

The simplest examples of cancellation categories are groupoids. In these categories the cancellation law holds trivially because every morphism is invertible. Subcategories of groupoids are also cancellation categories, however unlike in the case of abelian groups and semigroups it is not true that every cancellation category embeds in a groupoid.

If we return to the section of toric orders, it is easy to check that $\mathsf{Mat}_n(T)$ is the category algebra of the groupoid with n objects, that is equivalent to the group $\mathbb{Z}^k$. By consequence, if A is a toric order, then $\mathcal{C}_A$ is a cancellation category because it is a subcategory of that groupoid.

Observation 5.4. *Every toric order is a cancellation algebra.*

5.2. **Bimodule resolutions.** Let A be a category algebra with corresponding category $\mathcal{C}$. A (bi)-module M of A is called $\mathcal{C}$-graded if $M = \bigoplus_{p \in \mathsf{Mor}\mathcal{C}} M_p$ such that $M_p \subset h(p)Mt(p)$ and $\forall q : qM_p \subset M_{qp}$ and $M_p q \subset M_{pq}$, where we used the convention that $M_? = 0$ to cover the case where pq or qp is not defined. A homogeneous map is a morphism $\phi : M \to N$ such that $\phi M_p \subset N_p$ and the kernel and image of a homogeneous map are clearly $\mathcal{C}$-graded. For every p in $\mathcal{C}$ we define the projective bimodule $F_p = Ah(p) \otimes p \otimes t(p)A$ with the obvious grading $q_1 \otimes p \otimes q_2 \in (F_p)_{q_1 p q_2}$ and analogously the projective left module $P_p = Ah(p)$ with grading $q \in (P_p)_{qp}$.

If A is positively graded then the category of $\mathcal{C}$-graded bimodules with bihomogeneous morphisms is a perfect category in the sense of Eilenberg [12] and hence we can construct a minimal projective bimodule resolution of A as a bimodule over itself with the obvious grading. The first terms of this map are

$$\bigoplus_{s \in \mathcal{S}} F_s \xrightarrow{\delta_3} \bigoplus_{r \in \mathcal{R}} F_r \xrightarrow{\delta_2} \bigoplus_{b \in Q_1} F_b \xrightarrow{\delta_1} \bigoplus_{i \in Q_0} F_i \xrightarrow{m} A$$

where

$$m(q_1 \otimes i \otimes q_2) = q_1 q_2$$
$$\delta_1(q_1 \otimes b \otimes q_2) = q_1 b \otimes t(b) \otimes q_2 - q_1 \otimes h(b) \otimes b q_2$$
$$\delta_2(q_1 \otimes r \otimes q_2) = \sum_k q_1 a_1 \dots \otimes a_k \otimes \dots a_n q_2 - \sum_k q_1 b_1 \dots \otimes b_k \otimes \dots b_m q_2$$
$$\delta_3(q_1 \otimes s \otimes q_2) = q_1 s q_2.$$

and $r = a_1 \dots a_n - b_1 \dots b_m$, $F_r := F_{a_1 \dots a_n} = F_{b_1 \dots b_m}$ and $\mathcal{S}$ is a minimal set of homogeneous generators of Kerδ_2. This set might be infinite.

For every vertex $i \in Q_0$ we can tensor this resolution over A on the right with the one-dimensional left module $S_i = Ai/\mathcal{J}i$ concentrated in $\mathcal{C}$-degree i. This gives us the minimal graded left module resolution of S_i.

6. Quiver polyhedra

6.1. **Definition.** The last ingredient we need is quiver polyhedra.

Definition 6.1. A *quiver polyhedron* $\mathcal{Q}$ is a strongly connected quiver Q enriched with 2 disjoint sets of cycles Q_2^+ and Q_2^- such that

- PO **Orientability condition.** Every arrow is contained exactly once in one cycle in Q_2^+ and once in one in Q_2^-.
- PM **Manifold condition.** The incidence graph of the cycles and arrows meeting a given vertex is connected.

A quiver polyhedron is called *weighted* if there is a map $\mathtt{E} : Q_2 = Q_2^+ \cup Q_2^- \to \mathbb{N}_{>0}$ such that $\forall c \in Q_2 : \mathtt{E}_c|c| > 2$. We will use the symbol $\mathcal{Q}$ to denote a weighted quiver polyhedron. If $\mathtt{E}$ is the constant map to 1 we say $\mathcal{Q}$ is an unweighted quiver polyhedron.

A weighted quiver polyhedron is *positively graded* if there is an R-charge $\mathtt{R} : Q_1 \to \mathbb{R}_{>0}$ such that the expression $\mathtt{E}_c\mathtt{R}_c$ is the same for all cycles in Q_2.

Remark 6.2. For a number of examples of quiver polyhedra we refer to section 9. There we also discuss the connection with dimer models.

Remark 6.3. Not every weighted quiver polyhedron can be given a grading. In [6][Remark 2.3.5] a combinatorial condition is given for this to be true in the case of trivially weighted quiver polyhedra. At the end of this section we will state this condition for weighted quiver polyhedra.

The grading that one can assign to a weighted quiver polyhedron is also far from unique, however the algebraic properties that we will discuss further on do not depend on it. They depend merely on the existence of a grading.

From a quiver polyhedron we can build a topological space X by associating to every cycle of length k a k-gon. We label the edges of this k-gon cyclicly by the arrows of the quiver and identify edges of different polygon labeled with the same arrow.

Lemma 6.4. *If $\mathcal{Q}$ is a quiver polyhedron then X is a compact orientable surface.*

Proof. We need to show that every point in X has a neighborhood that is homeomorphic to an open disk. For the internal points of the polygons this is trivially true. If p lies on an edge of a polygon but not on a corner then this is true because by condition PO a small enough neighborhood of p will consist of two half disks glued together. If p is a corner of a polygon, a neighborhood of p consists of triangles glued together over common edges. The result in general will be a set of disks glued together at p and there is just one disk if and only if PM holds.

Using the condition PO, this surface can be oriented by assigning an anticlockwise direction to the cycles in Q_2^+ and a clockwise direction for those in Q_2^-. □

Conversely, if Q is a strongly connected quiver drawn on an orientable surface such that the complement of the quiver consists of simply connected pieces bounded by cycles, then we can give Q the structure of a quiver polyhedron by taking as Q_2^+ the cycles that bound pieces anticlockwise and as Q_2^- the cycles that bound pieces clockwise. It can easily be checked that PO and PM hold.

If $\mathcal{Q}$ is a weighted quiver polyhedron, it is interesting to give the topological space X the structure of an orbifold. We can do this by substituting the k-gon corresponding to a cycle c with the orbifold obtained by quotienting an kr-gon by the rotation group of order $r = \mathtt{E}_c$. If we do this for all cycles we get an orbifold that contains an orbifold singularity of order $\mathtt{E}_c$ for every cycle c. We will denote this orbifold by $|\mathcal{Q}|$. It is clear from this construction that the orbifold $|\mathcal{Q}|$ of a trivially weighted quiver is just the compact surface X.

For any weighted quiver polyhedron it makes sense to define its Euler characteristic as the Euler characteristictic of its orbifold $|\mathcal{Q}|$.

$$\chi_{\mathcal{Q}} := \#Q_0 - \#Q_1 + \sum_{c \in Q_2} \frac{1}{\mathtt{E}_c}.$$

For a weighted quiver polyhedron $\mathcal{Q}$, we can define a superpotential

$$W = W^+ - W^- := \sum_{c \in Q_2^+} \frac{c^{\mathtt{E}_c}}{\mathtt{E}_c} - \sum_{c \in Q_2^-} \frac{c^{\mathtt{E}_c}}{\mathtt{E}_c} + [\mathbb{C}Q, \mathbb{C}Q].$$

Here $c^{\mathtt{E}_c}$ stands for a cycle obtained by running through c $\mathtt{E}_c$ times. This superpotential gives rise to a Jacobi algebra $A_{\mathcal{Q}} := A_W$. Note that $W \in \mathcal{J}^3$ because for every cycle $\mathtt{E}_c|c| > 2$.

Lemma 6.5. *For any (positively graded) weighted quiver polyhedron $\mathcal{Q}$ the Jacobi algebra $A_{\mathcal{Q}}$ is a (positively graded) category algebra.*

Proof. For any arrow a the partial derivative $\partial_a W$ is $\mathtt{E}_c$ times the sum of two paths with opposite signs and therefore every relation is of the form $p - q$.

If $\mathcal{Q}$ is positively graded then the superpotential is homogeneous so the Jacobi algebra is positively graded. □

Remark 6.6. It is important to note that $A_{\mathcal{Q}}$ is a category algebra but not always a cancellation algebra. We will come back to this issue in section 8.

6.2. **Galois covers.** A morphism between weighted quiver polyhedra $\mathcal{Q}^A$ and $\mathcal{Q}^B$ is a pair of maps $\phi : Q_0^A \to Q_0^B$ and $\phi : Q_1^A \to Q_1^B$ respecting head and tails ($\phi(h(a)) = h(\phi(a))$ and $\phi(t(a)) = t(\phi(a))$) such that if $c \in Q_2^{A+}$ (Q_2^{A-}) we can find a $d \in Q_2^{B+}$ (Q_2^{B-}) such that $\phi(c^{\mathtt{E}_c}) = d^{\mathtt{E}_d}$. One can check easily from the definition that a morphism between quiver polyhedra gives corresponding morphisms between their orbifolds and their path algebras.

Let G be a group of automorphisms of a weighted quiver polyhedron $\mathcal{Q}$ such that no nontrivial element $g \in \mathsf{G}$ fixes a vertex of $\mathcal{Q}$. The quotient quiver Q/G is defined as the quiver with as vertices and arrows the orbit classes of vertices and arrows in $\mathcal{Q}$. There is a projection map $\pi : Q \to Q/\mathsf{G}$ that maps each vertex and arrow to its orbit. Under π every cycle $c \in Q_2$ is mapped to a cycle in $\mathcal{Q}/\mathsf{G}$. This cycle can sometimes be the power of a smaller primitive cycle: $\pi(c) = d^k$ for some k. The unique way to equip Q/G with a polyhedral structure is

$$(Q/\mathsf{G})_2^{\pm} := \{d | \exists c \in Q_2^{\pm} : d^k = \pi(c) \text{ and } d \text{ is primitive}\},$$

The weighting has the following form

$$\mathtt{E}_d := k\mathtt{E}_c.$$

The following theorem is straightforward:

Theorem 6.7.

- *If G is a group of automorphisms of a weighted quiver polyhedron $\mathcal{Q}$ such that no nontrivial element $g \in \mathsf{G}$ fixes a vertex of $\mathcal{Q}$ then the quotient morphism $\pi : \mathcal{Q} \to \mathcal{Q}/\mathsf{G}$ induces a cover morphism between the two orbifolds $\tilde{\pi} : |\mathcal{Q}| \to |\mathcal{Q}/\mathsf{G}|$ and the group of cover automorphisms of $\tilde{\pi}$ is G.*
- *On the level of path algebras we have a surjective map $\pi : \mathbb{C}\mathcal{Q} \to \mathbb{C}\mathcal{Q}/\mathsf{G}$ such that if q is a path in $\mathcal{Q}/\mathsf{G}$ then for every vertex $v \in \pi^{-1}(h(q))$ there is a unique lifted path $\mathsf{lift}_v q \in \pi^{-1}(q)$ such that $h(\mathsf{lift}_v q) = v$.*
- *Two paths in $q_1, q_2 \in \mathbb{C}\mathcal{Q}/\mathsf{G}$ are equivalent in $A_{\mathcal{Q}/\mathsf{G}}$ if and only if there is a $v \in \pi^{-1}h(q_1)$ such that $\mathsf{lift}_v q_1$ is equivalent with $\mathsf{lift}_v q_2$ in $A_{\mathcal{Q}}$.*

Proof. The first statement follows because by construction $|\mathcal{Q}/G|$ is the same orbifold as the quotient orbifold $|\mathcal{Q}|/\mathsf{G}$.

Suppose q is a path in $\mathcal{Q}/\mathsf{G}$ and choose any lift $p \in \pi^{-1}(q)$. There is a unique $g \in \mathsf{G}$ that maps $h(p)$ to $v \in \pi^{-1}(h(q))$ and therefore the lift of q starting in v is $g \cdot p$.

The last statement follows from the easy to check facts that $\pi(\partial_a W_{\mathcal{Q}}) = \partial_{\pi a} W_{\mathcal{Q}/\mathsf{G}}$ and $\pi^{-1}\partial_b W_{\mathcal{Q}/\mathsf{G}} = \{\partial_a W_{\mathcal{Q}} | \pi(a) = b\}$. □

This theorem states in fact that $A_{\mathcal{Q}}$ is a Galois cover of $A_{\mathcal{Q}/\mathsf{G}}$ in the sense of [29]. It implies a close relationship between the two Jacobi algebras and many nice properties will either hold in both or in none. An interesting example of such a property is the cancellation property.

Theorem 6.8. *The Jacobi algebra $A_{\mathcal{Q}}$ is a cancellation algebra if and only if $A_{\mathcal{Q}/\mathsf{G}}$ is a cancellation algebra.*

Proof. Let p and q be paths and a an arrow in $\mathcal{Q}$ such that $pa = qa$. In the quotient we have that $\pi(p)\pi(a) = \pi(q)\pi(a)$ and $\pi(p) \neq \pi(q)$ because these paths cannot be in the same orbit as they start in the same vertices and G acts freely on the vertices.

Suppose on the other hand that r, s are paths and b is an arrow in $\mathcal{Q}/\mathsf{G}$ with $rb = sb$. Fix a vertex $v \in \pi^{-1}(h(r))$. By the lifting property $\mathsf{lift}_v(rb) = \mathsf{lift}_v(sb)$ and both must end in the same arrow $a \in \pi^{-1}(b)$, so $\mathsf{lift}_v(r)a = \mathsf{lift}_v(s)a$ but $\mathsf{lift}_v(r) \neq \mathsf{lift}_v(s)$ because their projections to $A_{\mathcal{Q}/\mathsf{G}}$ are different. □

The existence of a grading is compatible with the notion of Galois covers.

Lemma 6.9. *$\mathcal{Q}$ admits a grading if and only if $\mathcal{Q}/\mathsf{G}$ does.*

Proof. If $\mathcal{Q}$ is positively graded, we can also give the new polyhedron a grading:

$$\mathtt{R}_{\pi(a)} := \frac{1}{|\mathsf{G}|}\sum_{b \in \mathsf{G}a} \mathtt{R}_b.$$

If $\mathcal{Q}/\mathsf{G}$ is graded we transfer the grading as follows:

$$\mathtt{R}_a := \mathtt{R}_{\pi(a)}.$$

□

The technique of Galois covers can be used to simplify the structure of the polyhedron, without changing the important properties of the cancellation algebra.

Theorem 6.10. *A weighted quiver polyhedron can be covered by a quiver polyhedron with trivial weighting if and only if it is not of the following forms:*

- *It has the topology of a sphere and 1 face with non-trivial weight.*
- *It has the topology of a sphere and 2 faces with different non-trivial weights.*

Proof. Given an orbifold X with a weighted quiver polyhedron $\mathcal{Q}$ on it, we can use every orbifold cover $\tilde{X} \to X$ to obtain a Galois cover $\tilde{\mathcal{Q}} \to \mathcal{Q}$. If $\tilde{X}$ is a manifold then $\tilde{\mathcal{Q}}$ is unweighted. From theorem 13.3.6 in [32] we know that in dimension 2 the only orientable orbifolds that cannot be covered by a manifold are the sphere with 1 or 2 different orbifold points. These correspond to the quiver polyhedra described above. □

In accordance with the theory of orbifolds we call $\mathcal{Q}$ *developable* if it has an unweighted galois cover. We will denote the unweighted cover of a weighted quiver polyhedron $\mathcal{Q}$ by $\mathcal{Q}^u$.

This cover can be used to check whether $\mathcal{Q}$ admits a grading.

Lemma 6.11. *A weighted quiver polyhedron $\mathcal{Q}$ admits a grading if and only if it is developable and its unweighted cover admits a grading.*

Proof. Suppose $\mathcal{Q}$ is not developable. Then it has the topology of a sphere and has 2 cycles u_1, u_2 such that all other cycles are unweighted. For any grading compatible with the Jacobi relations we have

$$\mathtt{R}_{u_1} = \mathtt{R}_{u_2} \mod \mathtt{R}_u$$

with u an unweighted cycle. Indeed the cycles u_1, u_2 have the same homology class in the union of all unweighted faces. Because all these faces have the same degree $\mathtt{R}_u$, the difference in degree between u_1 and u_2 must be a multiple of $\mathtt{R}_u$. But $\mathtt{R}_{u_i} = \frac{\mathtt{R}_u}{\mathtt{E}_{u_i}}$ so the weights of u_1 and u_2 must be the same and hence a positive grading is impossible.

A developable quiver polyhedron admits a positive grading if and only if its unweighted cover admits a positive grading by lemma 6.9. □

For unweighted quiver polyhedra we can use Hall's theorem (see [6][Remark 2.5.5]) to check whether a grading exists.

Theorem 6.12 (Hall). *An unweighted quiver polyhedron $\mathcal{Q}$ admits a grading if and only if for any subset $S^+ \subset (Q^u)_2^+$ we have that if $S^- \subset (Q^u)_2^-$ is the set of cycles connected to cycles of S^+ then*

$$|S^+| \geq |S^-|$$

with equality only happening if S^+ is not a proper subset.

If a weighted quiver polyhedron $\tilde{\mathcal{Q}}$ is developable, then the pullback of $\tilde{\mathcal{Q}}$ under the universal cover map is called the *universal cover* of $\tilde{\mathcal{Q}}$. This quiver is infinite if the Euler characteristic of $|\mathcal{Q}|$ is zero or negative. It still makes sense to define the corresponding category and category algebra, however one must take care that the latter is not a unital algebra any more. We will denote the universal cover of $\mathcal{Q}$ by $\tilde{\mathcal{Q}}$.

7. The CY-3 property and quiver polyhedra

Jacobi algebras coming from quiver polyhedra appear naturally in the context of CY-3 algebras.

Theorem 7.1. *If a positively graded cancellation algebra A is CY-3 then it comes from a graded weighted quiver polyhedron.*

To prove this theorem we need a lemma which is an adaptation of a theorem from [3].

Lemma 7.2. *If a positively graded cancellation algebra $A = \mathbb{C}Q/\mathcal{I}$ is CY-3 then it is a Jacobi algebra of some superpotential W and there exist a coefficients $\lambda_a \in \mathbb{C}$ depending on $a \in Q_1$ such that $\partial_a W = \lambda_a(p-q)$ for some $p-q \in \mathcal{R}$.*

Proof. We adapt the proof in [3][Theorem 3.1] which worked for an $\mathbb{N}$-graded algebra generated in degree 1, to this setting (where arrows can have different $\mathtt{R}$-degree).

As the global dimension of A must be 3, we know from section 5.2 the minimal projective $\mathcal{C}$-graded resolution of the trivial module $S_i = Ai/\mathcal{J}$ with $\mathcal{C}$-degree i looks like

$$P_\omega \xhookrightarrow{(f_r)} \bigoplus_{t(r)=i} P_r \xrightarrow{(r\not{b})} \bigoplus_{t(b)=i} P_b \xrightarrow{(\cdot b)} P_i \twoheadrightarrow S_i.$$

In the diagram above the $r's$ are elements of the minimal set of relations $\mathcal{R}$ and the $b's$ are arrows. Note that the last term in the resolution P_ω must be isomorphic to P_i because $\dim \mathsf{Ext}^3(S_i, S_j) \overset{CY}{=} \dim \mathsf{Hom}(S_j, S_i) = \delta_{ij}$. This P_i is shifted in $\mathcal{C}$-degree, and we let ω be the path that corresponds to $i \in P_\omega$.

Consider the finite dimensional quotient algebra

$$M = A/(f_r : r \in \mathcal{R}, A_n : n \geq N) \text{ where } \forall r : N > \mathtt{R}_{f_r}.$$

The Calabi-Yau property allows us to calculate the dimension of iMj:

$$\mathsf{Dim}\, iMj = \mathsf{Dim}\,\mathsf{Hom}(P_i, Mj) = \mathsf{Dim}\,\mathsf{Ext}^3(S_i, Mj) \overset{CY}{=} \mathsf{Dim}\,\mathsf{Hom}(Mj, S_i) = \delta_{ij},$$

and conclude that M must be isomorphic to the degree zero part of A. There are only as many f_r as there are arrows ($\dim \mathsf{Ext}^2(S_i, S_j) \overset{CY}{=} \dim \mathsf{Ext}^1(S_j, S_i)$). An f_r (with $r = p - q$) cannot be a linear combination of different arrows a and b because this would imply that ω, ap and bp have the same $\mathcal{C}$-degree which contradicts the cancellation property. Hence, we can conclude that the f_r must all be scalar multiples of arrows.

By rescaling our original relations, we can assume that the f_r can be identified with the arrows. Let r_a be the (nonzero) relation for which $f_{r_a} = a$.

Because the resolution of S_i is a complex we have that $\sum_a ar_a\not{b} \in \mathcal{I}$ so we can write it as

$$\sum_{h(a)=i} ar_a\not{b} = \sum g_{bc} r_c + rest \text{ with } rest \in \mathcal{J}\mathcal{I} + \mathcal{I}\mathcal{J}.$$

If we apply property C3 we can conclude that $g_{bc} \neq 0 \iff b = c$. The terms $ar_a\not{b}$ all have the same $\mathcal{C}$-degree which is equal to the degree of r_b. As r_b is a minimal relation, there are no $\mathcal{C}$-homogeneous elements in $\mathcal{J}\mathcal{I} + \mathcal{I}\mathcal{J}$ with the same $\mathcal{C}$-degree as r_b and hence $rest = 0$.

By introducing an appropriate rescaling of the relations we can assume that $g_{bb} = 1$ and

$$\sum_{h(a)=i} ar_a = \sum_{t(b)=i} r_b b.$$

If we sum these equations we get a superpotential $W := \sum_a ar_a = \sum_b r_b b$ and it is clear that iW is $\mathcal{C}$-homogeneous. Note that ar_a and $r_a a$ have the same $\mathtt{R}$-degree but sit in different parts of W ($h(a)W$ and $Wt(a)$). We can use this together with the fact that $\mathcal{Q}$ is strongly connected to show that W is $\mathtt{R}$-homogeneous.

Finally, because of the rescalings $r_a = \lambda_a(p-q)$ for some $\lambda_a \in \mathbb{C}$ and some $p - q \in \mathcal{R}$. □

Proof of theorem 7.1. Because A is positively graded and CY-3 we know from lemma 7.2 that $A = A_W$ for some superpotential W and $\partial_a W = \lambda_a(p_a - q_a)$ for some scalar λ_a and some relation $p_a - q_a \in \mathcal{R}$.

Every arrow occurs exactly in two cycles in W (ap_a and aq_a). If an arrow a occurs in a cycle c it can occur only once in this cycle or c is a power of a smaller cyclic path containing just one a. If this were not the case, the partial derivative to a of this cycle would contain more than one term with the same sign which is impossible.

Let Q_2 be the set of all cycles c such that a power c^k occurs in W and which are not powers of smaller cycles. The grading $\mathtt{R}$ on A gives a grading $\mathtt{R}$ on the arrows and we define $\mathtt{E}_c = k$ if and only if c^k sits in W.

This data turns Q into a weighted quiver polyhedron:

PM Fix a single vertex i and consider the following graph G_i: its nodes correspond to the arrows which have a head or a tail equal to i. There is an edge between two arrows a, b with $t(a) = h(b) = i$ if ab is contained in a cycle of W.

For every connected component $C \subset G_i$ we can construct a syzygy:

$$z_C = \sum_{a \in C, h(a)=i} a \otimes \partial_a W \otimes 1 - \sum_{a \in C, t(a)=i} 1 \otimes \partial_a W \otimes a.$$

Indeed for every vertex i the expression $\sigma_i := \sum_{h(a)=i} a \otimes \partial_a W \otimes 1 - \sum_{t(a)=i} 1 \otimes \partial_a W \otimes a$ is a syzygy. We can split this syzygy in parts because the sets of arrows occurring in z_{C_1} and z_{C_2} for two different components C_1 and C_2 are disjoint. By the CY-3 property C2 we know that the third syzygies are in one to one correspondence with the vertices. We can conclude that G_v consist of one component.

PO Define the map $\mathrm{cf} : Q_2 \to \mathbb{C}$ such that $W = \sum_{c \in Q_2} \frac{\mathrm{cf}(c)}{\mathtt{E}_c} c^{\mathtt{E}_c}$. We will show that the image of this map is $\{\lambda, -\lambda\}$ for some $\lambda \in \mathbb{C}$. We take $Q_2^{\pm}$ the preimage of $\pm\lambda$. Clearly if two cycles share an arrow a then $\mathrm{cf}(c_1) = -\mathrm{cf}(c_2) = \lambda_a$. So $\operatorname{Im} \mathrm{cf} = \{\lambda, -\lambda\}$ if we can go from one cycle to every other cycle by hopping over joint arrows. This follows from condition PM and the fact that Q is strongly connected.

The fact that Q is strongly connected also implies that the $c^{\mathtt{E}_c}$ have the same $\mathtt{R}$-degree and because $W \subset \mathcal{J}^3$ we must also have that $\mathtt{E}_c|c| > 2$. This implies that $\mathtt{E}$ is a weighting for the quiver polyhedron and $\mathtt{R}$ is a compatible grading. □

8. Toric orders and dimer models

In the previous section we proved that the CY-3 property for positively graded cancellation algebras implies the existence of a weighted quiver polyhedron. Now we will prove that if we restrict to cancellation algebras that are toric orders, we obtain that the quiver polyhedron must be unweighted and its underlying manifold $|\mathcal{Q}|$ must be a torus.

In order to prove this result we must first have a look at the cancellation property for quiver polyhedra.

8.1. **Cancellation for quiver polyhedra.**

As we noted in section 6 not all graded weighted quiver polyhedra give cancellation algebras.

The relations in the Jacobi algebra $A_{\mathcal{Q}}$ imply that all cycles in Q_2 are equivalent: ${c_1}^{\mathtt{E}_{c_1}} p = p {c_2}^{\mathtt{E}_{c_2}}$ for every p with $h(p) = t(c_1)$ and $t(p) = h(c_2)$. This implies that the algebra A has a central element: $\sum c^{\mathtt{E}_c}$ where we sum over a subset representatives of Q_2 that contains just one cyclic path c with $h(c) = i$ for every $i \in Q_0$. We will denote this central element by ℓ. For every arrow a we can find a path p such that $ap = h(a)\ell$ and $pa = t(a)\ell$: just take $p = \partial_a c^{\mathtt{E}_c}$ where c is a cycle in Q_2 containing a.

The cancellation property states that the map

$$A_{\mathcal{Q}} \to A_{\mathcal{Q}} \otimes_{\mathbb{C}[\ell]} \mathbb{C}[\ell, \ell^{-1}]$$

is an embedding. This tensor product is the algebra obtained by making every arrow invertible (i.e. for every a we have an a^{-1} such that $aa^{-1} = h(a)$ and $a^{-1}a = t(a)$). This algebra is the localization of $A_{\mathcal{Q}}$ by the Ore set $\{\ell^k | k \in \mathbb{N}\}$ and we denote it by $\hat{A}_{\mathcal{Q}}$. This definition makes also sense if $A_{\mathcal{Q}}$ does not satisfy the cancellation property, the map $A_{\mathcal{Q}} \to \hat{A}_{\mathcal{Q}}$ is no more injective but $\hat{A}_{\mathcal{Q}}$ is still a flat $A_{\mathcal{Q}}$-module.

Lemma 8.1. *If $A_{\mathcal{Q}}$ is CY-3 then $\hat{A}_{\mathcal{Q}}$ is also CY-3.*

Proof. Let $P^\bullet$ be the bimodule resolution of $A_{\mathcal{Q}}$ as a module over itself. The complex $\hat{A}_{\mathcal{Q}} \otimes_{A_{\mathcal{Q}}} P^\bullet \otimes_{A_{\mathcal{Q}}} \hat{A}_{\mathcal{Q}}$ is still exact because $\hat{A}_{\mathcal{Q}}$ is a flat $A_{\mathcal{Q}}$-module. This implies that $\hat{A}_{\mathcal{Q}}$ has a selfdual resolution and is hence CY-3. □

It is important to note that $\hat{A}_{\mathcal{Q}}$ always is a cancellation algebra even when $A_{\mathcal{Q}}$ is not. It is not always a CY-3 algebra, but in the case that $\mathcal{Q}$ is graded and $\chi_{\mathcal{Q}} \le 0$ it will be even CY-3 if $A_{\mathcal{Q}}$ is not. To prove this statement we first need to recall a well known lemma.

Lemma 8.2. *Let $\mathcal{Q}$ be a weighted quiver polyhedron and $\mathtt{R} : Q_1 \to \mathbb{R}$ be any (not necessarily positive) grading such that $\mathtt{R}_\ell \neq 0$. Two paths in $\hat{A}_{\mathcal{Q}}$ are equivalent if and only if they are homotopic and have the same $\mathtt{R}$-degree.*

Proof. It is clear that the relations $\partial_a W$ imply that equivalent paths are homotopic and must have the same $\mathtt{R}$-charge. Because homotopies in the quiver polyhedron are generated by substituting paths $p \to q$ such that $pq^{-1} = \ell$, homotopic paths can only differ by a factor ℓ^k. The degree of ℓ is not zero, so if homotopic paths have the same degree they must be equal in $\hat{A}_{\mathcal{Q}}$. □

Remark 8.3. By homotopic we mean homotopic as paths in $|\mathcal{Q}|$ considered as an orbifold, not merely as a topological space.

Theorem 8.4. *For any positively graded weighted quiver polyhedron $\mathcal{Q}$,*

$$\hat{A}_{\mathcal{Q}} \cong \mathsf{Mat}_n(\mathbb{C}[\Pi])$$

where n is the number of vertices and $\mathbb{C}[\Pi]$ is the group algebra of the fundamental group of some compact three-dimensional manifold.

Proof. Note that because of the gradedness $\mathcal{Q}$ is developable. Let $|\tilde{\mathcal{Q}}| \to |\mathcal{Q}|$ be the universal cover of the orbifold $|\mathcal{Q}|$ and fix a vertex $i \in Q_0$. To every path in $p \in i\hat{A}_{\tilde{\mathcal{Q}}}i$ corresponds an element in the fundamental group of $|\mathcal{Q}|$, which gives a cover automorphism $\phi_p : |\tilde{\mathcal{Q}}| \to |\mathcal{Q}|$. Conversely, every element in the fundamental group can be represented by a path in $\mathcal{Q}$.

Now consider the simply connected space $|\tilde{\mathcal{Q}}| \times \mathbb{R}$ and consider the group of diffeomorphisms

$$\Pi = \{\psi_p : |\tilde{\mathcal{Q}}| \times \mathbb{R} \to |\tilde{\mathcal{Q}}| \times \mathbb{R} : (x, a) \mapsto (\phi_p(x), a + \mathtt{R}_p) | p \in \mathrm{Hom}_{\mathcal{C}_{\hat{A_{\mathcal{Q}}}}}(i, i)\}$$

By lemma 8.2, every element in $\mathrm{Hom}_{\mathcal{C}_{\hat{A}_{\mathcal{Q}}}}(i, i)$ gives a different diffeomorphism and none of these diffeomorphisms has fixpoints. The quotient of $|\tilde{\mathcal{Q}}| \times \mathbb{R}/\Pi$ is thus a manifold and $i\hat{A}_{\mathcal{Q}}i \cong \mathbb{C}[\Pi] = \mathbb{C}[\pi_1(|\tilde{\mathcal{Q}}| \times \mathbb{R}/\Pi)]$.

For every vertex j, fix a path $p_j : i \leftarrow j$. Construct the following morphism

$$\mathsf{Mat}_n(i\hat{A}_{\mathcal{Q}}i) \to \hat{A}_{\mathcal{Q}} : qE_{uv} \mapsto p_u^{-1} q p_v$$

where E_{uv} is the matrix with one on the entry (u, v) and zero everywhere else. This morphism has an inverse

$$\hat{A}_{\mathcal{Q}} \to \mathsf{Mat}_n(i\hat{A}_{\mathcal{Q}}i) : q \mapsto p_{h(q)} q p_{t(q)}^{-1} E_{h(q)t(q)}.$$

In general the fundamental group algebra of a compact manifold is CY-n if it is orientable and its universal cover is contractible (see [15] Corollary 6.1.4). □

We recall a theorem by Kontsevitch.

Theorem 8.5 (Kontsevich, see [15] Corollary 6.1.4)**.** *The fundamental group algebra of a compact manifold is CY-n if it is orientable and its universal cover is contractible.*

This theorem can be used to relate the Euler characteristic of the dimer model with the CY-3 property.

Corollary 8.6. *Let $\mathcal{Q}$ be any positively graded weighted quiver polyhedron.*

- *$\hat{A}_{\mathcal{Q}}$ is CY-3 if and only if $\chi(\mathcal{Q}) \leq 0$.*
- *if $A_{\mathcal{Q}}$ is CY-3 then $\chi(\mathcal{Q}) \leq 0$.*

Proof. We know that $\hat{A}_{\mathcal{Q}}$ is Morita equivalent to the fundamental group algebra of some 3-manifold. This manifold has as universal cover $|\tilde{\mathcal{Q}}| \times \mathbb{R}$. This is contractable when $\chi_{\mathcal{Q}} \leq 0$. So by Kontsevich theorem $\hat{A}_{\mathcal{Q}}$ is CY-3. If the quiver polyhedron $\mathcal{Q}$ has positive Euler characteristic, then its universal cover $\tilde{\mathcal{Q}}$ has the topology of a sphere and the quotient manifold of the cover is $\mathbb{S}_2 \times \mathbb{S}_1$. The fundamental group is $\mathbb{Z}$ so $\hat{A}_{\tilde{\mathcal{Q}}}$ is Morita equivalent to $\mathbb{C}[\ell, \ell^{-1}]$. This last algebra is not CY-3.

If $A_{\mathcal{Q}}$ is CY-3 then $\hat{A}_{\mathcal{Q}}$ is also CY-3 so by the previous paragraph $\chi(\mathcal{Q}) \leq 0$. □

Theorem 8.7. *If a toric order A is CY-3 then it comes from a positively graded unweighted quiver polyhedron on a torus (in other words a dimer model on a torus)*

Proof. Because a toric order is cancellation we already know it comes from a weighted quiver polyhedron, so we only need to show that the weights are trivial and $\chi_{\mathcal{Q}} = 0$.

If $A_{\mathcal{Q}}$ is a toric order then $A_{\mathcal{Q}} \subset \hat{A}_{\mathcal{Q}} \subset \mathsf{Mat}_n(\mathbb{C}[\mathbb{Z}^3])$ because ℓ is invertible in $\mathsf{Mat}_n(\mathbb{C}[\mathbb{Z}^3])$. Hence, for every vertex v, the algebra $v\hat{A}_{\mathcal{Q}}v$ is commutative. This means that the fundamental group of the 3-manifold is commutative and by construction the orbifold fundamental group of the 2-orbifold must also be commutative.

As $\chi_{\mathcal{Q}} \leq 0$, this is only the case if $|\mathcal{Q}|$ is a torus. Indeed, if the fundamental group of $|\mathcal{Q}|$ is abelian then the deck transformations of $|\tilde{\mathcal{Q}}| \to |\mathcal{Q}|$ cannot have fixpoints (as such transformations do not commute with the fixpointless ones) this means $|\mathcal{Q}|$ is a manifold (not an orbifold) and $\mathcal{Q}$ is unweighted. The only compact surface with nonpositive Euler characteristic and abelian fundamental group is the torus. □

8.2. **Cancellation and Calabi-Yau.** The cancellation property and the Calabi-Yau property are very closely related. For quiver polyhedra with $\chi_{\mathcal{Q}} \leq 0$, Ben Davison in [11] proved that cancellation implies CY-3.

Theorem 8.8 (Davison)**.** *The Jacobi algebra of a graded weighted quiver polyhedron with nonpositive Euler characteristic is CY-3 if it is a cancellation algebra.*

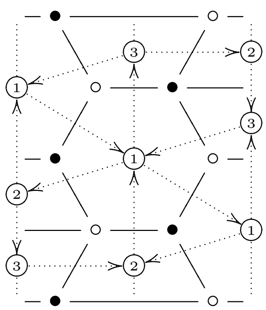
Although Davison proved this only in the case of dimer models (which are in our terminology the trivially weighted quiver polyhedra) his proof generalizes to the weighted case because we can cover any graded weighted quiver polyhedron by a graded unweighted quiver polyhedron. Davison's work was a generalization of work by Mozgovoy and Reineke [26] which used an extra consistency condition. This extra condition turned out to be a consequence of the cancellation property.

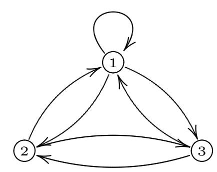
It is not clear whether for quiver polyhedra with $\chi_{\mathcal{Q}} \leq 0$ the cancellation property is really equivalent to the CY-3 property. There are no known examples of noncancellation CY-3 quiver polyhedra for which $\mathcal{Q}$ is finite. There are however examples where $\mathcal{Q}$ is infinite. We refer to the follow-up paper [4] which discusses the different notions of consistency for quiver polyhedra.

9. Examples

The most studied examples of quiver polyhedra come from dimer models. Dimer models are dual to unweighted quiver polyhedra. a dimer model consists of a bipartite graph on a Rieman surface. The bipartiteness implies that vertices are coloured black and white in such a way that no vertices of the same colour share an edge. Given a unweighted quiver polyhedron we construct the dimer model by using the centers of the cycles in Q_2^+ as black vertices and the centers of the cycles in Q_2^- as white vertices, two vertices are connected if their cycles share a face.

Example 9.1. The suspended pinchpoint [13][section 4.1] is an example of a CY-3 algebra given by the following dimer model and quiver:



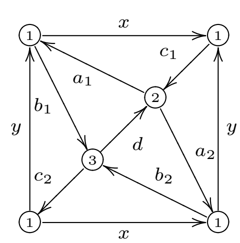


On the left we drew the tiling of the torus as a periodic tiling of the plane. The quiver is represented twice, once periodically on the left(the dotted lines) and once on the right. There are three vertices in the quiver corresponding to the three tiles on the torus (one hexagon and two trapezia). The sets of cycles are $Q_2^+ = \{a_{31}a_{13}a_{11}, a_{21}a_{12}a_{23}a_{32}\}$ and $Q_2^- = \{a_{21}a_{12}a_{11}, a_{13}a_{31}a_{32}a_{23}\}$ and the superpotential for this example is

$$W = a_{31}a_{13}a_{11} + a_{21}a_{12}a_{23}a_{32} - a_{21}a_{12}a_{11} - a_{13}a_{31}a_{32}a_{23} + [\mathbb{C}Q, \mathbb{C}Q]$$

The arrows are indexed according to their head and tail: $h(a_{ij}) = i$ and $t(a_{ij}) = j$.

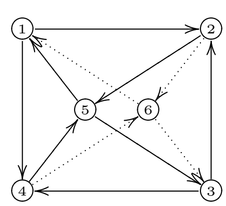
Example 9.2. An example of a quiver polyhedron that does not give a cancellation algebra is



We identify all vertices labeled ① to obtain an unweighted quiver polyhedron on a torus. The grading is done by giving all arrows degree 1. This is not a cancellation algebra because one can check that

$$xy \neq yx \text{ but } xy\ell = xa_1db_1y = ya_2db_2x = yx\ell.$$

Example 9.3. The same quiver polyhedron can be weighted differently to obtain different Jacobi algebras.



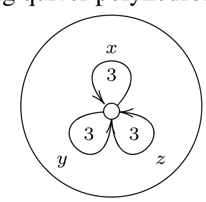
Use the same indexing convention as in the first example and set

$$Q_2^+ = \{a_{15}a_{52}a_{21}, a_{43}a_{35}a_{54}, a_{41}a_{16}a_{64}, a_{23}a_{36}a_{62}\}$$
$$Q_2^- = \{a_{16}a_{62}a_{21}, a_{43}a_{36}a_{64}, a_{41}a_{15}a_{54}, a_{23}a_{35}a_{52}\}.$$

This polyhedron is an octahedron and has the topology of a sphere.

- It can be given a trivial weighting if we give the arrows degree 1. In this case you get a cancellation algebra (there is at most one path between every pair of vertices of a given degree) but it is not CY-3 because χ_Q is 2.
- It can be equipped with a nontrivial weighting by giving cycles containing a_{41} or a_{23} weight 2 and the rest weight 1. The grading gives a_{21} or a_{43} degree 4 and the rest degree 1. This weighting gives rise to an orbifold with Euler characteristic 0. One can check using results from [4] (i.e. intersecting zigzag paths) that this does not give you a cancellation algebra.
- We can equip this with a nontrivial weighting by giving all cycles weight 2 and all arrows degree 1. This weighting gives rise to an orbifold with Euler characteristic -2. The nonintersection of zigzag paths tells us that this is a cancellation algebra and a CY-3 algebra.

Example 9.4. Weighted quiver polyhedra can also be used to describe certain Artin-Schelter regular algebras. Take the following quiver polyhedron on the sphere



where the backside is a triangle bounded by x, y and z. Then $A_Q = \mathbb{C}\langle x, y, z\rangle/\langle x^2 - yz, y^2 - zx, z^2 - xy\rangle$ which is a well-known three-dimensional Artin Schelter regular ring [1]. The center of this ring is isomorphic to $\mathbb{C}[u^3, v^3, w^3, uvw]$, which is the quotient singularity of the $\mathsf{G} = \mathbb{Z}_3 \times \mathbb{Z}_3$-action $(i, j) \cdot u = \eta^i u$, $gv = \eta^{i+j}v$, $gw = \eta^{i+2j}w$ with η the third root of unity. It is a cancellation algebra and CY-3, but take care, this algebra A_Q cannot be seen as a noncommutative crepant resolution over its center because a commutative crepant resolution has a rank 9 K-group, while the K-group of A_Q is only rank 1.

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