A SLICE THEOREM FOR QUIVERS WITH AN INVOLUTION

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Abstract. We construct an analogon to the notion of a local quiver setting in the case of quivers with an involution, or supermixed quivers as introduced by Zubkov in [?]. We use this technique to determine dimension vectors of simple supermixed representations.

1. Introduction

Given a reductive algebraic group G and a G-representation or a G-variety V we can construct the algebraic quotient $V/\!\!/G$, which is the affine variety corresponding to the ring of invariant polynomial functions $\mathbb{C}[V]^G$. The embedding $\mathbb{C}[V]^G \subset \mathbb{C}[V]$ gives rise to a quotient map $V \to V/\!\!/G$.

The main problem in invariant theory is to describe the geometry of such a quotients. There are several questions that one can try to answer: What is the dimension of $V/\!\!/G$, How do the fibers of $V/\!\!/G$ look like, is $V/\!\!/G$ a smooth variety.

In complete generality a solution for these problems is unattainable but given restrictions on the groups or the representations one can expect some interesting partial results. In many cases one can find a certain class of couples (V, G) sharing the same properties for the quotients. More precisely one can try to find class that are closed under local behaviour. By this we mean that if we have a couple (V, G) and a point $p \in V/\!\!/G$ that we can find another couple (V_p, G_p) of the same class such that there is an étale neighborhood of p that is locally isomorphic to an étale neighborhood of the zero in $V_p/\!\!/G_p$. Such a result simplifies the questions a lot because we can use this local result to reduce the questions to simpeler representations or varieties.

For (G, V) a representation space of a quiver this was done by Procesi and Le Bruyn in [?]. Similar results have been obtained fro representation spaces of prepojective algebras by Crawley-Boevey.

In this paper we will study the case of supermixed quivers these were introduced by Zubkov in [?] and are closely related to generalized quivers which were studied by Derksen en Weyman in [?]. First we will give some coordinate free description of a representation space of a supermixed quiver by means of involutions on semisimple algebras. Then we will extend the results by Derksen en Weyman to obtain a representation theoretic interpretation of the points in the representation spaces and in the quotient. This will enable us to formulate an extension of the result on local quivers by Procesi and Le Bruyn to super mixed quivers. To make full use of this result we will also determine which supermixed settings have simple representations. Finally we will give an application of this result for formally smooth algebras with an involution.

2. A quick review of quiver settings

We briefly recall that a quiver Q consists of a set of vertices Q_0, a set of arrows Q_1 and two maps $h, t : Q_1 \to Q_0$ which assign to each arrows its source and its tail. To a quiver we also associate its Euler form. This is the bilinear form $\chi_Q : \mathbb{Z}^{\#V} \times \mathbb{Z}^{\#V} \to \mathbb{Z}$ defined by the matrix

$$m_{ij} = \delta_{ij} - \#\{a | \textcircled{i} \xleftarrow{a} \textcircled{j}\},$$

where δ is the Kronecker delta. It is easy to see that that a quiver is uniquely defined by its Euler form.

A dimension vector of a quiver is a map $\alpha : Q_0 \to \mathbb{N}$. We will call a couple (Q, α) a quiver setting.

To every quiver setting we can associate a semisimple algebra

$$S_\alpha = \bigoplus_{v \in Q_0} \mathsf{Mat}_{\alpha_v \times \alpha_v}(\mathbb{C})$$

and an S_α-bimodule

$$\mathsf{Rep}(Q, \alpha) = \bigoplus_{a \in Q_1} \mathsf{Mat}_{\alpha_{h(a)} \times \alpha_{t(a)}}(\mathbb{C}).$$

Vice versa if S is a semisimple algebra and M an S-bimodule we can find a quiver setting (Q,α) unique up to isomorphism such that $(S,M)\cong(S_\alpha,\mathsf{Rep}(Q,\alpha))$. The vertices in Q correspond to the maximal set of orthogonal idempotents $\{v_1,\dots v_k\}$ in the center of S. The dimension vector is $\alpha_{v_i}=\sqrt{\dim v_i S}$ while the number of arrows from v_j to v_i is $\dim v_i M v_j/\alpha_{v_i}\alpha_{v_j}$.

We define the group GL_α as the group of invertible elements in S_α. This group has an action on $\mathsf{Rep}(Q,\alpha)$ by conjugation: $m\mapsto gmg^-1$. This action has a categorical quotient:

$$\mathsf{iss}(Q,\alpha)=\mathsf{Rep}(Q,\alpha)/\!\!/\mathsf{GL}_\alpha.$$

In algebraic terms we construct the quotient as follows: let $\mathbb{C}[\mathsf{Rep}(Q,\alpha)]$ be the ring of polynomial functions over $\mathsf{Rep}(Q,\alpha)$. On this ring we have an action of GL_α coming from conjugation action on $\mathsf{Rep}(Q,\alpha)$. The subring of functions that are invariant under this action is the ring of polynomial function over the categorical quotient:

$$\mathbb{C}[\mathsf{iss}(Q,\alpha)]=\mathbb{C}[\mathsf{Rep}(Q,\alpha)]^{\mathsf{GL}}_{\alpha}.$$

A path of length k in a quiver is a sequence of arrows $p=a_1\dots a_k$ with $t(a_i)=h(a_{i+1}$. We denote its head and tail as $h(p)=h(a_1)$ and $t(p)=t(a_k)$. A path the tail of which equals its head is called a cycle. A vertex is also called a path of length zero. The path algebra $\mathbb{C}Q$ is the vector space with as basis all paths and its multiplication is the concatenation of paths if possible and zero if not. The path algebra is Morita equivalent to the tensor algebra $T_S M=\oplus_i M^{\otimes_S i}$.

Any point $W\in M=\mathsf{Rep}(Q,\alpha)$ can be identified with a representation of $\mathbb{C}Q$ on the vector space $\oplus_v\mathbb{C}^\alpha_v$. Every arrow a will act as W_a between $\mathbb{C}^{\alpha_{t(a)}}$ and $\mathbb{C}^{\alpha_{h(a)}}$ and as zero between the rest. Therefore the points of $\mathsf{Rep}(Q,\alpha)$ will be called representations of Q with dimension vector α. In this way we can speak of simple, semisimple and indecomposable representations of Q. Two points W and W' will give isomorphic representations if and only if they are in the same GL_α-orbit.

If W is a representation we can evaluate the path p: $W_p=W_{a_1}\dots W_{a_k}$. To every cycle c we can associate the map $f_c:\mathsf{Rep}(Q,\alpha)\to\mathbb{C}:W\mapsto\mathsf{Tr}W_c$. This map is invariant under the GL_α-action and in general every invariant maps can be written in terms of these:

Theorem 2.1 (Le Bruyn-Procesi)**.** The ring $\mathbb{C}[\mathsf{iss}(Q,\alpha)]$ is generated by functions of the form f_c where c is a cycle in Q.

The representation theoretical interpretation of the quotient can be summarized as follows.

Theorem 2.2. The points in $\mathsf{iss}(Q,\alpha)$ are in one to one correspondence to isomorphism classes of the semisimple representations of Q in $\mathsf{Rep}(Q,\alpha)$. Two points in $\mathsf{Rep}(Q,\alpha)$ are mapped to the same point in $\mathsf{iss}(Q,\alpha)$ if and only if they have the same semisimplification.

3. Dualizing structures and supermixed settings

Let S be a finite dimensional semisimple algebra and let M be an S-bimodule.

Definition 3.1. A *dualizing structure* on (S,M) consists of two linear involutions $*: S \to S$ and $*: M \to M$, satisfying the following compatibility relations:

- $\forall a,b \in S : (ab)^* = b^*a^*$.
- $\forall a,b \in S : \forall m \in M : (amb)^* = b^*m^*a^*$.

The *dualizing group* of S is the group of elements for which the inverse and the involution coincide:

$$D(S) = \{g\ inS : gg^* = 1\}$$

while the *dualizing subspace* of M is the subspace

$$D(M) = \{v \in M : v^* = v\}$$

The group $D(S)$ has an action on $D(M)$ by conjugation: $v \mapsto gvg^*$ because $(gvg^*)^* = gvg^*$.

We can turn this data into the language of quivers. First we are going to impose the dualizing structure on a quiver setting. An involution on S_α restricts to an involution of the center which is a ring-automorphism. This maps idempotents to idempotents so we get an involution ϕ on the set of vertices $\{v_i\}$.

Given ϕ we can construct a standard involution on S_α:

$$s \mapsto s^\dagger \text{ with } (s^\dagger)_v = s_\phi(v)^T$$

where the transpose is taken according to the identification $S_\alpha = \oplus_{v\in Q_0}\mathsf{Mat}_{\alpha_v\times\alpha_v}(\mathbb{C})$. The composition of $*$ and $\dagger$ gives us an automorphism of S_α which is internal because all idempotents are fixed. Therefore we can say that there exists a $g \in S_\alpha$ such that

$$(s^*)_v = g_{\phi(v)} s^T_{\phi(v)} g^{-1}_{\phi(v)}$$

The fact that $*^2 = \mathsf{id}$ implies that

$$((s^*)^*)_v = g_v g^{-T}_{\phi(v)} s_v g^T_{\phi(v)} g^{-1}_{\phi(v)} = s_v$$

so $g_v = \epsilon_v g^T_{\phi(v)}$ for some scalars ϵ_v and $\epsilon_v\epsilon_{\phi(v)} = 1$. Because we only use the g_v for conjugation they are determined up to a scalar themselves. This allows us to chose $\epsilon_v = 1$ if $\phi(v) \neq v$ and $\epsilon_v = \pm 1$ if $\phi(v) = v$.

A base change in S_α by conjugation with h will transform the g_v as

$$(g')_v = h_v g_v h^T_{\phi(v)}.$$

Hence after a good base transformation we can suppose that g_v is the identity matrix if $\epsilon_v = 1$ and the standard symplectic antisymmetric matrix if $\epsilon_v = -1$.

The dualizing group of S_α will be of the following form

$$D_\alpha := D(S_\alpha) = \{g \in \mathsf{GL}_\alpha | gg^* = 1\} \cong \prod_{\{v,\phi(v)\}\subset Q_0} \begin{cases} O_{\alpha_v v} & v = \phi(v), \epsilon_v = 1 \\ Sp_{\alpha_v v} & v = \phi(v), \epsilon_v = -1 \\ GL_{\alpha_v} & v \neq \phi(v). \end{cases}$$

Now we will tackle the dualizing structures on $M = \mathsf{Rep}(Q,\alpha)$. We suppose that the involution on S_α is in its standard form and that ϕ is the corresponding involution on the vertices Q_0. Choose a second involution $\phi : Q_1 \to Q_1$ such that $h(\phi(a)) = \phi(t(a))$. Given this we can construct a 'standard' dualizing structure on $\mathsf{Rep}(Q,\alpha)$.

$$m \mapsto m^\dagger \text{ with } (m^\dagger)_a = f_a g^T_{t(a)} m^T_{\phi(a)} g_{h(a)}.$$

with $f_a f_{a^*} = \epsilon_{h(a)}\epsilon_{t(a)}$.

As was the case for S_α the composition $*\dagger$ is an automorphism of $\mathsf{Rep}(Q,\alpha)$ as an S_α-module. Therefore there are coefficients $\sigma_{ab},\ a,b \in Q_1$ such that $(m^{*\dagger})_a = \sum_{a\in Q_1}\sigma_{ab}m_b$. Also note that σ_{ab} is only nonzero if $h(a) = h(b)$ and $t(a) = t(b)$.

We can diagonalize σ and its eigenvalues σ_a can only be ± 1 because $*$ is an involution. This gives the following formula for $*$:

$$(m^*)_a = \sigma_a \epsilon_{h(a)} \epsilon_{t(a)} g_{t(a)} m^T_{\phi(a)} g_{h(a)}$$

If $a \neq \phi(a)$ we can again suppose that $\sigma_a = \epsilon_{h(a)} \epsilon_{t(a)} \sigma_{\phi(a)}$.

All this rewriting can be summarized in the terminology of supermixed quivers as introduced by Zubkov in [**?**].

Definition 3.2. A supermixed quiver consists of a quiver Q, two involutions $\dagger$ (one on the vertices and one on the arrows) and two sign maps $\epsilon : Q_0 \to \{\pm 1\}$ and $\sigma : Q_1 \to \{\pm 1\}$. which satisfy:

- $h(a^\dagger) = t(a)^\dagger$,
- $v^\dagger \neq v \Rightarrow \epsilon(v) = 1$.
- $\sigma_a \sigma_{\phi(a)} = \epsilon_{h(a)} \epsilon_{t(a)}$.

We say that (Q, α) is a supermixed setting if $\alpha_v = \alpha_{v^\dagger}$. Every supermixed quiver setting defines involutions $*$ on the algebra $\S_\alpha$ and on the space $\mathsf{Rep}(Q, \alpha)$

$$(s^*)_v = g_{v^\dagger} s^T_{v^\dagger} g_{v^\dagger}$$

$$(m^*)_a = \sigma_a g_{t(a)} m^T_{a^\dagger} g_{h(a)}$$

where $g_v = \mathsf{id}_{\alpha_v}$ if $\epsilon_v = 1$ and $g_v = \Lambda_{\alpha_v} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)^{\oplus \alpha_v/2}$ if $\epsilon_v = -1$. The couple $(S_\alpha, \mathsf{Rep}(Q, \alpha))$ is called the dualizing structure coming from the supermixed quiver setting.

Theorem 3.1. For every couple (S, M) with a dualizing structure is isomorphic to a dualizing structure coming from a supermixed quiver setting.

We will now try to interpret the space $\mathsf{DRep}(Q, \alpha)$ representation-theoretically. The discussion below closely mathes the discussion of symmetric quivers in [**?**].

First of all we will identify ϵ with the element $\sum_v \epsilon_v v \in \mathbb{C}Q$.

We now put an anti-automorphism $*$ on the path algebra

$$* : \mathbb{C}Q \to \mathbb{C}Q : \begin{cases} v \to v^\dagger \\ a \to \sigma_a a^\dagger \end{cases} \quad \text{and } \forall x, y \in \mathbb{C}Q : (xy)^* = y^* x^*.$$

This map is not an involution but it satisfies the following equality

$$\epsilon a^{**}\epsilon = a$$

This anti-automorphism allows us to turn left $\mathbb{C}Q$-modules into right $\mathbb{C}Q$-module. Therefore, given a left module V we can consider its complex dual again V^* again as a left module, the same can be done for morphisms: if $f : V \to W$ is a module morphism then f^* will be a morphism of $f : V^* \to W^*$. So taken the dual can be seen as an anti-equivalence in the category of left $\mathbb{C}Q$-modules. Because $eps^* = \epsilon = \epsilon^{-1}$ we also have that $V^{**} \cong V$.

A ϵ-mixed module is a left $\mathbb{C}Q$-module V together with a nondegenerate bilinear map $\langle,\rangle : V \times V \to \mathbb{C}$ satisfying

- $\forall v, w \in V : \langle v, w\rangle = \langle w, \epsilon v\rangle$ (ϵ-commuting),
- $\forall v, w \in V : \forall x \in \mathbb{C}Q : \langle xv, w\rangle = \langle v, x^* w\rangle$ ($*$-compatible).

In fact ϵ-mixed module are module that are isomorphic to their dual by the standard isomorphism induced by the bilinear form. If the $*$-compatiblity only holds for $\mathbb{C}Q_0$ instead of the whole $\mathbb{C}Q$ we will speak of almost ϵ-mixed modules.

For every (almost) ϵ-mixed $\mathbb{C}Q$-module we can define the dimension vector as $v \mapsto \dim vV$. Now we can choose bases (b_i^v) in every vV such that

- (b_i^v) is an orthogonal basis if $v^* = v$ and $\epsilon_v = 1$.
- (b_i^v) is a symplective basis if $v^* = v$ and $\epsilon_v = -1$.
- (b_i^v) and $(b_i^{v^*})$ are dual bases if $v^* \neq v$.

Such a basis is called a standard basis for V.

According to these bases we can express every a as a matrix V_a. If V is ϵ-mixed then (V_a) is in fact an element of $\mathsf{DRep}(Q, \alpha)$. Vice versa if $W \in \mathsf{DRep}(Q, \alpha)$ we can build an ϵ-mixed module out of it in the usual way.

Furthermore two (almost) ϵ-mixed modules are called isomorphic if there is a module morphism between them that preserves the bilinear form. It is also easy to check that $V, W \in \mathsf{DRep}(Q, \alpha)$ correspond to isomorphic modules if and only if they are in the same orbit under D_α. We can conclude:

Theorem 3.2. The D_α-orbits in $\mathsf{DRep}(Q,\alpha)$ classify the ϵ-mixed modules with dimension vector α up to *eps*-isomorphism.

A second theorem is less trivial, but its proof can easily be adapted from [?][Thm 2.6].

Theorem 3.3. Two ϵ-mixed modules are isomorphic as ϵ-mixed modules if and only if they are isomorphic as $\mathbb{C}Q$-modules.

$$\forall m \in \mathsf{DRep}(Q,\alpha) : \mathsf{DGL}_\alpha m = \mathsf{GL}_\alpha m \cap \mathsf{DRep}(Q,\alpha)$$

The ϵ-mixed structure is not only compatible with isomorphisms but it is also compatible with degenerations:

Theorem 3.4. If V is an ϵ-mixed module then its semisimplification as a $\mathbb{C}Q$-module is also an ϵ-mixed module.

Proof. We prove this by induction on the length of the composition serie of V. If V is simple then the statement is trivially true. Now suppose $S \subset V$ is a simple submodule then we also have a projection $V = V^* \to S^*$.

The kernel of this projection is $S^\perp = \{v \in V | \forall w \in S : \langle v, w\rangle = 0\}$. There are two possibilities: $S \cap S^\perp = 0$ or $S \subset S^\perp$. In the first case $\langle,\rangle_S$ and $\langle,\rangle_{S^\perp}$ are nondegenerate and $V = S \oplus S^\perp$. By the induction hypothesis $S^{\perp ss}$ admits an ϵ-mixed structure and hence $V^{ss} = S \oplus S^{\perp ss}$ as well.

In the second case $\langle,\rangle_{S^\perp}$ is degenerate but it becomes nondegenerate if we quotient out $(S^\perp)^\perp = S$. By the induction hypothesis $(S^\perp/S)^{ss}$ admits an ϵ-mixed structure we can use this structure to put an ϵ-mixed structure on $V^{ss} = S \oplus (S^\perp/S)^{ss} \oplus S^*$:

$$\langle (s_1,t_1,s_1^*),(s_2,t_2,s_2^*)\rangle = s_2^*(s_1) + s_1^*(s_2) + \langle t_1,t_2\rangle_{(S^\perp/S)^s s}$$

It is easy to see that the action of $\mathbb{C}Q$ is compatible with this form.

□

This implies that the closed DGL_α-orbits in $\mathsf{Diss}_(Q,\alpha)$ are exactly the intersections of closed GL_α-orbits with $\mathsf{Diss}_(Q,\alpha)$.

Apart from the orbits themselves we are also interested in the quotient space of the orbits.

$$\mathsf{Diss}_{(}Q,\alpha) = \mathsf{DRep}_{(}Q,\alpha)/\!\!/DS_\alpha$$

The points of this variety are in one to one correspondence with the closed DGL_α-orbits in $\mathsf{DRep}_{(}Q,\alpha)$. As all the closed orbits come from closed GL_α-orbits this means that we have an embedding $\mathsf{Diss}_{(}Q,\alpha) \subset \mathsf{iss}_{(}Q,\alpha)$ which sit inside a commutative diagram

$$\begin{array}{ccc} \mathsf{DRep}(Q,\alpha) & \hookrightarrow & \mathsf{Rep}(Q,\alpha) \\ \downarrow & & \downarrow \\ \mathsf{Diss}(Q,\alpha) & \hookrightarrow & \mathsf{iss}(Q,\alpha) \end{array}$$

This result can be restated in terms of invariants and as such it can also be found in [**?**]

Theorem 3.5. The DGL_α-invariant functions on $\mathsf{DRep}(Q,\alpha)$ come from GL_α-invariant function on $\mathsf{Rep}(Q,\alpha)$.

Representationtheoretically we can say that $\mathsf{Diss}(Q,\alpha)$ classifies the isomorphism classes of semisimple ϵ-mixed modules.

An ϵ-submodule is a submodule $W \subset V$ such that $\langle,\rangle|_W$ is non degenerate, an ϵ-mixed module is ϵ-irreducible if it has no nontrivial supermixed submodules. It is clear that it is possible to define the direct sum of two ϵ-mixed modules. If $W \subset V$ is an ϵ-submodule, we can define the submodule $W^\perp := \{v \in V | \forall w \in W : \langle v, w\rangle = 0\}$. The nondegeneracy of W makes that $W \cap W^\perp = 0$ and the compatibility of $\langle,\rangle$ with the involution makes that $W^\perp$ is also an ϵ-mixed representation, so $V = W \oplus W^\perp$.

It is not true that an ϵ-irreducible module is always semisimple as a $\mathbb{C}Q$-module, so its orbit might not be closed in $\mathsf{DRep}(Q,\alpha)$. Therefore it is more interesting to restrict our attention to ϵ-irreducible that are also semisimple. Such modules will be called ϵ-simple and every semisimple ϵ-mixed module is a direct sum of ϵ-simple modules.

Theorem 3.6. Suppose that V is *eps*-mixed simple. There are 3 possibilities:

(1) V is an ordinary simple module,

(2) $V \cong W \oplus W^*$ where W is an ordinary simple module and W and W^* are isomorphic modules. Furthermore W can be given the structure of an $-\epsilon$-mixed module.
(3) $V \cong W \oplus W^*$ where W is an ordinary simple module and W and W^* are non-isomorphic modules.

Proof. If we are not in the first case, there exists a proper simple submodule $W \subset V$. This module is perpendicular to itself because if $\exists w, w' \in W : \langle w, w' \rangle \neq 0$ then $\langle , \rangle_W$ would be nondegenerate. This follows from the fact all nonzero vectors in a simple module are cyclic: if $x \in W$ we can find $a \in \mathbb{C}Q$ such that $w = ax$ and hence $\langle x, a^* w' \rangle = \langle ax, w' >\neq 0$.

Because V is semisimple we know that $V \cong W \oplus W^\perp/W \oplus W^*$. But the bilinear form on $W^\perp/W$ is nondegenerate so $W^\perp/W$ is an ϵ-submodule of V and hence 0.

Now suppose that $W \cong W^*$. Every element in V can then be written as a couple (v, v') with $v, v' \in W$.

We know that $W^\perp = W$ and $W^{*\perp} = W^*$ therefore there are two bilinear forms on W such that

$$\langle (v, v'), (w, w') \rangle_V = \langle v, w' \rangle_1 + \langle v', w \rangle_2$$

The ϵ-commutativity of $\langle , \rangle_V$ implies that

$$\langle v', w \rangle_2 = \langle \epsilon w, v' \rangle_1$$

We denote the adjoint according to $\langle , \rangle_1$ by $-^\#$. If $a \in \mathbb{C}Q$ then the adjoint of $\rho_V(a) = \rho_W(a) \oplus \rho_W(a)$ according to $\langle , \rangle_V$ is

$$\epsilon \rho_W(a)^\# \epsilon \oplus \rho_W(a)^\#$$

Taking twice the V-adjoint shows us that $\epsilon \rho_W(a)^{\#\#} \epsilon = \rho_W(a)$.

It is not neccesarily so that $\langle , \rangle_1$ is ϵ-commuting, so define ϕ by the equation $\langle v, w \rangle = \langle \phi w, v \rangle$. This map has the property that $\phi\phi^\# = 1$. Now taking twice the adjoint here tell us that $\phi \rho_W(a)^{\#\#} \phi^- 1 = \rho_W(a) = \epsilon \rho_W(a)^{\#\#} \epsilon$. This holds for all a and as ρ_W is simple we can conclude that ϕ must be $\pm\epsilon$ so $\langle , \rangle_1$ is either ϵ-commuting or $-\epsilon$-commuting.

The first case is impossible because the diagonal module $\nabla = \{(v,v)|v \in V\}$ is nondegenerate:

$$\langle (v,v),(w,w)\rangle_V = \langle v,w\rangle_1 + \langle \epsilon w, v\rangle_1 = \langle v,w\rangle + \langle \epsilon w, \epsilon v\rangle_1 = 2\langle v,w\rangle.$$

□

We will call these types of ϵ-simple modules (a) orthogonal, (b) symplectic and (c) general, because the ϵ-automorphism groups of such modules are (a) O_1, (b) Sp_2, (c) GL_1.

The theorem implies that we can decompose a semisimple ϵ-mixed representation as follows

$$V = \bigoplus_{1\leq \ell \leq k_1} S_1^{e_i} \bigoplus_{k_1 < \ell \leq k_2} (T_j \oplus T_j^*)^{e_j} \bigoplus_{k_2 < \ell \leq k_3} (U_\ell \oplus U_\ell^*)^{e_\ell}$$

Where the S_i are orthogonal, the $(T_i \oplus T_i^*)$ symplectic and the $(U_i \oplus U_i^*)$ are general.

4. Local mixed quivers

Another tool we want to adapt to the ϵ-mixed quiver case is the Luna slice theorem and the construction of local quivers.

First of all let us recall the Luna slice theorem [**?**]. We will restrict to its use for group actions on a vector space. Let V be a vector space with a linear action from an algebraic group G. If $v \in V$ has an orbit Gv which is closed in V, we can approximate the quotient $V/\!\!/G$ in an étale neighborhood of V as follows. Construct the normal space which is the quotient of V by the tangent space to the orbit

$$N_v = V/T_vGv.$$

On this space there is an action of the stabilizer of v: G_v because G_v acts on both V and T_vGv.

Theorem 4.1 (Luna Slice)**.** There exists an étale neighborhoods U_v of $v \in V$ and U_0 of $0 \in N_v$ such that we have the following commutative diagram

$$\begin{array}{ccccc} U_v & \longrightarrow & U_v/\!\!/G & \hookrightarrow & V/\!\!/G \\ \downarrow & & \downarrow & & \\ U_0 \times_{G_v} G & \longrightarrow & U_0/\!\!/G_v & \hookrightarrow & N_v/\!\!/G_v. \end{array}$$

Which means that the quotient of V by G around v is locally isomorphic to the quotient of N_v by G_v around the zero.

Now suppose W is a semisimple representation of Q in $\mathsf{Rep}(Q,\alpha)$. We can write out the decomposition of W as a direct sum of simples:

$$W \cong S_1^{\oplus e_1} \oplus \cdots \oplus S_k^{\oplus e_k}$$

The stabilizer of W in GL_α is isomorphic to $\mathsf{GL}_{e_1} \times \cdots \times \mathsf{GL}_{e_k}$. Putting these things together we get the local quiver theorem:

Theorem 4.2 (Le Bruyn-Procesi)**.** For a point $p \in \mathsf{iss}_\alpha Q$ corresponding to a semisimple representation $W = S_1^{\oplus e_1} \oplus \cdots \oplus S_k^{\oplus e_k}$, there is a quiver setting (Q_p, α_p) called the *local quiver setting* such that we have an étale isomophism between an open neighborhood of the zero representation in $\mathsf{iss}_{\alpha_p} Q_p$ and an open neighborhood of p.

Q_p has k vertices corresponding to the set $\{S_i\}$ of simple factors of W and between S_i and S_j the number of arrows equals

$$\delta_{ij} + \sum_{a \in Q_1} \alpha^i_{h(a)} \alpha^j_{t(a)} - \sum_{v \in Q_0} \alpha^i_v \alpha^j_v$$

where α_i is the dimension vector of the simple component S_i.

The dimension vector α_p is defined to be $(e_1, \dots, e_k)$, where the e_i are the multiplicities of the simple components in W.

We can extend this theorem to supermixed settings. First we note that if $W = W^*$ then the stabilizer of W is closed under the involution.

$$g \in \mathsf{Stab}_W \Rightarrow gWg^{-1} = W \Rightarrow g^{-1*}Wg^* = W \Rightarrow g^{-1*} \in \mathsf{Stab}_W \Rightarrow g^* \in \mathsf{Stab}_W.$$

The same holds for the tangent space to the orbit $T_W \mathsf{GL}_\alpha W$

$$\begin{aligned} T_W \mathsf{GL}_\alpha W^* &= \{sW - Ws | s \in S_\alpha\}^* \\ &= \{-s^*W^* + W^*s^* | s \in S_\alpha\} \\ &= \{-s^*W^* + W^*s^* | -s^* \in S_\alpha\} = T_W \mathsf{GL}_\alpha W. \end{aligned}$$

And hence we can transport the involution on $\mathsf{Rep}(Q,\alpha)$ to the normal space $\mathsf{Rep}(Q,\alpha)/T_W \mathsf{GL}_\alpha W$.

Theorem 4.3. For a point $p \in \mathsf{Diss}(Q,\alpha)$ corresponding to a semisimple representation

$$V = \bigoplus_{1\le \ell\le k_1} S_1^{e_i} \bigoplus_{k_1<\ell\le k_2} (T_\ell \oplus T_\ell^*)^{e_\ell} \bigoplus_{k_2<\ell\le k_3} (U_\ell \oplus U_\ell^*)^{e_\ell}$$

there is a quiver setting (Q_p,α_p) called the *local quiver setting* such that we have an étale isomophism between an open neighborhood of the zero representation in $\mathsf{Diss}(Q_p,\alpha_p)$ and an open neighborhood of p.

Q_p has $k_1 + k_2 + 2 * k_3$ vertices corresponding to the set $Q_{p0} = \{S_i, T_i, U_i, U_i^*\}$ of isomorphism classes of simple factors of W. The new mixing factor will be $\epsilon_p(S_i) = \epsilon(U_i) = \epsilon(U_i^*) = -\epsilon(T_i) = 1$. Between X and $Y \in Q_{p0}$ the number of arrows equals

$$\sum_{a\in Q_1} \alpha^X_{h(a)}\alpha^Y_{t(a)} - \sum_{v\in Q_0} \alpha^X_v \alpha^Y_v + \delta_{XY}$$

If $X = Y^*$ then we need to find the number of symmetric and antisymmetric arrows. This equals

$$\sum_{a=sa*} \alpha^X_{h(a)}) - \sum_{v=v^*,\epsilon_v=s} \alpha^X_v + \delta_{sX}$$

The s is $+1$ for symmetric arrows and -1 for antisymmetric arrows. By δ_{sX} we mean that this is 1 if $s = +1$ and X is orthogonal or $s = -1$ and X is symplectic and in all other cases it is zero.

The rest of the arrows are general, which is the same as saying that half of them are symmetric and the other half is antisymmetric.

Proof. Suppose $W = W^*$ is a semisimple representation of (Q,α) with decomposition

$$\begin{aligned} W &= \bigoplus_{1\le \ell\le k_1} S_\ell^{e_\ell} \bigoplus_{k_1<\ell\le k_2} (T_\ell \oplus T_\ell^*)^{e_\ell} \bigoplus_{k_2<\ell\le k_3} (U_\ell \oplus U_\ell^*)^{e_\ell} \\ &= \bigoplus_{1\le \ell\le k_1} \mathbb{C}^{e_\ell}\otimes S_\ell \bigoplus_{k_1<\ell\le k_2} \mathbb{C}^{e_\ell}(T_\ell \oplus T_\ell^*)\otimes \mathbb{C}^{e_\ell} \bigoplus_{k_2<\ell\le k_3} \mathbb{C}^{e_\ell}\otimes (U_\ell \oplus U_\ell^*). \end{aligned}$$

Denote the standard basis for each $\mathbb{C}^{e_i}$ by b_{ellmu} and chose a standard basis $c_{\ell v\nu}$ for every S_ℓ, $T_\ell \oplus T_\ell^*$ and $U_\ell \oplus U_\ell^*$. Note that the basis depends on two extra indices v and ν, the first one runs through the vertices of Q the second one runs for each vertex v from 1 to α^ℓ_v (if $\ell \le k_1$) and $\alpha^\ell_v + \alpha^\ell_{v^*}$ if $\ell > k_1$.

We can combine these to a basis for W: $b_{\ell\mu}\otimes c_{\ell v\nu}$. These basisses also define projections $\beta_{\ell\mu}\mathbb{C}^{e_\ell} \to_{\ell\mu}$ and $\gamma_{\ell v\nu} S_\ell \to_{\ell v\mu}$ (or with T or U). The $\beta_{\ell\mu}\otimes\gamma_{\ell v\nu}$ define a set of orthogonal idempotents in S_α and because we work with standard basisses there are involutions on the indices $\mu \to \mu^*$ and $\nu \to \nu^*$ such that

$$(\beta_{\ell\mu} \otimes \gamma_{\ell v \nu})^* = \beta_{\ell\mu^*} \otimes \gamma_{\ell v^* \nu^*}.$$

We will also need the projections $\mathsf{id}_\ell : T_\ell \oplus T_\ell^* \to T_\ell$ (same with U) and $\mathsf{id}_\ell^* : T_\ell \oplus T_\ell^* \to T_\ell^*$. For the S's these two projections coincide. In this way we can write the identity in S_α as

$$1 = \oplus_{\bar\ell,\mu} \beta_{\ell\mu} \otimes \mathsf{id}_{\bar\ell}$$

By the notation $\bar\ell$ we mean that for $\ell > k_1$ we sum twice over ℓ once with id_ℓ and once with id_ℓ^*.

Let us denote by S_{α_p} the subalgebra of S_α such that commutes with W:

$$S_{\alpha_p} = \{a \in S_\alpha | aW = Wa\}$$

It is easy to see that the units inside this algebra form the stabilizer group GL_{α_p}.

By Schur's lemma and the fact that $\mathbb{C}Q$ only acts on the right hand part of the basis $b_{\ell\mu} \otimes c_{\ell\nu}$, we can conclude that S_{α_p} looks like

$$\prod_{1 \le \ell \le k_1} \mathsf{Mat}_{e_\ell} \mathsf{id}_{S_\ell} \prod_{k_1 < \ell \le k_2} \mathsf{Mat}_{2e_\ell} \mathsf{id}_{T_\ell} \prod_{k_2 < \ell \le k_3} \mathsf{Mat}_{e_\ell} \mathsf{id}_{U_\ell} \times \mathsf{Mat}_{e_\ell} \mathsf{id}_{U_\ell^*}$$

The bilinear form on the other hand looks like $(v, w) \mapsto v^T g w$ with

$$g = \bigoplus_{1 \le \ell \le k_1} \mathsf{id}_{e_\ell} \otimes g_{S_\ell} \bigoplus_{k_1 < \ell \le k_2} \mathsf{id}_{e_\ell} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes g_{T_\ell} \bigoplus_{k_2 < \ell \le k_3} \mathsf{id}_{e_\ell} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes g_{U_\ell}$$

As the stabilizer only works on the left side of the tensor product we can easily deduce that

$$D_{\alpha_p} = \prod_{1 \le \ell \le k_1} O_{e_\ell} \mathsf{id}_{S_\ell} \prod_{k_1 < \ell \le k_2} \mathsf{Sp}_{2e_\ell} \mathsf{id}_{T_\ell} \prod_{k_2 < \ell \le k_3} \{g \mathsf{id}_{U_\ell} \times g^{-1} \mathsf{id}_{U_\ell^*} | g \in \mathsf{GL}_{e_\ell}\}.$$

This also implies that we can take ϵ_p to be $\epsilon_p(S_i) = \epsilon(U_i) = \epsilon(U_i^*) = -\epsilon(T_i) = 1$.

To calculate the arrows in the local quiver we first need a lemma that deals with restriction of dualizing structures

Lemma 4.4. Let (S, M) be any dualizing structure such that S is isomorphic to the S_α from above and denote the mixed quiver to which M corresponds Q_M. The structure of the restricted dualizing structure (S_{α_p}, M) corresponds to a new quiver

Q'_M with $k_1 + k_2 + 2 * k_3$ vertices and every arrow a in Q_M transforms for every pair of vertices $\bar{\ell}_1, \bar{\ell}_2$ into $\alpha_h^{\bar{\ell}_1}(a)\alpha_t^{\bar{\ell}_2}(a)$ arrows between $\bar{\ell}_2$ and $\bar{\ell}_1$. If a is a (anti)-symmetric arrow and $\bar{\ell}_2 = \bar{\ell}_1^*$ then $\alpha_h^{\bar{\ell}_1}(a)^2 - \alpha_h^{\bar{\ell}_1}(a)$ are of general type and $\alpha_h^{\bar{\ell}_1}(a)$ are

- the same type as a if neither or both $h(a)$ and ℓ_1 are of symplectic type.
- the opposite type of a if either $h(a)$ or ℓ_1 but not both are of symplectic type.

Proof. Let (S, M) be any dualizing structure such that S is isomorphic to the S_α from above. We are interested in the structure of the restricted dualizing structure (S_{α_p}, M). Let a be an arrow in M and denote its corresponding simple sub-bimodule by M_a. As and S_{α_p} bimodule M_a decompose as a direct sum of S_{α_p}-bimodules

$$M_a = 1M_a1^*$$

$$= \bigoplus_{\bar{\ell}_1,\bar{\ell}_2,\mu_1,\mu_2} \beta_{\ell_1 h(a)\mu_1} \otimes \mathsf{id}_{\bar{\ell}_1} M_a(\beta_{\ell_2 t(a)\nu_2} \otimes \mathsf{id}_{\bar{\ell}_2})^*$$

Note that we only need the β's for which the vertex $v = h(a), t(a)$ because the others act as zero on M_a. All these components are simple S_{α_p}-bimodules and hence represent arrows from the vertex $\bar{\ell}_2$ to $\bar{\ell}_1$. So in total there are

$$\sum_a \alpha_h^{\bar{\ell}_1}(a)\alpha_t^{\bar{\ell}_2}(a)$$

between $\bar{\ell}_2$ and $\bar{\ell}_1$ in Q'.

Under the involution the only components which are mapped onto themselves are the ones for which $a = a^*, \bar{\ell}_1 = \bar{\ell}_2$ and $\mu_1 = \mu_2$.

The action under the involution on $x \in \beta_{\ell_1 h(a)\mu_1} \otimes \mathsf{id}_{\bar{\ell}_1} M_a(\beta_{\ell_1 h(a)\nu_1} \otimes \mathsf{id}_{\bar{\ell}_1})^*$ is given by

$$\begin{aligned} x \mapsto & \sigma_a g^T(\beta_{\ell_1 h(a)\mu_1} \otimes \mathsf{id}_{\bar{\ell}_1} x(\beta_{\ell_1 h(a)\nu_1} \otimes \mathsf{id}_{\bar{\ell}_1})^*)^T g \\ &= \sigma_a \underbrace{(\beta_{\ell_1 h(a)\mu_1} \otimes \mathsf{id}_{\bar{\ell}_1}) g^T (\beta_{\ell_1 h(a)\mu_1} \otimes \mathsf{id}_{\bar{\ell}_1})^{*T}}_{\pm G^T} x^T (\beta_{\ell_1 h(a)\mu_1} \otimes \mathsf{id}_{\bar{\ell}_1})^T g (\beta_{\ell_1 h(a)\mu_1} \otimes \mathsf{id}_{\bar{\ell}_1})^*{}_G \\ &= \pm\sigma_a G^T x G \end{aligned}$$

Where G is either the identity matrix or the standard symplectic matrix depending on whether ℓ is a general, orthogonal or a symplectic vertex. The $\pm$ sign depends on the nature of the vertices in Q and Q': If $h(a)$ or ℓ is symplectic an extra minus sign is added, if both are symplectic these minus signs annihillate each other. $\square$

As S_{α_p}-bimodules with dualizing structure we have that $N_p = \mathsf{Rep}(Q,\alpha)/T_W\mathsf{GL}_\alpha W$. Also we can identify $T_W\mathsf{GL}_\alpha W$ with S_α/S_{α_p} but this identification only works as bimodules without the dualizing structure. In order to make it work with the dualizing structure we must put a new dualizing structure on S_α as an S_α-bimodule. This is done by putting $\star : S_\alpha \to S_\alpha : x \to -x^*$. This new structure turns the morphism

$$\pi : S_\alpha \to T_W\mathsf{GL}_\alpha W : x \mapsto xW - Wx$$

into a $*$-morphism: $\pi(x^\star) = x^\star W - Wx^\star = -x^*W + Wx^* = -x^*W^* + W^*x^* = \pi(x)^*$. In quiver terminology Q_{S_α} is a quiver with the same number of vertices as Q but with a unique loop in every vertex. The loops in selfdual vertices are antisymmetric.

So to determine the number of arrows in N_p from $\bar\ell_2$ to $\bar\ell_1$ we can use the following formula

$$\begin{aligned}\#\{\bar\ell_1\bar\ell_2 \text{ in } N_p\} &= \#\{\bar\ell_1\bar\ell_2 \text{ in } \mathsf{Rep}(Q,\alpha)\} - \#\{\bar\ell_1\bar\ell_2 \text{ in } S_\alpha\} + \#\{\bar\ell_1\bar\ell_2 \text{ in } S_{\alpha_p}\}\\ &= \sum_{a\in Q_1} \alpha_{h(a)}^{\bar\ell_1}\alpha_{t(a)}^{\bar\ell_2} - \sum_{v\in Q_0} \alpha_v^{\bar\ell_1}\alpha_v^{\bar\ell_2} + \delta_{\bar\ell_1\bar\ell_2}\end{aligned}$$

To determine the number of symmetric arrows from $\bar\ell_1^*$ to $\bar\ell_1$ we can do the same thing:

$$\begin{aligned}\#\{\bar\ell_1\bar\ell_2 \text{ in } N_p\} &= \#\{\bar\ell_1\bar\ell_2 \text{ in } \mathsf{Rep}(Q,\alpha)\} - \#\{\bar\ell_1\bar\ell_2 \text{ in } S_\alpha\} + \#\{\bar\ell_1\bar\ell_2 \text{ in } S_{\alpha_p}\}\\ &= \sum_{a=\sigma_a a^*\in Q_1, \sigma_a\epsilon_{h(a)}\epsilon_{\ell_1}=1} \alpha_{h(a)}^{\bar\ell_1} - \sum_{v\in Q_0,\epsilon_v\epsilon_{\ell_1}=-1} \alpha_v^{\bar\ell_1}.\end{aligned}$$

for the antisymmetric arrows we obtain

$$\sum_{a=\sigma_a a^*\in Q_1, \sigma_a\epsilon_{h(a)}\epsilon_{\ell_1}=-1} \alpha_{h(a)}^{\bar\ell_1} - \sum_{v\in Q_0,\epsilon_v\epsilon_{\ell_1}=1} \alpha_v^{\bar\ell_1} + 1$$

$\square$

5. Simples

In this section we are going to determine for which S there are *eps*-simples of orthogonal, symplectic and general type.

Lemma 5.1. If there exists a simple S of dimension vector α with $S^* \ncong S$ then there exists a general ϵ-simple of dimension vector $\alpha+\alpha^*$. If there exists an orthogonal ϵ-simple of dimension vector α then there exists a symplectic $-\epsilon$-simple of dimension vector 2α

Proof. If there exists simples in $\mathsf{Rep}(Q,\alpha)$, they form an open dense part. This implies that we can find a simple $W \in \mathsf{Rep}(Q,\alpha) \setminus \mathsf{DRep}(Q,\alpha)$. Now form the representation $W \oplus W^*$. If we can put a bilinear form on it that is compatible with the $\mathbb{C}Q$-action and the involution we are done. This bilinear form is

$$\langle x, y\rangle = \sum_{v \in Q_0} (x_{1v}y_{2v} + \epsilon_v x_{2v} y_{1v})$$

□

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