Chapter 1

A short review of Algebraic Geometry

1.1 Affine Varieties

In algebraic geometry one studies the connections between algebraic varieties, which are sets of solutions of polynomial equations, and complex algebras.

An affine variety is a subset $X \subset \mathbb{C}^n$ that is defined by a finite set of polynomial equations.

$$X := \{x \in \mathbb{C}^n | f_1(x) = 0, \dots, f_k(x) = 0\}$$

A morphism between two varieties $X \in \mathbb{C}^n$ an $Y \in \mathbb{C}^m$ is a map $\phi : X \to Y$ such that there exist a polynomial map $\Phi : \mathbb{C}^n \to \mathbb{C}^m$ such that $\phi = \Phi|_X$. Such a morphism is an isomorphism if ϕ is invertible and ϕ^{-1} is also a morphism.

We can consider $\mathbb{C}$ as a variety, so it makes sense to look at the morphisms from a variety X to the variety $\mathbb{C}$, these maps are also called the regular functions on X. They are closed under pointwise addition and multiplication so they form a commutative $\mathbb{C}$-algebra: $\mathbb{C}[X]$.

$$\mathbb{C}[X] := \{\phi : X \to \mathbb{C} | \rho \text{ is a morphism of varieties}\}$$

This algebra can be described with generators and relations. To every variety $X \in \mathbb{C}^n$ the set of polynomial functions that are zero on X form an ideal in $\mathbb{C}[x_1, \cdots, x_n]$. If we divide out this ideal we get the ring of polynomial functions on X.

$$\mathbb{C}[X] := \mathbb{C}[x_1, \dots, x_n]/(f | \forall x \in X : f(x) = 0)$$

This algebra is finitely generated by the x_i and it also has no nilpotent elements because $f(x)^n = 0 \Rightarrow f(x) = 0$.

A morphism between varieties, $\phi : X \to Y$, will also give an algebra morphism between the corresponding rings but the arrow will go in the opposite direction:

$$\phi^* : \mathbb{C}[Y] \to \mathbb{C}[X] : g \mapsto g \circ \phi.$$

On the other hand if R is a finitely generated commutative $\mathbb{C}$-algebra without nilpotent elements, by definition we will call this an *affine algebra*. Every affine algebra can be written as a quotient of a polynomial ring $\mathbb{C}[x_1, \dots, x_n]$ with an ideal $\mathfrak{i}$. Because polynomial rings are Noetherian, $\mathfrak{i}$ is finitely generated by f.i. $f_1, \dots, f_k$. Therefore we can associate to R the variety $V(R)$ in $\mathbb{C}^n$ defined by the f_i.

Although this variety depends on the choice of generators of R and $\mathfrak{i}$, there is a more intrinsic description of $V(R)$. Indeed the points of $V(R)$ are in one to one correspondence with the algebra morphisms from R to the algebra $\mathbb{C}$ (or equivalently the maximal ideals of R.)

$$V(R) := \{\rho : R \to \mathbb{C} | \rho \text{ is an algebra morphism}\}$$

A morphism between algebras, $\phi : R \to S$, will also give an algebra morphism between the corresponding rings but the arrow will go in the opposite direction:

$$\phi_* : V(S) \to V(R) : \rho \mapsto \rho \circ \phi.$$

The main theorem of algebraic geometry now states that the operations $V(-)$ and $\mathbb{C}[-]$ are each other's inverses:

Theorem 1.1. *The category of commutative affine algebras and the category of affine varieties are anti-equivalent. So working with affine varieties is actually the same as working with affine algebras but all maps are reversed. The anti-equivalence is given by the contravariant functors $V(-)$ and $\mathbb{C}[-]$, so*

$$\mathbb{C}[V(R)] \cong R \text{ and } V(\mathbb{C}[X]) \cong X$$

This theorem enables us to translate every geometrical statement about affine varieties into an algebraic about affine algebras and vice versa.

Memo 1.2. Affine algebraic geometry is affine commutative algebra with the arrows reversed.

1.2 Projective Varieties

Given a set of variables $x_1, \dots, x_n$ with degree 0, a set of variables $y_0, \dots, y_m$ of degree 1, and a set of homogeneous polynomials $f_1, \dots, f_k$. We define a subset of $\mathbb{C}^n \times \mathbb{P}^m$ where $\mathbb{P}^m$ is the projective m-space.

$$V := \{((x_1, \dots, x_n), (y_0 : \dots : y_m)) \,|f_i = 0\}$$

This is well defined because the polynomials are homogeneous. To this we can associate the ring

$$R = \mathbb{C}[x_1, \dots, x_n, y_0, \dots, y_m]/(f_1, \dots, f_k).$$

This ring is graded by giving the x_i degree 0 and the y_i degree 1.

$Y = \mathsf{Proj}\ R := \{$maximal graded ideals in R that do not contain the ideal $(y_0, \dots, y_n)\}$

The ring R contains a subring R_0 of all degree 0 elements (the ring generated by the x_i) and there is a map from Y to $X := V(R_0)$:

$$\pi : Y \mapsto V(R_0) : ((x_1, \dots, x_n), (y_0 : \dots : y_m)) \to (x_1, \dots, x_n).$$

We call Y a variety that is projective over X. If $R_0 \cong \mathbb{C}$ then $V(R_0)$ is a point and we call Y projective. Such a variety is the union of affine varieties:

$$Y_i := \{((x_1, \dots, x_n), (y_0 : \dots : y_m)) \in Y|x_i = 1\} = V(R/(y_i - 1)).$$

A morphism between two such varieties Y and Y' is a map $\phi : Y \to Y'$ such that $\phi_{ij} : \phi^{-1}(Y'_j) \cap Y_j \to Y'_j : x \mapsto \phi(x)$ is a morphism between affine varieties for all i, j. Such a morphism is an isomorphism if ϕ is invertible and ϕ^{-1} is also a morphism.

An example of an isomorphism of two projective varieties is the following. Let $X = \mathbb{P}_1$ and $Y := \{(x : y : z)|xy - z^2 = 0\}$ and define

$$\phi : X \to Y : (x : y) \mapsto (x^2 : y^2 : xy).$$

However unlike in the affine case there are non-isomorphic rings that give isomorphic projective varieties: in the example above $\mathsf{Proj}\ \mathbb{C}[x, y] = X$ and $\mathsf{Proj}\ \mathbb{C}[x, y, z]/(xy - z^2) = Y$.

The advantage of using projective varieties is that as topological spaces they are compact, while the only affine varieties that are compact are points. Using these varieties makes some geometry usual simpler (affine, there is a differents between the hyperbola and parabola but there projective versions are isomorphic).

Apart from projective and affine varieties there are lots of other varieties, but the general definition of a variety is quite complicated, so we will not introduce it. Intuitively a variety is a topological set which is the union of overlapping affine varieties.

1.3 Smoothness

If $V \subset \mathbb{C}^n$ is an affine variety defined by $f_1, \ldots, f_k$ then we define the tangent space in $p \in V$ as the vector space

$$T_pV := \{(y_1, \ldots, y_n) | \left(\frac{\partial f_i}{\partial x_j}\right)_p y_j = 0\}$$

Again this definition depends on the choice of the $f's$ but one can show that a different choice of generators gives an isomorphic vector space.

The dimension of a variety V is the minimal dimension of all its tangents spaces.

$$\dim V := \min_{p \in V} \dim T_pV$$

A point p is called a smooth point if $\dim V = \dim T_pV$ and otherwise it is called singular. A vbariety is called smooth if all its points are smooth and singular otherwise. The subset of all singular points of a variety is called the singular locus.

Smooth varieties are in many ways the nicest and easiest varieties and singular varieties are a lot more difficult to study. Therefore one wants to find a method to turn a singular variety V into a closely related smooth variety $\tilde{V}$. This process is called resolving the singularities. More precisely one wants to construct a surjective morphism $\tilde{V} \to V$ that is almost everywhere bijective (i.e. on an open dense part) such that all the fibers are compact and $\tilde{V}$ is smooth.

Chapter 2

Resolving singularities

2.1 Resolving singular curves

The first example we do is a singular elliptic curve. Let R be the ring $\mathbb{C}[X,Y]/(Y^2 - X^3 + X^2)$. This curve has a singular point in $(0,0)$.

The problem seems that there are two tangent directions in $(0,0)$, and the curve intersects itself. To get rid of this singularity one should split to point $(0,0)$ in two. This can be done by adding the function Y/X (the slope of the tangent in 0) to the ring

$$\tilde{R} := \mathbb{C}[X,Y,\frac{Y}{X}]/(Y^2 - X^3 - X^2) = \mathbb{C}[X,Y,Z]/(Y^2 - X^3 + X^2, ZX - Y)$$

What is special about the element Z? it is an element of the quotient field $Q(R) := \{\frac{a}{b}|a,b \in \mathbb{R}\}$ and it is integral over R i.e. it satisfies a monic polynomial with coefficients in R:

$$Z^2 - (X-1) = 0.$$

In fact one can check that the ring $\tilde{R}$ is the integral closure of R: it is the subring of $Q(R)$ consisting of all elements that are integral over R. A ring that equals its integral closure is called integrally closed or normal.

Theorem 2.1 (normal curves are smooth)**.** *If R is a normal ring and $V(R)$ has dimension 1 then $V(R)$ is smooth.*

Proof. See Janos Kollar, Lectures on resolution of singularities. Theorem 1.30 □

If R is the coordinate ring of a singular curve then we can look at the embedding of R in its integral closure. The embedding $\iota : R \to \tilde{R}$ gives us a map $\pi : V(\tilde{R}) \to$

$V(R)$. One can prove that this map is surjective and a bijection on the smooth points of $V(R)$.

Memo 2.2. Resolving a singular curve can be done by going to the integral closure.

2.2 Resolving surface singularities

In dimension 2 going to the integral closure is not sufficient because there are singular varieties for which the ring is integrally closed. The standard example of this is $\mathbb{C}[x, y, z]/(xy - z^2)$.

If we to generalize the constructions of dimension 1, we would like to add one point for each tangent line through a singularity. This procedure cannot be done by affine geometry alone because the space of lines through a point is a projective variety.

An interesting way of constructing resolutions is by using blow-ups. Suppose V is an affine variety and $\mathfrak{n} \lhd \mathbb{C}[V]$ is an ideal corresponding to the closed subset X. The blow-up of X is then defined as

$$\tilde{V} = \mathsf{Proj}\ \mathbb{C}[V] \oplus \mathfrak{n}t \oplus \mathfrak{n}^2 t^2 \oplus \cdots .$$

The standard projection $\pi : \tilde{V} \to V$ is at least a partial resolution because if $p \in V \setminus X$ and $y_0 t, \dots y_m t$ are the generators of $\mathfrak{n}t$ is there must be at least one y_i that is not zero on p, so the preimage of p will only contain the point

$$(x_1(p), \dots, x_n(p), y_0(p), \dots, y_m(p))$$

We will now do some examples. The ring $\mathbb{C}[X, Y, Z]/(XY - Z^2)$ has a unique singularity in the point $(0, 0, 0)$ because

$$(\partial_X, \partial_Y, \partial_Z)r = (Y, X, 2Z) = 0 \Leftrightarrow (X, Y, Z) = (0, 0, 0).$$

The blow-up is (using the convention $x = Xt, y = Yt, z = Zt$)

$$\begin{aligned}
&\mathsf{Proj}\ \mathbb{C}[X, Y, Z]/(r) \oplus (X, Y, Z)t \oplus (X, Y, Z)^2 t^2 \oplus \cdots \\
&= \frac{\mathbb{C}[X, Y, Z, x, y, z]}{(XY - Z^2, Xy - xY, xZ - Xz, Yz - yZ, Xy - Zz, xy - z^2)} \\
&= \{(X, Y, Z, X, Y, Z) \in \mathbb{C}^3 \setminus \{0\} \times \mathbb{P}^2 | XY - Z^2\} \cup \{(0, 0, 0, x, y, z) \in \mathbb{P}^2 | xy - z^2 = 0\}
\end{aligned}$$

where the last bit is the exceptional fiber, it is a conic and hence as a variety it is isomorphic to $\mathbb{P}_1$. We can cover the blow-up variety by two parts corresponding to

$x \neq 0$ we can put $x = 1$ and then the ring becomes

$$R/(x-1) = \frac{\mathbb{C}[X, Y, Z, y, z]}{(XY - Z^2, Xy - Y, Z - Xz, Yz - yZ, Xy - Zz, y - z^2)} = \mathbb{C}[X, z]$$

which is smooth.

$y \neq 0$ we can put $y = 1$ and then the ring $\mathbb{C}[Y, z]$.

$z \neq 0$ is not necessary because it implies that both $x, y \neq 0$.

The ring $A_n = \mathbb{C}[X, Y, Z]/(XY - Z^n), n \geq 3$ has a unique singularity in the point $(0, 0, 0)$ because

$$(\partial_X, \partial_Y, \partial_Z)r = (Y, X, 3Z^2) = 0 \Leftrightarrow (X, Y, Z) = (0, 0, 0).$$

The blow-up is

$$\frac{\mathbb{C}[X, Y, Z, x, y, z]}{(XY - Z^n, Xy - xY, \ldots, Xy - Z^{n-1}z, xy - Z^{n-2}z^2)}$$
$$= \{(X, Y, Z, X, Y, Z) \in \mathbb{C}^3 \setminus \{0\} \times \mathbb{P}^2 | XY - Z^2\} \cup \{(0, 0, 0, x, y, z) \in \mathbb{P}^2 | xy = 0\}$$

where the last bit is the exceptional fiber, it is a union of 2 projective lines that intersect in the point $(0, 0, 0, 0, 0, 1)$.

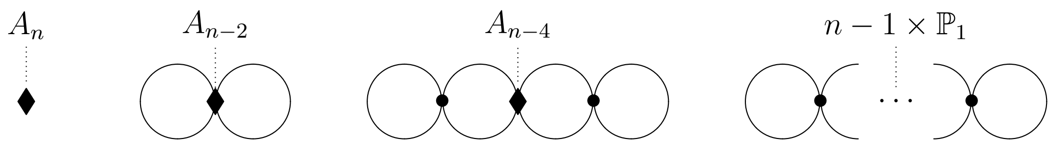
We can cover the blow-up variety by three parts corresponding to

$x \neq 0$ gives $\mathbb{C}[X, z]$ which is smooth.

$y \neq 0$ gives $\mathbb{C}[Y, z]$ which is smooth.

$z \neq 0$ gives $\frac{\mathbb{C}[Z,x,y]}{(xy-Z^{n-2})}$ which has a singularity if $n > 3$, but this singularity is 'smaller' so we can blow it up again.

Diagramatically we get the following



Memo 2.3. The singularities $\mathbb{C}[X, Y, Z]/(XY - Z^n)$ can be resolved by blowing up several times.

Now have a look at $D_n := \mathbb{C}[X,Y,Z]/(X^{n+1}+XY^2+Z^2)$

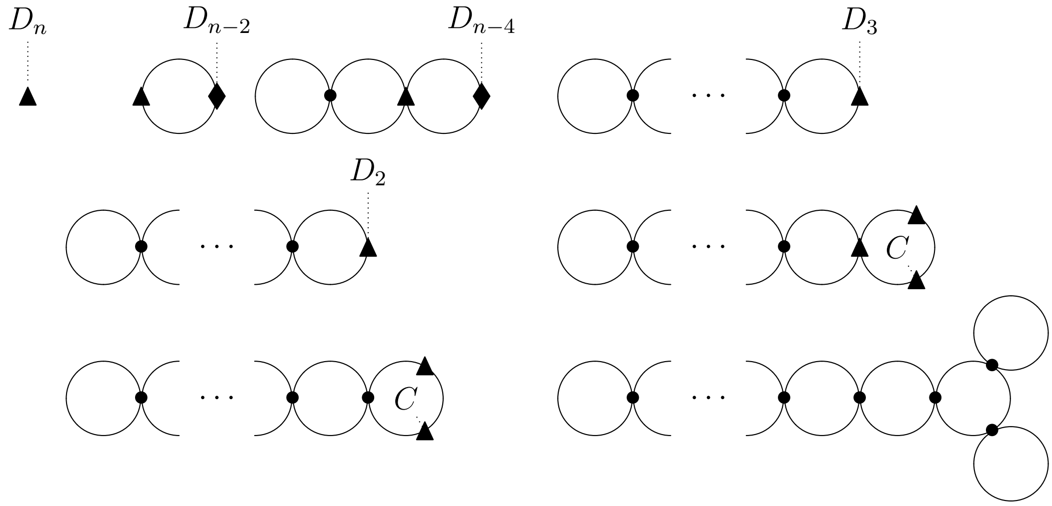
If $n=2$ The first blow-up has exceptional fiber $z=0$ because the relation becomes $Xx^2+Xy^2+z^2=0$. If we look at the chart for $y\neq 0$ we get the relation

$$x^3Y+xY+z^2$$

which has three singularities, for $x=0,\pm i$ if we blow these 3 up, we get 3 exceptional fibers of the form $\bar{\xi}y+\bar{\zeta}^2$ with $\bar{\zeta}=zt$, $\bar{\xi}=t(x+0,\pm i)$ (depending on the point blown up). One can check easily that there are no further singularities.

If $n>2$ then the exceptional fiber is $z=0$. There are two singularities in the blow-up, the one corresponding to $(1,0,0)$ which is of the type D_{n-2} and the one in $(0,1,0)$ which has local equation $x^{n+1}Y^{n-1}+xY+z^2$. The blow-up of this last singularity has as exceptional fiber a conic (look at the degree 2-part) and there are no further singularities.

The diagram looks as



These two examples indicate that in dimension 2 the way to resolve singularities, is by blowing up singular points. This is indeed the case in general:

Memo 2.4. Resolving a singular surface can be done by consecutively going to the intgral closure and blowing up singular points.

In general this process can be quite cumbersome and tedious, so we want to search for a new way of constructing the resolution. To do this we need to have a look at group actions.

Chapter 3

Group actions and the affine quotient

3.1 Quotients and rings of invariants

Suppose we have a finite group G and let V be a finite dimensional complex vector space with a linear G-action (i.e. for every $g \in \mathsf{G}$ the map $v \mapsto g \cdot v$ is linear.

V can also be considered as the variety $\mathbb{C}^n$. For every point $v \in V$ we can define the orbit $\mathsf{G} \cdot v := \{g \cdot v | g \in \mathsf{G}\}$. Orbits never intersect so we can partition V into its orbits. We will denote the set of all orbits by V/G.

A natural question one can ask if whether this set can also be given the structure of an affine variety. In the case of finite groups it will be possible, but for general groups there will be extra complications.

We can take a closer look at the problem by looking at the algebraic side of the story. The ring of polynomial functions over V is $R = \mathbb{C}[V] \cong \mathbb{C}[X_1, \ldots, X_k]$ is a graded polynomial ring if we give the X_i degree 1.

On R we have an action of G:

$$\mathsf{G} \times \mathbb{C}[V] \to \mathbb{C}[V] : (g, f) \mapsto g \cdot f := f \circ \rho_V(g^{-1}).$$

This action is linear and compatible with the algebra structure: $g \cdot f_1 f_2 = (g \cdot f_1)(g \cdot f_2)$.

The set of elements of R that are invariant under the group form a sub ring of R, which we call the ring of invariants and we denote it by

$$R^{\mathsf{G}} := \{f | g \cdot f = f\}$$

The G-action maps homogeneous elements of to homogeneous elements with the same degree and therefore the ring of invariants is alsoo a graded ring.

We are now ready to state the main theorem:

Theorem 3.1. *If* G *is a finite group with a linear group action* V *then the ring of invariants* $S = \mathbb{C}[V]^{\mathsf{G}}$ *is finitely generated.*

Proof. To prove that R^{G} is finitely generated we first prove that this ring is noetherian. Suppose that

$$\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \mathfrak{a}_3 \subset \cdots$$

is an ascending chain of ideals in R^{G}. Multiplying with R we obtain a chain of ideals in R:

$$\mathfrak{a}_1 R \subset \mathfrak{a}_2 R \subset \mathfrak{a}_3 R \subset \cdots .$$

This chain is stationary because R is a polynomial ring and hence noetherian.

Now construct the map

$$\varrho : R \mapsto R^{\mathsf{G}} : f \mapsto \frac{1}{|\mathsf{G}|} \sum_{g \in \mathsf{G}} g \cdot f.$$

This map is called the reynolds operator and has the property that $\pi(f) = f$ if and only if $f \in R^{\mathsf{G}}$ and $\pi(f_1 f_2) = f_1 \pi(f_2)$ if $f_1 \in R^{\mathsf{G}}$. Therefore $\pi(\mathfrak{a}_i R) = \mathfrak{a}_i$ and hence the chain $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \mathfrak{a}_3 \subset \cdots$ is also stationary.

Now let $S_+ =$ denote the ideal of R^{G} generated by all homogeneous elements of nonzero degree. Because R^{G} is Noetherian, S_+ is generated by a finite number of homogeneous elements: $S_+ = f_1 \mathrm{R}^{\mathsf{G}} + \ldots + \mathrm{f}_r \mathrm{R}^{\mathsf{G}}. We will show that these \mathrm{f}_i$ also generate S as a ring.

Now $R^{\mathsf{G}} = \mathbb{C} + S_+$ so $S_+ = \mathbb{C}f_1 + \cdots + \mathbb{C}f_r + S_+^2$, $S_+^2 = \sum_{i,j} \mathbb{C} f_i f_j + S_+^3$ and by induction

$$S_+^t = \sum_{i_1 \ldots i_t} \mathbb{C} f_{i_1} \cdots f_{i_t} + S_+^{t+1}.$$

So $\mathbb{C}[f_1, \ldots, f_r]$ is a graded subalgebra of R^{G} and $R^{\mathsf{G}} = \mathbb{C} + S_+ = \mathbb{C}[f_1, \ldots, f_r] + S_+^t$ for every t. If we look at the degree d-part of this equation we see that

$$R_d^{\mathsf{G}} = \mathbb{C}[f_1, \ldots, f_r]_d + (S_+^t)_d.$$

Because S_+^t only contains elements of degree at least t, $(S_+^t)_d = 0$ if $t > d$. As the equation holds for every t we can conclude that

$$R_d^{\mathsf{G}} = \mathbb{C}[f_1, \ldots, f_r]_d \text{ and thus } R^{\mathsf{G}} = \mathbb{C}[f_1, \ldots, f_r]$$

□

Now because R^{G} is finitely generated and does not have nilpotent elements, it corresponds to a variety $V(R^{\mathsf{G}})$ and the embedding $R^{\mathsf{G}} \subset R$ gives a

$$V(R) \to V(R^{\mathsf{G}}) : \mathfrak{m} \mapsto \mathfrak{m} \cap R^{\mathsf{G}}$$

This map is a surjection because if $\mathfrak{s}$ is a maximal ideal in R^{G} then $\mathfrak{s}R$ is not equal to R because then $\pi(\mathfrak{s}R) = \mathfrak{s}\pi(R) = \mathfrak{s} \neq R^{\mathsf{G}}$. Therefore $\mathfrak{s}R$ will be contained in a maximal ideal $\mathfrak{m} \lhd R$ (there may be more) so $\pi(\mathfrak{m}) = \mathfrak{s}$.

Furthermore points of $V(R)$ in the same orbit are mapped to the same point in $V(R^{\mathsf{G}})$ because $f(g \cdot p) = (g \cdot f)(p) = f(p)$ if $f \in R^{\mathsf{G}}$. The reverse implication is also true. Remember that for a finite number of points $p_1, \ldots, p_k$ and a set of complex numbers $a_1, \ldots a_k$ we can always find a polynomial function $f \in R$ such that $f(p_i) = a_i$. So if p and q have disjoint orbits we can chose the values such that

$$\sum_{g \in \mathsf{G}} f(gp) \neq \sum_{g \in \mathsf{G}} f(gq).$$

therefore the function $\varrho(f) \in R^{\mathsf{G}}$ will be different on the orbits of p and q.

Memo 3.2. We can construct a quotient of a finite group by taking the variety corresponding to the ring of invariants.

3.2 Infinite groups

For infinite groups the situation is a bit more complicated. In full generality the ring of invariants is not finitely generated, but for a large class of groups it still is. Such groups are call reductive groups. Examples of infinite reductive groups ar compact groups, general linear groups, orthogonal groups and symplective groups and finite cartesian products of them. There are however nonreductive groups of which $\mathbb{C}, +$ is the most important.

For reductive groups we still get a surjection from $V(R)$ to $V(R^{\mathsf{G}})$ but more than one orbit can be mapped to the same point. Because the surjection is a continuous map $\pi^{-1}(x)$ must be a closed subset, so if there exists an orbit $\mathcal{O}$ that is not closed and $w = \pi(\mathcal{O})$ we know that $\pi^{-1}(w)$ must contain points outside $\mathcal{O}$ (Note that if G is finite then this problem does not occur because all orbits contain only a finite number of points and are hence closed). However it is still true that different closed orbits are mapped to different points.

We can summarize all this in a theorem

Theorem 3.3. *If V is a finite dimensional representation of a reductive group, then there exists a unique variety $V/\!\!/\mathsf{G} = V(\mathbb{C}[V]^{\mathsf{G}})$ such that*

1. *The points are in one-to-one correspondence with the closed orbits in V.*
2. *The projection $V \to V/\!\!/\mathsf{G}$ is a categorical quotient.*
3. *If G is finite then as a set $V/\!\!/\mathsf{G} = V/\mathsf{G}$.*

3.3 Examples

In this section we will determine generators and relations for the rings of invariants of some group actions. We will do two examples consisting where the group is a group of 2×2-matrices which act on $V = \mathbb{C}^2$.

I is generated by $g = \begin{pmatrix} e^{\frac{2\pi i}{n}} & 0 \\ 0 & e^{-\frac{2\pi i}{n}} \end{pmatrix}$

II is generated by $g = \begin{pmatrix} e^{\frac{\pi i}{n}} & 0 \\ 0 & e^{-\frac{\pi i}{n}} \end{pmatrix}$ and $s = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

In order to find the generator and relations of the rings of invariants, we will use the Reynolds operator

$$\varrho(f) = \frac{1}{|G|} \sum_{g \in G} g \cdot f.$$

This map is a projection $\varrho^2 = \varrho$ and it is the identity operation on $\mathbb{C}[V]^G$. So to get a basis for the ring of invariants we can look at the set of images of all the monomials in $\mathbb{C}[V]$.

$$\varrho X^i Y^j.$$

We will now consider the different types

I If g is the generator of the cyclic group then $g \cdot X = \zeta X, g \cdot Y = \zeta^{-1} Y$ with $\zeta = e^{2\pi/n}$. Therefore

$$\begin{aligned} \varrho X^i Y^j &= \sum_{k=1}^{n} g^k \cdot X^i Y^j \\ &= \sum_{k=1}^{n} \zeta^{k(i-j)} X^i Y^j \\ &= \begin{cases} 0 & i \neq j \mod n \\ n X^i Y^j & i = j \mod n \end{cases}. \end{aligned}$$

From this one can deduce that all invariants are generated by $\xi = X^n, \eta = Y^n$ and $\zeta = XY$. One can easily see that there is a relation between the three invariants $\xi\eta - \zeta^n$.

II Because $s^2 = g^n = -1$ and $gs = -sg$, the elements of this group can be written as $s^i g^j$ with $i = 0, 1$ and $j = 1, \ldots, 2n$.

Therefore

$$\begin{aligned}
\varrho X^i Y^j &= \sum_{k=1}^{2} ng^k \cdot X^i Y^j + sg^k \cdot X^i Y^j \\
&= \sum_{k=1}^{n} \zeta^{k(i-j)} (X^i Y^j + i^{i+j} X^j Y^i) \\
&= \begin{cases} 0 & i \neq j \mod 2n \text{ or} \\ X^i Y^j - Y^j X^i & i = j \mod 2n \text{ and } i \text{ is odd} \\ X^i Y^j + Y^j X^i & i = j \mod 2n \text{ and } i \text{ is even} \end{cases} .
\end{aligned}$$

From this one can deduce that all invariants are generated by $\xi = X^2 Y^2$, $\eta = XY(X^{2n} - Y^{2n})$ and $\zeta = X^{2n} + Y^{2n}$. One can easily see that there is a relation between the three invariants $\xi^{n+1} - \xi\eta^2 + \zeta^2$.

In both cases, the dimension of the quotient space must be two because the map $\mathbb{C}^2 \to V/G$ has finite fibers. If the ideal would be generated by more than one generator $\mathbb{C}[\xi, \eta, \zeta]/\mathfrak{p}$, its corresponding variety would not be two-dimensional.

As we notice here, these singularities are precisely the ones we tried to blow up in the previous chapter. In the following chapters we will use this viewpoint for a new way to construct resolutions for the singularity.

Chapter 4

Smash Products and Quivers

4.1 Smash Products

As we know from the previous chapter, the action of G on V gives rise to an action on the polynomial ring $C[V] \cong \mathbb{C}[X,Y]$. Construct the vector space $\mathbb{C}[V]^{|}G|$. We can identify the standard basis elements with the elements in the group, such that every element of this space can be written uniquely as a sum of $f(X,Y)g$ where $f(X,Y)$ is a polynomial function and g is an element of G. We can now define a product on this vector space

$$f_i(X,Y)g_i \times f_j(X,Y)g_j = (f_i g_i \cdot f_j)g_i g_j,$$

in this expression the $\cdot$ denotes the action of G on $\mathbb{C}[V]$. One can easily check that this product is associative and by linearly extending it to the whole vector space one obtains an algebra: the smash product of $\mathbb{C}[V]$ and G. In symbols we write $\mathbb{C}[V]\#G$. The center of this algebra can be easily determined: if $z = \sum_g f_g g \in \mathcal{Z}$ then

$$\forall f \in \mathbb{C}[V] : [z,f] = f_g(f - g \cdot f)g = 0 \text{ and } \forall h \in g : [z,h] = (h \cdot f_g - f_g)gh$$

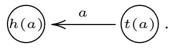
The first equation implies that $f_g = 0$ if $g \neq 1$ and the second implies that f_1 must be a G-invariant function so we can conclude that

$$\mathcal{Z}(\mathbb{C}[V]\#G) \cong \mathbb{C}[V]^G.$$

This algebra has a nice description using quivers.

4.2 Quivers

A *quiver* $Q = (Q_0, Q_1, h, t)$ consists of a set of vertices Q_0, a set of arrows Q_1 between those vertices and maps $h, t : A \to V$ which assign to each arrow its head and tail vertex. We also denote this as



A sequence of arrows $a_1 \dots a_p$ in a quiver Q is called a *path of length* p if $t(a_i) = h(a_{i+1})$, this path is called a cycle if $t(a_p) = h(a_1)$. A path of length zero will be defined as a vertex. A quiver is strongly connected if for every couple of vertices (v_1, v_2) there exists a path p such that $s(p) = v_1$ and $t(p) = v_2$.

If we take all the paths, including the one with zero length, as a basis we can form a complex vector space $\mathbb{C}Q$. On this space we can put a noncommutative product, by concatenating paths. By the concatenation of two paths $a_1 \dots a_p$ and $b_1 \dots b_q$ we mean

$$a_1 \dots a_p \cdot b_1 \dots b_q := \begin{cases} a_1 \dots a_p b_1 \dots b_q & s(a_p) = t(b_1) \\ 0 & t(a_p) \neq h(b_1) \end{cases}$$

For a vertex v and a path p we define vp as p if p ends in v and zero else. On the other hand pv is p if this path starts in v and zero else.

The vector space $\mathbb{C}Q$ equipped with this product is called the *path algebra*. The set of vertices $Q_0 = \{v_1, \dots, v_k\}$ forms a set of mutually orthogonal idempotents for this algebra. The subalgebra generated by these vertices is isomorphic to $\mathbb{C}Q_0 = \mathbb{C}^{\oplus k}$ and this is also the degree zero part if we give $\mathbb{C}Q$ a gradation using the length of the paths.

An algebra A is called a path algebra with relations A is isomorphic to the quotient of a path algebra by an ideal sitting inside $\mathbb{C}Q_{\geq 2}$, which is the space spanned by all paths of length at least 2.

Theorem 4.1. *If G is a finite abelian group with a linear action on V, then the smash product is a path algebra of a quiver with relations.*

Proof. Sketch. Let $\hat{\mathsf{G}}$ be the set of group morphisms from G to $\mathbb{C}^*$. for each element $\phi \in \hat{\mathsf{G}}$ we define

$$e_\phi := \frac{1}{\mathsf{G}} \sum_{g \in \mathsf{G}} \phi(g) g$$

One can prove that $e_\phi e_\psi = 0$ if $\phi \neq \psi$ and $e_\phi^2 = e_\phi$. Moreover $\sum_{\phi \in \hat{\mathsf{G}}} e_\phi = 1$.

This means that the e_ϕ play the role of the vertices.

Because G is abelian we can find a basis $X_1, \dots, X_k$ for V such that the X_i are eigenvectors for all of the $g \in \mathsf{G}$. This means that every X_i there is a $\phi_i \in \hat{\mathsf{G}}$ such that $g \cdot X_i = \phi_i(g) X_i$.

One can check that $e_\phi X_i = X_i e_{\phi_i \phi} = e_\phi X_i e_{\phi_i \phi}$ and therefore $e_\phi X_i$ can be seen as an arrow with head e_ϕ and tail $e_{\phi_i \phi}$.

This means that $X_i = \sum_\phi e_\phi X_i$ splits as the sum of $|G|$ arrows one for each e_ϕ.

In a similar way we can split the relations $X_i X_j - X_j X_i$ as a sum of paths of length two in these arrows. □

remark 4.2. If G is not abelian this is not true anymore. But something very similar is going on. The statement now becomes that the smash product $A = \mathbb{C}[V]\#\mathsf{G}$ contains an idempotent e such that eAe is a path algebra with relations and $A = AeA$ (which implies that the idempotent does not map anything essential to zero).

4.3 representations of quivers

A *dimension vector* of a quiver is a map $\alpha : Q_0 \to \mathbb{N}$, the size of a dimension vector is defined as $|\alpha| := \sum_{v \in Q_0} \alpha_v$. A couple (Q, α) consisting of a quiver and a dimension vector is called a *quiver setting* and for every vertex $v \in Q_0$, α_v is referred to as the dimension of v. A setting is called *sincere* if none of the vertices has dimension 0. For every vertex $v \in Q_0$ we also define the dimension vector

$$\epsilon_v : V \to \mathbb{N} : w \mapsto \begin{cases} 0 & v \neq w, \\ 1 & v = w. \end{cases}$$

An α*-dimensional complex representation* W of Q assigns to each vertex v a linear space $\mathbb{C}^{\alpha_v}$ and to each arrow a a matrix

$$W_a \in \mathsf{Mat}_{\alpha_{h(a)} \times \alpha_{t(a)}}(\mathbb{C})$$

The space of all α-dimensional representations is denoted by $\mathsf{Rep}(Q, \alpha)$.

$$\mathsf{Rep}(Q, \alpha) := \bigoplus_{a \in A} \mathsf{Mat}_{\alpha_{h(a)} \times \alpha_{t(a)}}(\mathbb{C})$$

To the dimension vector α we can also assign a reductive group

$$\mathsf{GL}_\alpha := \bigoplus_{v \in V} \mathsf{GL}_{\alpha_v}(\mathbb{C}).$$

An element of this group, g, has a natural action on $\mathsf{Rep}(Q,\alpha)$:

$$W := (W_a)_{a\in A},\ W^g := (g_{t(a)}W_a g_{s(a)}^{-1})_{a\in A}$$

The quotient of this action will be denoted as

$$\mathsf{iss}(Q,\alpha) := \mathsf{Rep}(Q,\alpha)/\!\!/\mathsf{GL}_\alpha.$$

A representation W is called *simple* if the only collections of subspaces $(\mathsf{V}_v)_{v\in V},\ \mathsf{V}_v \subseteq \mathbb{C}^{\alpha_v}$ having the property

$$\forall a \in A : W_a\mathsf{V}_{s(a)} \subset \mathsf{V}_{t(a)}$$

are the trivial ones (i.e. the collection of zero-dimensional subspaces and $(\mathbb{C}^{\alpha_v})_{v\in V}$).

The direct sum $W \oplus W'$ of two representations W, W' has as dimension vector the sum of the two dimension vectors and as matrices $(W \oplus W')_a := W_a \oplus W'_a$. A representation equivalent to a direct sum of simple representations is called *semisimple*.

A very important theorem is:

Theorem 4.3. *A representation of a quiver is semisimple if and only if its orbit is closed in* $\mathsf{Rep}(Q,\alpha)$.

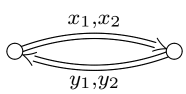
Proof. omitted □

If we have a path algebra with relations A we define $\mathsf{Rep}(A,\alpha)$ as the subset of $\mathsf{Rep}(Q,\alpha)$ which respect the relations. We will denote this variety as $\mathsf{Rep}(A,\alpha)$. Because this is a closed subset of $\mathsf{Rep}(Q,\alpha)$, the quotient $\mathsf{Rep}(A,\alpha)/\!\!/\mathsf{GL}_\alpha$ can be seen as the image of $\mathsf{Rep}(A,\alpha)$ under the quotient map $\mathsf{Rep}(Q,\alpha) \to \mathsf{iss}(Q,\alpha)$. Again it classifies the semisimple α-dimensional representations up to isomorphism and hence we denote it by $\mathsf{iss}(A,\alpha)$.

4.4 The representation space of an abelian smash product

In this section we do some examples in the case of abelian smash products

- Let G be $\mathbb{Z}_2$ which acts on $\mathbb{C}^2$ by $gX = -X$ and $gY = -Y$. The quiver looks like



with relations $x_1y_2 - x_2y_1$ and $x_1y_2 - x_2y_1$.

The representation space with dimension vector $(1,1)$ is

$$\{(a_1, a_2, b_1, b_2) \in \mathbb{C}^4 | a_1b_2 - a_2b_1\}.$$

This space is threedimensional.

The invariant functions on $\mathsf{Rep}(A, \alpha)$ are generated by $X = a_1b_1$, $Y = a_2b_2$ and $Z = a_1b_2$ which is the same as a_2b_1. The relation between these invariants is $XY - Z^2$ so the ring of invariants is indeed $\mathbb{C}[\mathsf{iss}(A,)] = \mathbb{C}[X, Y, Z]/(XY - Z^2)$.

For each point $(x, y, z) \neq (0, 0, 0)$ we can construct a simple representation

- if $z \neq 0$ then we take $(1, y/z, x, z)$.
- for $(x, y, z) = (1, 0, 0)$ we take $(1, 0, 1, 0)$.
- for $(x, y, z) = (0, 1, 0)$ we take $(0, 1, 0, 1)$.

For the point $(x, y, z) = (0, 0, 0)$ there are several orbits:

- $\{(0, 0, 0, 0)\}$ which is a point
- For each $(x : y) \in \mathbb{P}^1$ we have an orbit $\{(ax, ay, 0, 0) | a \in \mathbb{C}^*\}$.
- For each $(x : y) \in \mathbb{P}^1$ we have an orbit $\{(0, 0, ax, ay) | a \in \mathbb{C}^*\}$.

Chapter 5

Semi-invariants and Moduli spaces

5.1 Semi-invariants

As we have seen in the previous chapter it is possible to get a resolution of an affine variety by constructing the Proj of a graded ring of which the degree zero part is the ring of regular functions of the original variety. The method we used for this was blow-ups. In invariant theory it is also possible to do a different construction using semi-invariants.

If G is a reductive group then a *multiplicative character* of G is a group morphism $\theta : G \to \mathbb{C}^* : g \mapsto g^\theta$. We will write the action of θ exponentially because it will be very handy later on. The characters of G form an additive group if we define $g^{\theta_1+\theta_2} := g^{\theta_1} g^{\theta_2}$, we will also use the shorthand $n\theta = \theta + \cdots + \theta$.

If G acts on a variety V then a function $f \in \mathbb{C}[V]$ is called a *θ-semi-invariant* if

$$\forall g \in G : g \cdot f = g^\theta f.$$

The subspace of θ-semi-invariants will be denoted by $\mathbb{C}[V]_\theta$. This space does not form a ring, it is only a module over the ring of invariants $\mathbb{C}[V]^G$.

We can construct an $\mathbb{N}$-graded ring by taking the direct sum of all $n\theta$-semi-invariants with $n \in \mathbb{N}$:

$$\mathsf{SI}_\theta[V] = \bigoplus_{n \in \mathbb{N}} \mathbb{C}[V]_{n\theta}.$$

It is easy to extend the proof of theorem to show that $\mathsf{SI}_\theta[V]$ is also finitely generated as an algebra over $\mathsf{SI}_\theta[V]_0 = \mathbb{C}[V]^G$.

A point $p \in V$ is called θ-semi-stable if there is an $f \in \mathbb{C}[V]_{n\theta}$ such that $f(p) \neq 0$. The set of θ-semi-stable points will be denoted by V_θ^{ss}. Note that V_θ^{ss} itself is not necessarily an affine variety but if $f_1, \cdots, f_k$ forms a set of homogeneous generators of $\mathsf{SI}_\theta[V]$ over $\mathbb{C}[V]^G$ then V_θ^{ss} can be covered with affine varieties corresponding to the rings

$$R_i = \mathbb{C}[V][f_i^{-1}] \; V(R_i) = \{p \in V | f_i(p) \neq 0\}$$

These varieties and rings have G-actions on them coming from the G-action on V and one can take the categorical quotient of these varieties. Their ring are of the form

$$\mathsf{SI}_\theta[V][f_i^{-1}]_0$$

and hence one can cover $\mathsf{Proj}\ \mathsf{SI}_\theta[V]$ with these quotients varieties. Out of this one can conclude

Theorem 5.1. *The variety* $\mathsf{Proj}\ \mathsf{SI}_\theta[V]$ *classifies the closed orbits in* V_θ^{ss}. *If there exists a θ-semi-invariant that is non-zero in a point of V then V_θ^{ss} is open and dense in V and the image of the map $V_\theta^{ss}/\!\!/G \to V/\!\!/G$ is dense.*

5.2 semi-stable representations of quivers

If Q is a quiver and α a dimension vector then we can look at the θ-semi-invariants of the GL_α-action on $\mathsf{Rep}(Q, \alpha)$. We will denote this set by $\mathsf{Rep}_\theta^{ss}(Q, \alpha)$, the quotient of this set by the GL_α-action we be denoted by $\mathsf{M}_\theta^{ss}(Q, \alpha)$ and is called the moduli space of θ-semistable representations.

First of all we have to look at the multiplicative characters of GL_α. For the general linear group GL_n the characters are given by powers of the determinant, so the group of characters is isomorphic to $\mathbb{Z}$. As GL_α consists of $k = \#Q_0$ components each one isomorphic to a general linear group, the group of characters will be isomorphic to $\mathbb{Z}^k$:

$$\theta = (\theta_1, \ldots, \theta_k) : \mathsf{GL}_\alpha \to \mathbb{C}^* : (M_1, \ldots, M_k) \mapsto \det M_1^{\theta_1} \cdots \det M_k^{\theta_k}.$$

In the case of invariants we had a nice description using traces of cycles, for semi-invariants we can do a similar thing. A way to construct a θ-semi invariant is the following: let $i_1, \ldots, i_s$ be the vertices for which θ_{i_ℓ} is positive, while $j_1, \ldots, j_t$ be the ones with a negative θ_{j_ℓ}. Now chose for each i and j $|\theta_i\theta_j|$ elements in $j\mathbb{C}Qi$

and put all these in a $\sum_j \theta_j \times \sum_i \theta_i$-matrix D over $\mathbb{C}Q$.

$$D := \begin{bmatrix} j_1\leftarrow i_1 & \dots & j_1\leftarrow i_1 & & j_1\leftarrow i_s & \dots & j_1\leftarrow i_s \\ \vdots & |\theta_{j_1}\theta_{i_1}|\times & \vdots & \dots & \vdots & |\theta_{j_1}\theta_{i_s}|\times & \vdots \\ j_1\leftarrow i_1 & \dots & j_1\leftarrow i_1 & & j_1\leftarrow i_s & \dots & j_1\leftarrow i_s \\ \vdots & & & \ddots & & & \vdots \\ j_t\leftarrow i_1 & \dots & j_t\leftarrow i_1 & & j_t\leftarrow i_s & \dots & j_t\leftarrow i_s \\ \vdots & |\theta_{j_t}\theta_{i_1}|\times & \vdots & \dots & \vdots & |\theta_{j_t}\theta_{i_s}|\times & \vdots \\ j_t\leftarrow i_1 & \dots & j_t\leftarrow i_1 & & j_t\leftarrow i_s & \dots & j_t\leftarrow i_s \end{bmatrix}$$

Now if $W \in \mathsf{Rep}(Q,\alpha)$ then we can substitute each entry in D to its corresponding matrix-value in W. In this way we obtain a block matrix D_W with dimensions $\sum_i \alpha_i|\theta_i| \times \sum_j \alpha_j|\theta_j|$. One can easily check that

$$D_{g\cdot W} = \begin{bmatrix} g_{j_1} & & & & \\ & \ddots & & & \\ & & g_{j_1} & & \\ & & & \ddots & \\ & & & & g_{j_t} \\ & & & & & \ddots \\ & & & & & & g_{j_t} \end{bmatrix} D_W \begin{bmatrix} g_{i_1}^{-1} & & & & \\ & \ddots & & & \\ & & g_{i_1}^{-1} & & \\ & & & \ddots & \\ & & & & g_{i_s}^{-1} \\ & & & & & \ddots \\ & & & & & & g_{i_s}^{-1} \end{bmatrix}$$

So if D_W is a square matrix the determinant of D_W is a θ-semi-invariant:

$$\det D_{g\cdot W} = \det g_{j_1}^{\theta_{j_1}} \cdot \det g_{j_t}^{\theta_{j_t}} \det D_W \det g_{i_1}^{-|\theta_{i_1}|} \cdot \det g_{i_s}^{-|\theta_{i_s}|} = g^\theta \det D_W.$$

We will call these semi-invariants *determinantal semi-invariants*

Theorem 5.2. *As a* $\mathbb{C}[\mathsf{Rep}_\alpha Q]^{\mathsf{GL}_\alpha}$*-module* $\mathbb{C}[\mathsf{Rep}_\alpha Q]_\theta$ *is generated by determinantal semi-invariants. As a ring* $\mathsf{SI}_\theta[\mathsf{Rep}_\alpha Q]$ *is generated by invariants (i.e. traces of cycles) and determinantal* $n\theta$*-semi-invariants with* $n \in \mathbb{N}$*.*

Note that this implies that there are only θ-semi-invariants if D_W is a square matrix so $\sum_i \alpha_i|\theta_i| = \sum_j \alpha_j|\theta_j|$ or equivalently $\theta \cdot \alpha = 0$.

Now we can use this special form for the semi-invariants to get a nice interpretation for the covering of $\mathsf{Rep}(Q,\alpha)^s s_\theta /\!\!/ \mathsf{GL}_\alpha$.

As we have seen $\mathsf{Rep}(Q,\alpha)$ describes the α-dimensional representations of the pathalgebra $\mathbb{C}Q$. Now if $W \in \mathsf{Rep}(Q,\alpha)$ is θ-semistable then there exists a $\sum_j |\theta_j| \times \sum_i |\theta_i|$-matrix D with entries in $\mathbb{C}Q$ such that $\det D_W \neq 0$, so D_W is an invertible matrix:

$$\exists E_W : D_W E_W = 1_{\sum |\theta_j|\alpha_j} \text{ and } D_W E_W = 1_{\sum |\theta_i|\alpha_i}$$

So W is also a representation of a new algebra for which D is is indeed an invertible matrix. To be more precise we need a good interpretation a the identity matrices that appeared in the equations above. Recall that for an α-dimensional representation W the vertex i can considered as an idempotent in $\mathbb{C}Q$ and i_W will correspond to the identity matrix on the α_i-dimensional space iW. The identity matrix $\mathsf{1}_{\sum|\theta_j|\alpha_j}$ can hence be considered as the evaluation in W of the matrix

$$\mathsf{1}_j = \begin{bmatrix} j_1 & & & & & \\ & \ddots & & & & \\ & & j_1 & & & \\ & & & \ddots & & \\ & & & & j_t & \\ & & & & & \ddots & \\ & & & & & & j_t \end{bmatrix} = \begin{bmatrix} h(D_{11}) & & \\ & \ddots & \\ & & h(D_{pp}) \end{bmatrix}$$

Now we define $E = E_{\mu\nu}$ to be the matrix such that $DE = \mathsf{1}_j$ and $ED = \mathsf{1}_i$ so

$$\sum_\kappa D_{\mu\kappa}E_{\kappa\nu} = \delta_{\mu\nu}h(D_\mu\mu) \text{ and } \sum_\kappa E_{\mu\kappa}D_{\kappa\nu} = \delta_{\mu\nu}t(D_\mu\mu).$$

Now we define the universal localization of $\mathbb{C}Q$ at D to be the algebra

$$\mathbb{C}Q[D^{-1}] = \mathbb{C}Q_E/(\sum_\kappa D_{\mu\kappa}E_{\kappa\nu} - \delta_{\mu\nu}h(D_\mu\mu), \sum_\kappa E_{\mu\kappa}D_{\kappa\nu} = \delta_{\mu\nu}t(D_\mu\mu))$$

Here Q_E is a new quiver consisting of Q together with extra arrows $E_{\mu\nu}$ such that $h(E_{\mu\nu}) = t(D_{\mu\nu})$ and $t(E_{\mu\nu}) = h(D_{\mu\nu})$.

There is a natural map $\mathbb{C}Q \to \mathbb{C}Q[D^{-1}]$ so we also have a map

$$\mathsf{Rep}(\mathbb{C}Q[D^{-1}], \alpha) \to \mathsf{Rep}(Q, \alpha)$$

This map is an (open) embedding because $(D^{-1})_W$ is uniquely defined by D_W, its image consist precisely of these representations of $\mathbb{C}Q$ for which $\det D_W \neq 0$.

Theorem 5.3. $\mathsf{Rep}^{ss}_\theta(Q, \alpha)$ *can be covered by representation spaces of universal localizations of* $\mathbb{C}Q$*. This covering is compatible with the* GL_α*-action, so* $\mathsf{M}^{ss}_\theta(Q, \alpha)$ *can be covered by quotient spaces of universal localizations of* $\mathbb{C}Q$*.*

This theorem also holds for quotients of path algebras. We will work this out in the next section for the preprojective algebras.

5.3 Moduli space for the smash product

So lets now take a closer look at the case of the singularity $\mathbb{C}[X, Y, Z]/(XY - Z^n)$. As we already know we can consider the singularity as the quotient space of the preprojective algebra over the McKay quiver with the standard dimension vector.

$$V/\!\!/G = \mathsf{iss}(\Pi_G, \alpha_G.$$

In order to construct a nice desingularization of this space we have to find a good character. Let $e_1, \ldots, e_n$ be the vertices of the quiver and let e_1 correspond to the trivial representation. For every vertex $e_i \neq e_1$ there exists a character θ_i mapping e_1 to $-\alpha_{Gi} = -\dim S_i$, e_i to 1 and all the other vertices to zero. We denote the sum of all these θ_i as θ and this will be the character under consideration:

$$\theta(e_1) = -|\alpha_G| + 1 \text{ and } \theta_i = 1 \text{ if } i \neq 0.$$

Theorem 5.4. *If $V/\!\!/G$ is a kleinian singularity and $\tilde{V}/\!\!/G \to V/\!\!/G$ is its minimal resolution, then $\tilde{V}/\!\!/G = \mathsf{M}^{ss}_\theta(\Pi_G, \alpha)$.*

The proof of this theorem can be found in [**?**]. We will just show how one can see the equivalence in the A case.

The semi-invariants are constructed using matrices D of which the entries are all paths starting from e_1. If we construct D_W, then every column of D_W corresponds to a column of D because the dimension of e_1 is 1. The determinant is linear in the columns so we can chose D up to linear combinations of the columns. In this way we can turn D into a form such that D_W is block diagonal with the dimension of every block corresponding to the dimension of a vertex. This means that the θ-semi-invariants are generated by products of the θ_i-semi-invariants. Therefore we can conclude that a representation is θ-semistable if and only if it is θ_i-semistable for every i.

The θ_i-semi-invariants are generated over $\mathbb{C}[\mathsf{iss}(\Pi, \alpha)]$ by D's that are $1 \times \dim S_i$-matrices whose entries are paths from e_1 to e_i. Using the preprojective relation and the relations from matrix identities one can find a finite number of generating paths. For instance, these paths cannot run twice through a vertex with dimension 1 otherwise we could split of a trace of a cycle (and this is contained in $\mathbb{C}[\mathsf{iss}(\Pi, \alpha)]$). If there are k_i such paths there are $C^{k_i}_{\alpha_i}$ generators for the semi-invariants.

This means that we can embed $M^{ss}_\theta(\Pi, \alpha)$ in

$$\mathbb{C}^3 \times \mathbb{P}^{C^{k_2}_{\alpha_2}} \times \cdots \times \mathbb{P}^{C^{k_n}_{\alpha_n}}.$$

The first factor is for the 3 invariants, the others for the θ_i-semi-invariants for every $i > 1$.

In the A-case, up to multiplication with invariants there are for every e_i exactly two paths from e_1: a clockwise p_i and a counterclockwise q_i. Each of these paths gives a θ_i-semi-invariant. The projection map $\mathsf{Rep}^{ss}_\theta(\Pi, \alpha) \to M^{ss}_\theta(\Pi, \alpha)$ can now be seen as

$$W \mapsto [(X_W, Y_W, Z_W), (p_2, q_2)_W, \ldots, (p_n, q_n)_W].$$

To calculate the exceptional fiber we must look at the semistable representation that have zero invariants (X, Y, Z). Because X is zero there must be an i such that $p_{iW} \neq 0$ but $p_{i+1W} = 0$. Semistability then implies that $q_{i+1W} \neq 0$. Also q_{i-1W} must be zero otherwise $Z = p_i/p_{i-1}q_{i-1}/q_i$ would not be zero. This means that a point P comes from a point in the exceptional fiber if it is of the form

$$[(0, 0, 0), (1, 0), \cdots, (1, 0), (p_i, q_i)_W, (0, 1), \cdots, (0, 1)].$$

From this we can conclude that the exceptional fiber is indeed the union of $n - 1$ $\mathbb{P}_1$ intersection each other consecutively.

Task

- Choose your favorite finite abelian group of at least 3 elements.
- Construct a linear action of this group on either $\mathbb{C}^2$ of $\mathbb{C}^3$ that is faithfull (i.e. no element of the group acts trivially).
- Calculate the generators and relations of the ring of invariants R^{G} for this action.
- Find a presentation of the smash product as a path algebra of a quiver with relations.
- Describe the representation space for dimension vector $(1, \ldots, 1)$ and check that its quotient corresponds indeed the ring of invariants. Describe the orbits of this space.
- Describe the moduli space for dimension vector $(1, \ldots, 1)$ and character $(-n+1, 1, \ldots, 1)$
- Describe the exceptional fiber over the zero. itemize