THE POWER OF SLICING IN NONCOMMUTATIVE GEOMETRY

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Abstract. In this note we will show how one can describe the representation varieties of certain Fuchsian groups using the Luna Slice theorem and the technique of local quivers. We will illustrate with one specific example: the Fuchsian group $\mathbb{Z}_2 * \mathbb{Z}_4$.

1. Introduction

Fuchsian groups are the crystallographic groups of the hyperbolic plane, i.e. the discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$. The orientation-preserving Fuchsian groups can be presented in the following form

- generators

$$\begin{aligned} \text{hyperbolic: } & a_1, b_1, \dots, a_g, b_g \\ \text{elliptic: } & x_1, \dots, x_e \\ \text{parabolic: } & y_1, \dots, y_p \end{aligned}$$

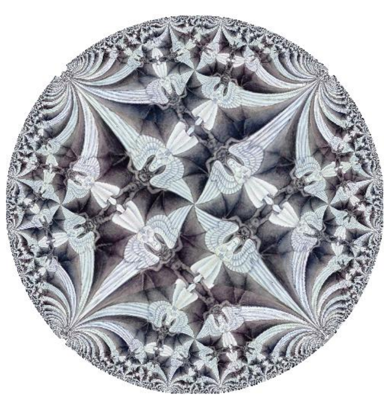
- relations

$$\begin{aligned} & x_1^{m_1} = \dots x_e^{m_e} = 1 \\ & x_1 \cdots x_e y_1 \cdots y_p [a_1, b_1] \cdots [a_g, b_g] = 1 \end{aligned}$$

If the group has parabolic generators we can use one to get rid of the last relation, so that it can also be written as a free product of cyclic groups.

$$\mathbb{Z}_{m_1} * \cdots * \mathbb{Z}_{m_e} * \mathbb{Z}^{*(p+2g-1)}$$

For instance the orientation-preserving symmetry group of the Escher-like picture below is $\mathsf{G} = \mathbb{Z}_2 * \mathbb{Z}_4$.



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2. THE REPRESENTATION SPACES OF G

To G we can assign the variety of n-dimensional representations

$$\begin{aligned}\mathtt{rep}(\mathsf{G},n) &:= \{\rho:\mathsf{G}\to \mathrm{GL}_n : \rho(gh)=\rho(g)\rho(h)\}\\ &\cong \{(X,Y)\in \mathrm{GL}_n\times \mathrm{GL}_n | X^2=Y^4=1\}\end{aligned}$$

On this space we have a GL_n-action by conjugation: $g{\cdot}(X,Y)=(gXg^{-1},gYg^{-1})$. Orbits under this action correspond to isomorphism classes of representations and if one takes the algebraic quotient of this action one obtains a new variety $\mathtt{iss}(\mathsf{G},n)$ which classifies the semisimple representations of G up to isomorphism (see [9]). The projection map $\pi:\mathtt{rep}(\mathsf{G},n)\to\mathtt{iss}(\mathsf{G},n)$ maps every representation to the isomorphism class of its semisimplification.

The main goal of geometric representation theory is now the study this map and find the answer to the following questions:

- what is the geometry of $\mathtt{rep}(\mathsf{G},n)$, does it contain singularities and if so what is their nature?
- what is the geometry of $\mathtt{iss}(\mathsf{G},n)$, does it contain singularities and if so what is their nature?
- what do the fibers of π look like?

In this case the first question can be answered quickly using only the theory of representations of finite groups, because $\mathtt{rep}(\mathsf{G},n)=\mathtt{rep}(\mathbb{Z}_2,n)\times\mathtt{rep}(\mathbb{Z}_4,n)$. The two factors contain each a finite number of disjoint conjugation classes of diagonal matrices with eigenvalues ±1 resp. $\pm1,\pm i$.

This fact can be used to split $\mathtt{rep}(\mathsf{G},n)$ into disjoint components corresponding to a dimension vector, which is an element in $\mathbb{N}^{2+4}$ consisting of the dimensions of the eigenspaces of X (with eigenvalues ±1) and Y (with eigenvalues $\pm1,\pm i$). Each of these components is smooth because it is the product of 2 conjugation classes of matrices and we will denote these varieties by $\mathtt{rep}(\mathsf{G},\delta)$ with $\delta\in\mathbb{N}^{2+4}$. We can also split the quotient map π and its target to obtain quotient varieties denoted as $\mathtt{iss}(\mathsf{G},\delta)$.

3. The Luna Slice Theorem

The fact that the representation spaces $\mathtt{rep}(\mathsf{G},\delta)$ are smooth provides us with a very powerful tool to study both the representation spaces and their quotient spaces $\mathtt{iss}(\mathsf{G},\delta)$, the *Luna Slice Theorem.* This theorem is an algebraic version of the classical slice theorem in differential geometry, using étale morphisms instead of diffeomorphisms.

Recall that a morphism $\varphi: X\to Y$ between algebraic varieties is called *étale* if it is a *smooth morphism with finite fibers*. If both X and Y are smooth, this is equivalent to the fact that $d_x\varphi: T_xX\to T_{\varphi(x)}Y$ is an isomorphism. The Luna Slice Theorem now states

Theorem 1 (Luna, [12])**.** *Let $x\in\mathtt{rep}(\mathsf{G},\delta)$ be a semisimple representation and let H be the stabilizer of x in $GL_n(\mathbb{C})$. Let N be the normal space to the small tangent space $T_xGL_n(\mathbb{C}).x$ in the big tangent space $T_x\mathtt{rep}(\mathsf{G},\delta)$. Then there exists a locally closed affine subset $S\subset\mathtt{rep}(\mathsf{G},\delta)$, called a* slice*, such that*

1. *$x\in S$;*
2. *S is stable under the action of H;*

(3) *the action map* $GL_n(\mathbb{C}) \times S \to \mathit{rep}_\alpha \mathsf{G} : (g, s) \mapsto g.s$ *induces an* étale *and* $\mathrm{GL}_n(\mathbb{C})$-equivariant *morphism*

$$\psi : GL_n(\mathbb{C}) \times^H S \to \mathit{rep}(\mathsf{G}, \delta)$$

for which the induced morphism

$$\psi /\!\!/ GL_n(\mathbb{C}) : S /\!\!/ H \to \mathit{iss}(\mathsf{G}, \delta)$$

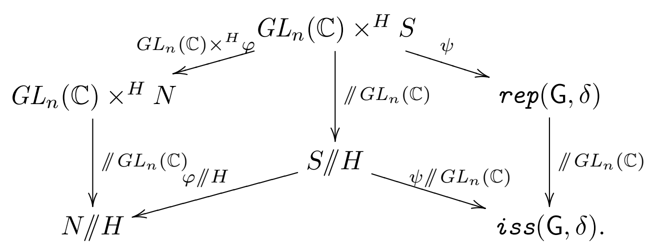
is étale;

(4) *there exists a* H*-equivariant map* $\varphi : S \to N = T_x S$ *with affine image and* $\varphi(x) = 0$ *such that the induced map*

$$\varphi /\!\!/ H : S /\!\!/ H \to N /\!\!/ H$$

is étale;

(5) *we have a commutative diagram*



4. Local Quivers

To get a combinatorial description of the Luna Slice Theorem we need quivers.

A *quiver* $Q = (V, A, h, t)$ consists of a set of vertices V, a set of arrows A between those vertices and maps $h, t : A \to V$ which assign to each arrow its head and tail.

A *dimension vector* of a quiver is a map $\alpha : V \to \mathbb{N}$ and a couple (Q, α) consisting of a quiver and a dimension vector is called a *quiver setting* and for every vertex $v \in V$, α_v is refered to as the dimension of v. In the examples the dimensions are written inside the vertices.

An α*-dimensional complex representation* W of Q assigns to each vertex v a linear space $\mathbb{C}^{\alpha_v}$ and to each arrow a a matrix

$$W_a \in \mathsf{Mat}_{\alpha_{h(a)} \times \alpha_{t(a)}}(\mathbb{C})$$

The space of all α-dimensional representations is denoted by $\mathtt{rep}_\alpha Q$.

$$\mathtt{rep}(Q, \alpha) := \bigoplus_{a \in A} \mathsf{Mat}_{\alpha_{h(a)} \times \alpha_{t(a)}}(\mathbb{C})$$

To the dimension vector α we can also assign a reductive group

$$\mathrm{GL}_\alpha := \prod_{v \in V} \mathrm{GL}_{\alpha_\mathsf{v}}(\mathbb{C}).$$

An element of this group, g, has a natural action on $\mathtt{rep}(Q, \alpha)$:

$$W := (W_a)_{a \in Q_1},\ W^g := (g_{h(a)} W_a g_{t(a)}^{-1})_{a \in Q_1}$$

Using this definitions the normal space N from the previous section can be described as follows:

Theorem 2. *Let x be a semisimple representation in $\mathtt{rep}(\mathsf{G}, n)$ with decomposition in simples*

$$x = s_1^{\oplus e_1} \oplus \cdots \oplus s_k^{e_k},$$

and let Q_x be the quiver with

- *vertex set $\{1, \dots, k\}$;*
- *the number of arrows between vertex i and vertex j given by* $\dim \mathrm{Ext}^1_A(s_i, s_j)$.

Let $\alpha_x = (e_1, \dots, e_k)$, then

$$N \cong \mathtt{rep}(Q_x, \alpha_x) \quad \textit{and} \quad H \cong GL_{\alpha_x}(\mathbb{C}),$$

so

$$N /\!\!/ H \cong \mathtt{iss}(Q_x, \alpha_x).$$

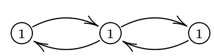
Combining this description with the Luna Slice Theorem, we are able to study étale invariant properties such as the existence of simple representations and smoothness by passing from the representation spaces of G to the representation space of a quiver, where the latter has as a great advantage that a lot of properties can be computed combinatorally.

5. Applications

5.1. **Which $\mathtt{rep}(\mathsf{G}, \delta)$ contain simple representations?** To solve this problem we start with the set of 8 one-dimensional simples. These elementary simples $S_{\xi\eta}$ let X and Y act as scalars ξ, η with $\xi^2 = \eta^4 = 1$ and have dimension vectors $(1, 0|1, 0, 0, 0), \dots, (0, 1|0, 0, 0, 1)$.

As every dimension vector can be seen as a sum of these basic dimension vectors, we can find in every $\mathtt{iss}(\mathsf{G}, \delta)$ a semisimple representation whose factors are these one-dimensional simples. If we fix such a representation R, we can look at its local quiver setting (Q_R, α_R). The fact that simplicity is a Zariski-open condition and $\mathtt{rep}(\mathsf{G}, \delta)$ is irreducible implies that either every (étale) neighborhood of R will contain simples or $\mathtt{rep}(\mathsf{G}, \delta)$ does not contain simples. Therefore $\mathtt{rep}(\mathsf{G}, \delta)$ will contain simple representations if and only if $\mathtt{rep}(Q_R, \alpha_R)$ does.

For instance, the representation space with dimensionvector $\alpha = (2, 1|1, 1, 1, 0)$ contains the representation $S_{11} \oplus S_{1-1} \oplus S_{-1i}$. The local quiver looks like



because one can easily check that the space of extensions between two elementary simples is zero if they have an eigenvalue in common and one-dimensional otherwise.

The local quiver setting has a simple representation that assigns to all arrows a nonzero scalar, hence $\mathtt{rep}(\mathsf{G}, \delta)$ contains simple representations.

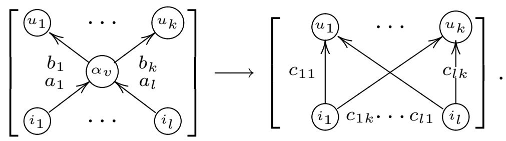
Using the classification of quiver settings with simple representations by Le Bruyn and Procesi in [7], one can obtain a criterium that determines whether $\mathtt{rep}_\alpha G$ contains simples

Theorem 3. [2] *Let $\delta = (a_1, a_2, b_1, b_2, b_3, b_4)$ with $a_1 + a_2 = b_1 + \cdots + b_4 = n$, then $\mathtt{rep}(\mathsf{G}, \delta)$ contains simples if and only if $\forall i, j : a_i + b_j \leq n$.*

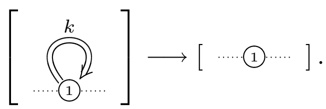
5.2. **Which points in $\mathtt{iss}(\mathsf{G},\delta)$ are smooth?** To check whether a given point is smooth, one only has to check whether the corresponding local quiver setting has a smooth quotient space. This can be done using the technique of reduction moves.

To a local quiver one can apply the following reduction moves which keep the quotient space intact up to a product with an affine space.

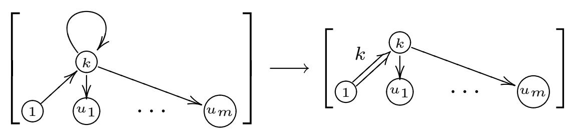
$\mathcal{R}_I$ If $\sum_{j=1}^{k} i_j \leq \alpha_v$ or $\sum_{j=1}^{l} u_j \leq \alpha_v$ we delete the vertex v and connect all the arrows.



$\mathcal{R}_{II}$ Remove the loops on a vertex with dimension 1.

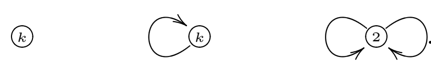


$\mathcal{R}_{III}$ Remove the only loop on a vertex with dimension $k > 1$ which has a neighborhood like the picture below or its dual with the arrows reversed.

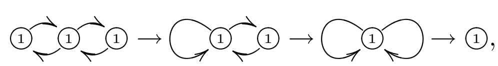


After applying all possible reduction step one obtains a *reduced local quiver.* To check whether its quotient space is smooth one can use the following theorem

Theorem 4. [3] $\mathtt{rep}(Q,\alpha)$ *is smooth if and only if the strongly connected components of the corresponding reduced setting are listed below:*



As an example let us again take the representation $R = S_{11} \oplus S_{1-1} \oplus S_{-1i}$. The local quiver can be reduced as follows:



so the image of R in $\mathtt{rep}(\mathsf{G},\delta)$ is a smooth point.

In general there are only a finite number of local quiver settings that can occur in $\mathtt{rep}(\mathsf{G},\delta)$. By studying all of these one can also obtain a list of dimension vectors whose quotient spaces are smooth.

Theorem 5. [1] *Let* $\delta = (a_1, a_2, b_1, b_2, b_3, b_4)$ *with* $a_1 + a_2 = b_1 + \cdots + b_4 = n$, *then* $\mathtt{iss}(\mathsf{G},\delta)$ *is a smooth space, if up to permutation of the a's and the b's, the dimension vector is one of the following forms*

$$(1,0|1,0,0,0)\ (1,1|1,1,0,0)\ (2,1|1,1,1,0)$$
$$(3,1|1,1,1,1)\ (2,2|2,1,1,0)\ (4,2|2,2,2,0).$$

5.3. **Which singularities can occur?** The notion of reduced quiver settings can also be used in principle to give a list of all possible singularities that can occur in a given dimension. This follows from the fact the for a fixed n there are only a finite number of reduced quivers whose $\mathtt{iss}(Q, \alpha)$ has dimension n. A classification up to dimension 6 has been given in [4].

5.4. **What fibers can occur?** Using the Luna Slice Theorem the question of the fibers can agian be tracked back to calculating the nullcone of the local quiver $\mathtt{Null}(Q, \alpha) = \{W \in \mathtt{rep}(Q, \alpha) | 0 \in \overline{\mathrm{GL}_n \cdot W}\}$. In [6], the authors obtained a list of the local quivers whose nullcone has minimal dimension, which allows one to calculate the fibers with generic dimension.

6. The Future

The methods explained above work not only for the specific group we chose as example but more in general for formally smooth algebras (see [11]). Other examples of these algebras include group algebras of trees of groups, universal localizations of path algebra and free products of coordinate rings of curves.

The framework however breaks down if the representation space is not smooth. To conclude this note, we propose two possible pathways to extend the results to these algebras.

- The first path is to use the Luna Slice theorem for singular points to extend the local description to a broader class of algebras. A first step in this approach was made by W. Crawley-Boevey in [8] in which he obtained a local description of representation spaces of deformed preprojective algebras.
- A second approach is to first blow up the singularities in the representation space and then to take the quotient. The formalism of quivers then extends to quivers with an automorphism action on it. First steps in this direction have been explored by L. Le Bruyn, S. Symens and the first author in [5] and [10].

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