The Flat Locus of Brauer-Severi fibrations of smooth orders

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Abstract

Given a Cayley-Hamilton smooth order A in a central simple algebra Σ, we determine the flat locus of the Brauer-Severi fibration of A. Moreover, we give a classification of all (reduced) central singularities where the flat locus differs from the Azumaya locus and show that the fibers over the flat, non-Azumaya points near these central singularities can be described as fibered products of graphs of projection maps. This generalizes an old result of Artin on the fibers of the Brauer-Severi fibration of a maximal order over a ramified point. Finally, we show these fibers are also toric quiver varieties and use this fact to compute their cohomology.

1 Introduction

To a finitely generated algebra A over $\mathbb{C}$ (or any other algebraically closed field of characteristic zero) one can associate the set of all left ideals of codimension n. This set can be turned into a scheme, called the Brauer-Severi Scheme $\mathtt{BS}_n(A)$.

In [13] M. Van den Bergh showed how one can construct $\mathtt{BS}_n(A)$ using geometric invariant theory. Consider the variety of couples of a representation and a

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Preprint submitted to *20 June 2005*

cyclic vector

$$\mathtt{brauer}_n A := \{(\rho, v) | \rho : A \to \mathrm{Mat}_{n\times n}(\mathbb{C}), \rho(A)v = \mathbb{C}^n\} \subset \mathtt{rep}_n A \times \mathbb{C}^n.$$

To every point (ρ, v) in this variety we can associate an ideal of codimension n, namely the kernel of the map $a \mapsto \rho(a)v$. on the other hand, given an ideal $\mathfrak{m}$ we can choose a basis for $A/\mathfrak{m}$ to obtain a representation and the associated cyclic vector will be $1 \mod \mathfrak{m}$.

All these identifications depend on the choice of a basis, so there is a natural action of GL_n on $\mathtt{brauer}_n A$ given by

$$g.(\rho, v) := (g\rho g^{-1}, gv).$$

The orbits for this action are closed in $\mathtt{brauer}_n A$ but not $\mathtt{rep}_n A \times \mathbb{C}^n$ (see [12]) and we can make the geometrical quotient of this action to obtain the Brauer-Severi variety

$$\mathtt{BS}_n(A) := \mathtt{brauer}_n A/\mathrm{GL}_n.$$

The GL_n-action on $\mathtt{rep}_n A$ on the other hand has only a categorical quotient as not all orbits are closed. We will denote this quotient by $\mathtt{iss}_n A$. It is a well-known fact that it classifies the isomorphism classes of semi simple n-dimensional representations of A, i.e.

$$\mathtt{iss}_n(A) := \mathtt{rep}_n A /\!\!/ \mathrm{GL}_n.$$

Because of the compatibility of the actions of $\mathtt{brauer}_n(A)$ and $\mathtt{rep}_n A$, there is a natural map

$$\pi : \mathtt{BS}_n(A) \to \mathtt{iss}_n A.$$

This fibration carries a lot of information about our original algebra A and therefore it is an interesting question to ask to what extent we can describe the geometry of the map and its fibers.

The construction described above can also be transferred to the setting of orders and more generally Cayley-Hamilton algebras. Recall that a n^{th} Cayley Hamilton algebra is a finitely generated algebra A equipped with a trace, that is a $Z(A)$-linear map $\mathrm{tr} : A \to Z(A)$, such that for all a, b in A

- $\mathrm{tr}(ab) = \mathrm{tr}(ba)$,
- $\mathrm{tr}1 = n$,
- $\chi_{n,a}(a) = 0$.

Where $\chi_{n,a}(X)$ is the n^{th} Cayley-Hamilton identity expressed in the traces of powers of a. The matrix algebra $\mathrm{Mat}_{n\times n}(\mathbb{C})$ with the natural trace is the simplest example of such a Cayley-Hamilton algebra. For a Cayley-Hamilton algebra, we can define the set of trace preserving representations as

$$\mathtt{trep}_n A := \{\rho \in \mathtt{rep}_n A : \mathrm{tr} \circ \rho = \rho \circ \mathrm{tr}\}.$$

which is a closed subset of $\mathtt{rep}_n A$. The quotient $\mathtt{triss}_n A := \mathtt{trep}_n A /\!\!/ \mathrm{GL}_n$ can in this case be identified with the spectrum of the center $Z(A)$. Changing $\mathtt{rep}_n A$ into $\mathtt{trep}_n A$ in the above definition of the Brauer-Severi variety, we obtain the trace preserving Brauer Severi variety, which we will again denote by $\mathtt{BS}_n(A)$, and the corresponding fibration $\pi : \mathtt{BS}_n A \to \mathtt{triss}_n A$. These notions were also introduced in [13].

Orders are special cases of Cayley-Hamilton algebras in the following way. Let R be a commutative Noetherian integrally closed domain with fraction field F and algebraic closure $\bar{F}$. If A is an R-order in the central simple algebra Δ then we can turn A into a Cayley-Hamilton algebra using the trace function of the matrix algebra $\Delta \otimes_R \bar{F}$. On the other hand if A is Cayley-Hamilton algebra such that $\mathtt{triss}_n A$ is an irreducible and normal variety and contains simple representations, then A is an $\mathbb{C}[\mathtt{triss}_n A]$-order in $\Delta = A \otimes_{\mathbb{C}[\mathtt{triss}_n A]} \mathbb{C}(\mathtt{azu}_n A)$, with

$$\mathtt{azu}_n A := \{\rho \in \mathtt{triss}_n A | \rho \text{ is simple}\}.$$

Cayley-Hamilton algebras are quite geometrical in nature as they can be reconstructed from their representation variety as the ring of covariant matrix valued functions (see [10]):

$$A \cong \{f : \mathtt{trep}_n A \to \mathrm{Mat}_{n \times n}(\mathbb{C}) | f(g \cdot x) = g f(x) g^{-1}\}$$

Because of this geometrical aspect, it is natural to call a Cayley-Hamilton algebra (or order) smooth if $\mathtt{trep}_n A$ is a smooth variety. However, the smoothness of $\mathtt{trep}_n A$ does not imply that the quotient is smooth and singularities that occur in $\mathtt{triss}_n A$ will be referred to as central singularities.

From now on we will assume that A is a smooth order. This class of algebras has the advantage that the étale locale structure of the natural maps $\mathtt{trep}_n \to \mathtt{triss}_n A$ and $\mathtt{BS}_n A \to \mathtt{triss}_n A$ can be described using (marked) quiver settings (see [8]). In Section 2 we will recall how this is done and then use these techniques for a closer study of the Brauer Severi fibration.

In Section 3, we describe the flat locus of a Brauer-Severi scheme associated to a smooth order, i.e. the points in $\mathtt{triss}_n A$ for which the fiber has minimal dimension. In Section 4 we determine all possible central singularities for which the Azumaya locus does not coincide with the flat locus and in Section 5 we give a description of the fibers over points in the flat, non-Azumaya locus near such central singularity. As an application we indicate how this description is an extension of a result of Artin in the case of maximal orders over a smooth curve. Finally, in Section 6, we show how these fibers can be seen as toric varieties and we use this to compute their cohomology.

2 Preliminaries

We begin by introducing the notions and results we will need throughout the rest of this paper.

2.1 Definitions and Notations

Definition 1 (Quivers)

- *A* quiver *is a four-tuple* $Q = (Q_0, Q_1, h, t)$ *consisting of a set of* vertices Q_0*, a set of* arrows Q_1 *and two maps* $t : Q_1 \to Q_0$ *and* $h : Q_1 \to Q_0$ *assigning to each arrow its tail resp. its head:*

$$\underset{h(a)}{\circ} \xleftarrow{\;a\;} \underset{t(a)}{\circ} \quad .$$

- *A marked quiver* $Q^\bullet$ *is a quiver together with a subset of the loops (these are the arrows for which* $h(a) = t(a)$*), these loops are called the marked loops and are labeled with a black dot.*
- *A* dimension vector *of a quiver* Q *is a map* $\alpha : Q_0 \to \mathbb{N}$ *and a* quiver setting *is a couple* (Q, α) *of a quiver and an associated dimension vector. The dimension vector which is equal to* 1 *on all vertices is denoted by* $\mathbf{1}$*.*
- *Fix an ordering of the vertices of* Q*. The* Euler form of a quiver Q *is the bilinear form*

$$\chi_Q : \mathbb{N}^{\#Q_0} \times \mathbb{N}^{\#Q_0} \to \mathbb{Z}$$

  defined by the matrix having $\delta_{ij} - \#\{a \in Q_1 \mid h(a) = j, t(a) = i\}$ *as element at location* (i, j)*.*
- *A quiver is called* strongly connected *if and only if each pair of vertices in its vertex set belongs to an oriented cycle.*
- *A path in a quiver setting will be called* quasiprimitive *if it does not run* $n + 1$ *times through a vertex* v *with* $\alpha(v) = n$*. A quasiprimitive path from vertex* v *to vertex* w *is depicted as* $v \rightsquigarrow w$*.*
- *A quiver* Q *is called a* connected sum *of two subquivers* R *and* S *in vertex* v *if* $Q_0 = R_0 \cup S_0$*,* $R_0 \cap S_0 = \{v\}$*,* $Q_1 = R_1 \cup S_1$ *and* $R_1 \cap S_1 = \emptyset$*.*
- *A quiver setting is called* prime *if it is not a connected sum of two quiver settings in a vertex with dimension* 1*, and the prime components of a quiver setting are its maximal prime subquiver settings.*

A quiver setting is graphically depicted by drawing the quiver and listing in each vertex v either the dimension $\alpha(v)$, in which case the vertex is encircled, or the name of the vertex itself.

Definition 2 (Representations of quivers)

- *An α-dimensional representation V of a quiver Q assigns to each vertex $v \in Q_0$ a linear space $\mathbb{C}^{\alpha(v)}$ and to each arrow $a \in Q_1$ a matrix $V(a) \in M_{\alpha(h(a))\times\alpha(t(a))}(\mathbb{C})$. We denote by $\boldsymbol{rep}(Q,\alpha)$ the space of all α-dimensional representations of Q. That is,*

$$\boldsymbol{rep}(Q,\alpha) = \bigoplus_{a\in Q_1} M_{\alpha(h(a))\times\alpha(t(a))}(\mathbb{C}).$$

  In the case of marked quivers we ask that the marked loops are represented by traceless matrices.
- *We have a natural action of the reductive group*

$$GL_\alpha := \prod_{v\in Q_0} GL_{\alpha(v)}(\mathbb{C})$$

  on a representation V defined by base change in the vector spaces. That is

$$(g_v)_{v\in Q_0}.(V(a))_{a\in Q_1} = (g_{h(a)}V(a)g_{t(a)}^{-1})_{a\in Q_1}.$$

- *The quotient space with respect to this action classifies all isomorphism classes of semi simple representations and is denoted by $\boldsymbol{iss}(Q,\alpha)$. The quotient map with respect to this action will be denoted by*

$$\pi_Q : \boldsymbol{rep}(Q,\alpha) \twoheadrightarrow \boldsymbol{iss}(Q,\alpha).$$

- *The fiber of π_Q in $\pi_Q(0)$ is called the* nullcone *of the quiver setting and is denoted by $\boldsymbol{Null}(Q,\alpha)$.*

2.2 The étale Local Structure of the Brauer-Severi Fibration

In this section we briefly recall how (marked) quivers can be used to describe the local structure of a smooth order and its Brauer-Severi fibration. The results stated in this section are taken from [8].

We want to determine the étale locale structure near a point $p \in \mathtt{triss}_n A$. The point p corresponds to a semi simple representation with a decomposition in simple representations

$$S_1^{\oplus e_1} \oplus \cdots \oplus S_k^{\oplus e_k}.$$

From this decomposition we construct the following data.

- A (marked) quiver Q_p with k vertices indexed by the different simple component. The number of (marked) arrows from the i^{th} to the j^{th} can be determined from the normal space $T_p\mathtt{rep}_n A/T_p\mathrm{GL}_n \cdot p$ (for a more detailed construction see [8]).
- A dimension vector α_p assigning e_i to the i^{th} vertex.
- A dimension vector γ_p assigning $\dim S_i$ to the i^{th} vertex.

From this data we obtain an tale local description of $\mathtt{triss}_n A$.

Theorem 1 *There is an étale neighborhood of $p \in \mathtt{triss}_n A$ that is isomorphic to an étale neighborhood of the zero in $\mathtt{iss}_\alpha Q$.*

We will call (Q_p, α_p) the *local quiver setting of* p and $(Q_p, \alpha_p, \gamma_p)$ the *local quiver data of* p.

To describe the Brauer Severi fibration we modify Q_p, α_p and γ_p in the following way:

- To Q_p we add a new vertex v_0 and $\gamma_p(v)$ arrows from v_0 to v for each $v \in (Q_p)_0$. The new quiver is denoted by $\tilde{Q}_p$.
- We extend α_p to $\tilde{\alpha}_p$ by giving v_0 dimension one.
- We extend γ_p to a character θ_p of $\mathrm{GL}_{\tilde{\alpha}_p}$ by putting $\theta_p(v_0) = -n$ and $\theta_p|_{Q_0} = \gamma_p$.

The character can now be used to define θ_p-semistable representations:

Definition 3 *An α_S-dimensional representation S of $\tilde{Q}_p$ is called θ_p-semistable if and only if $\alpha_S \cdot \theta_p = 0$ and for all subrepresentations $T \subset S$ we have that $\alpha_T \cdot \theta_p \leq 0$.*

The open subvariety of θ_p-semistable representations of dimension $\tilde{\alpha}_p$ will be denoted by $\mathtt{ress}_{\theta_p}(Q_p, \tilde{\alpha}_p)$.

Theorem 2 *Given a Brauer-Severi scheme $\mathtt{BS}(A)$ and a point $p \in \mathtt{triss}_n A$, we have*

$$\pi^{-1}(p) = (\mathtt{Null}(\tilde{Q}_p, \tilde{\alpha}_p) \cap \mathtt{ress}_{\theta_p}(\tilde{Q}_p, \tilde{\alpha}_p))/GL_{\tilde{\alpha}_p}$$

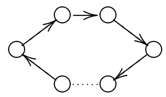
Moreover, the dimension vector α_p is such that there exist simple representations in $\mathtt{rep}(Q_p, \alpha_p)$.

Remark. As we are only considering representations in the nullcone (all traces are zero), the distinction between marked and unmarked loops is superfluous and will be omitted.

This theorem reduces the study of the fibers of the Brauer-Severi fibration to the study of moduli spaces of nullcones of quiver settings that have simple representations. In [7], a criterion for the existence of simple representations of dimension vector α was given.

Theorem 3 *Let (Q, α) be a quiver setting such that for all vertices v we have that $\alpha(v) \geq 1$. There exist simple representations of dimension vector α if and only if*

- *Q has exactly one vertex, at most one loop and $\alpha = 1$;*
- *Q is of the extended Dynkin form $\tilde{A}_n$*



  with $\alpha = \mathbf{1}$;
- *none of the above, but Q is strongly connected and*

$$\forall v \in Q_0 : \chi_Q(\alpha, \varepsilon_v) \leq 0 \text{ and } \chi_Q(\varepsilon_v, \alpha) \leq 0.$$

  Here $\varepsilon_v(w) := \delta_{vw}$ for all $w \in Q_0$.

If $\alpha(v) = 0$ for some vertices v, (Q, α) has simple representations if (Q', α') has simple representations, where (Q', α') is the quiver obtained by removing all vertices v with $\alpha(v) = 0$.

3 The Flat Locus of the Brauer-Severi Fibration

In this section, we will determine the flat locus of the Brauer-Severi fibration

$$\mathtt{BS}(A) \twoheadrightarrow X,$$

using the quiver description recalled in Section 2. We have

Theorem 4 *Let (Q, α, γ) be the local quiver data of a point $\xi \in X$, then*

$$\dim \pi^{-1}(\xi) = \dim \mathtt{Null}(Q, \alpha) + n - \dim GL(\alpha).$$

Proof. Let $\gamma = (d_1, \ldots, d_k)$ and let θ be the corresponding character for $\tilde{Q}$, then by Theorem 2 we know that

$$\pi^{-1}(\xi) = (\mathtt{Null}(\tilde{Q}, \tilde{\alpha}) \cap \mathtt{rep}_\theta(\tilde{Q}, \tilde{\alpha}))/\mathrm{GL}(\tilde{\alpha}).$$

Also

$$\mathtt{Null}(\tilde{Q}, \tilde{\alpha}) = \mathtt{Null}(Q, \alpha) \times \mathbb{A}^n$$

because any choice of a representation for an arrow with tail v_0 belongs to the nullcone by a straightforward application of Hilbert's criterion. Now for any irreducible component C of $\mathtt{Null}(Q,\alpha)$ we have that $\mathtt{rep}_\theta(\tilde{Q},\tilde{\alpha}) \cap (C \times \mathbb{A}^n) \neq \emptyset$ only contains θ-stable representations which have stabilizer $\mathbb{C}^*$ so when C is an irreducible component of maximal dimension we obtain

$$\begin{aligned}\dim(\mathtt{Null}(\tilde{Q},\tilde{\alpha}) \cap \mathtt{rep}_\theta(\tilde{Q},\tilde{\alpha}))/\mathrm{GL}(\tilde{\alpha}) &= \dim C + n \\ &\quad -(\dim \mathrm{GL}(\tilde{\alpha}) - 1) \\ &= \dim \mathtt{Null}(Q,\alpha) + n - \dim \mathrm{GL}(\alpha).\end{aligned}$$

□

Let us recall the definition of a cofree quiver setting from [3].

Definition 4 *A quiver setting (Q,α) is called* cofree *if its quotient space $\mathtt{iss}(Q,\alpha)$ is smooth and its nullcone $\mathtt{Null}(Q,\alpha)$ has minimal dimension i.e.*

$$\dim \mathtt{Null}(Q,\alpha) = \dim \mathtt{rep}(Q,\alpha) - \dim \mathtt{iss}(Q,\alpha).$$

We have

Corollary 5 *A point $\xi \in X$ belongs to the flat locus of the Brauer-Severi fibration of X if and only if its corresponding local quiver setting is cofree.*

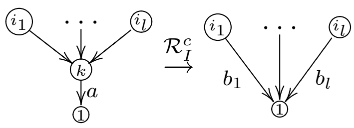
Proof. The fibers of the Brauer-Severi fibration over an Azumaya point are isomorphic to $\mathbb{P}^{n-1}$ so have dimension $n-1$. This means that in order to have minimal dimension we must have $\dim \pi^{-1}(\xi) = n-1$. This is exactly the case when

$$\dim \mathtt{Null}(Q,\alpha) = \dim \mathrm{GL}(\alpha) - 1 = \dim \mathtt{rep}(Q,\alpha) - \dim \mathtt{iss}(Q,\alpha).$$

In combination with the fact that the flat locus of the Brauer-Severi fibration is contained within the smooth locus of the Brauer-Severi fibration by an application of the Popov conjecture for quiver representations (see [14]) we obtain the claim. □

In combination with Theorem 2, this means that in order to find the flat locus of the Brauer-Severi fibration of X, we have to classify all cofree quiver representations that have simple representations.

A characterization of cofree representations was given in [3]. By *reduction step* $\mathcal{R}_I^c$ we mean the construction of a new quiver from a given quiver by removing a vertex (and connecting all arrows) in the situation illustrated below, where k is not smaller than the number of quasiprimitive cycles through ⓚ.



We now have

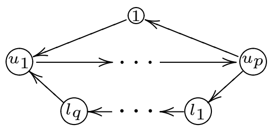
Theorem 5 *A quiver setting (Q, α) is cofree if and only if it can be reduced using $\mathcal{R}_I^c$ to a setting whose prime components are in the list below.*

(i) strongly connected quiver settings (P, ρ) for which

(1) There is a vertex $v \in P_0$ such that $\rho(v) = 1$ and through which all cycles run,

(2) $\forall w \neq v \in P_0 : \rho(w) \geq \#\{\, v \rightsquigarrow w \,\} + \#\{\, v \leftsquigarrow w \,\} - 1$,

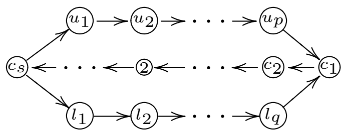
(ii) quiver settings (P, ρ) of the form



with $p \geq 1, q \geq 0$, such that there is at most one vertex x in the path $u_1 \rightsquigarrow u_p$ which attains the minimal dimension $\min\{u_1, \ldots, u_p, l_1, \ldots, l_q\}$.

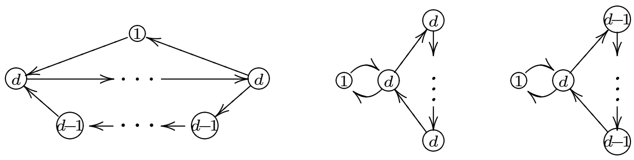
(iii) quiver settings of extended Dynkin type $\tilde{A}_n$ with cyclic orientation

(iv) quiver settings (P, ρ) of the form

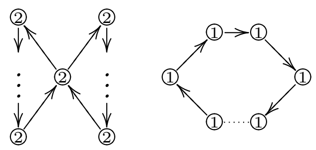


with $p, q \geq 0$ and $u_i, l_j \geq 2$ for all $1 \leq i \leq p$, $1 \leq j \leq q$ and all $c_k \geq 4$ except for a unique vertex with dimension 2.

Theorem 6 *A point $\xi \in X$ belongs to the flat locus of the Brauer-Severi fibration of X if and only if the prime components of its local quiver are of the form*



(1)



(2)

Proof. As ξ belongs to the flat locus of the Brauer-Severi fibration, we know its local quiver setting must have simple representations. Now note that for a dimension vector to have simple representations, the vertex k in reduction step $\mathcal{R}_I^c$ must have dimension $k = 1$. This means there runs only one quasiprimitive path through this vertex, which is only the case if the quiver setting to which the vertex belongs to, is a connected sum of cyclic quivers with dimension vector $\mathbf{1}$. All other cofree quiver settings with simple representations cannot be reduced by reduction step $\mathcal{R}_I^c$ and hence must have prime components as described in Theorem 5.

The condition $\chi_Q(\alpha, \varepsilon_v) \leq 0$ and $\chi_Q(\varepsilon_v, \alpha) \leq 0$ means that for any given vertex v,

$$\sum_{a \in Q_1, h(a)=v} \alpha(t(a)) \leq \alpha(v) \geq \sum_{a \in Q_1, t(a)=v} \alpha(h(a)). \quad (*)$$

Consider the prime components of Theorem 5 described in (iv). Only a unique $\textcircled{c_i}$ can have dimension 2, so $s = 1$ by condition $(*)$ and by the same condition $u_i = 2$ and $l_j = 2$ for all i, j. This yields a connected sum of cyclic quivers with dimension vector $2.\mathbf{1}$. The component described in (iii) (the cyclic quiver $\tilde{A}$) must obviously have dimension vector $\mathbf{1}$ by Theorem 3. The same argument used for the components listed in (iv) applies to the components described in (ii), which means they only have simple representations when they are of the forms in (1).

Finally, consider the components described in (i). Let v be a vertex with $\alpha(v) = 1$ through which all cycles run. Because of this condition, there is a vertex w_0 that has only incoming arrows from v. Should $\alpha(w_0) > 1$, the existence of simple representations implies there are $\alpha(w_0)$ arrows from v to w. The second condition on w then implies there is only one arrow leaving, and this to a vertex w_1 with dimension $\alpha(w_1) = \alpha(w)$. As all cycles run through v, this vertex also has only one arrow leaving to a vertex w_2 with dimension $\alpha(w_2) = \alpha(w)$. This argument can be repeated until we find a vertex w_k that is connected to v with exactly one arrow, so all vertices w_i for $0 \leq i \leq k$ must have $\alpha(w_i) = 1$. This means the quiver setting has a prime component that is a cyclic quiver setting with dimension vector $\mathbf{1}$. Removing this prime component then yields by induction on the number of vertices that the quiver setting we started with was a connected sum of cyclic quivers with dimension vector $\mathbf{1}$. $\square$

4 Central Singularities and the Flat Locus

In this section, we give a classification of all reduced (in the sense of [2]) central singularities where the flat locus of the Brauer-Severi fibration does not coincide with the Azumaya locus. This translates into finding all reduced non cofree quiver settings (Q, α), which have a local quiver setting that is cofree. Recall from [7] the construction of a local quiver setting (Q_p, α_p) from a quiver (Q, α): for a given semi simple representation

$$S_1^{\oplus a_1} \oplus \ldots \oplus S_k^{\oplus a_k}$$

of the quiver (Q, α), where S_i has dimension vector a_i (and a combinatorial property of simplicity described in Theorem 3), we construct the local quiver (Q_p, α_p) by the following data:

- $(Q_p)_0$ consist of k vertices.
- The number of arrows between vertices u and v is given by $\delta_{uv} - \chi_Q(\alpha_u, \alpha_v)$.
- $\alpha_p := (a_1, \ldots, a_k)$.

We will use this combinatorial description to obtain the desired classification. From now on, we will call a quiver setting simple if it has simple representations.

The following is a characterization of reduced quiver settings (with at least 2 vertices):

Definition 6 *A strongly connected quiver setting (Q, α) with at least 2 vertices is called reduced if and only if*

- *every vertex t with no loop has $\chi_Q(\alpha, \varepsilon_t) \leq -1$ and $\chi_Q(\varepsilon_t, \alpha) \leq -1$.*
- *every vertex t with one loop has $\chi_Q(\alpha, \varepsilon_t) \leq -2$ and $\chi_Q(\varepsilon_t, \alpha) \leq -2$.*
- *all vertices with dimension 1 do not have loops.*

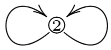
Lemma 1 *A non cofree quiver setting with a cofree local quiver has at least 2 vertices.*

Proof. From [3] we easily deduce that a non cofree quiver setting with one vertex has at least 2 loops and dimension at least 3 or at least 3 loops and dimension at least 2. In both cases one easily verifies that too many arrows appear in the local quivers. □

From now on we assume (Q, α) is a reduced quiver setting with at least 2 vertices (this implies that (Q, α) is not cofree). We first determine the possible local quiver settings.

Lemma 2 *A cofree local quiver setting of (Q, α) can never have 2 or more loops at a vertex v with dimension $\alpha(v) = k \geq 2$.*

Proof. The only situation of Theorem 6 where two loops occur is



(3)

Suppose one can find a reduced quiver setting (Q, α) with n vertices such that it has (3) as a local quiver. Then we know that 2 divides α_i for all i. Since (Q, α) is reduced, $-\chi(\alpha, \varepsilon_i) \geq 1$ and $-\chi(\alpha/2, \varepsilon_i) \geq 1$. But then the number of loops of the local quiver is

$$1-\chi_Q(\frac{\alpha}{2},\frac{\alpha}{2})=1-\sum_{i=1}^{n}\frac{\alpha}{2}\chi_Q(\frac{\alpha}{2},\varepsilon_i)\geq 3.$$

□

Lemma 3 *(Q,α) is simple and $\chi_Q(\alpha,\alpha)=0$ if and only if $Q=\tilde{A}_n$ and $\alpha=(1,\dots,1)$.*

Proof. By linearity of χ_Q, it is clear that $\chi_Q(\alpha,\alpha)=0$ is equivalent to $\chi_Q(\alpha,\varepsilon_t)=0$ for all t. Looking at the vertex with minimal dimension, one easily deduces that all dimensions must be the same. The only strongly connected quiver allowing this is an $\tilde{A}_n$. The simplicity condition gives us dimension vector $(1,\dots,1)$. □

Theorem 7 *Suppose (Q,α) has a cofree local quiver setting (Q_p,α_p), then (Q_p,α_p) is always a connected sum of $\tilde{A}_i$, where every vertex has dimension 1 (and a number of loops).*

Proof. Suppose we have a cofree local quiver with a vertex of dimension $k\geq 2$. This vertex v corresponds to a simple quiver setting (Q_v,α_v), with Q_v a subquiver of Q and α_v a sub-dimension vector of α, nonzero on Q_v and zero elsewhere. From Lemma 2 and the fact that the number of loops in vertex v is given by

$$1-\chi_Q(\alpha_u,\alpha_u)$$

(and therefore is at least 1), we know that vertex v has exactly one loop. This also implies by Lemma 3 that $(Q_u,\alpha_u)=(\tilde{A}_n,\mathbf{1})$. The number of non-loop arrows arriving in vertex v is given by

$$\begin{aligned}&\sum_{w\neq v}-\chi_Q(\alpha_w,\alpha_v)\\&=-\chi_Q(\alpha-k\alpha_v,\alpha_v)\\&=-\chi_Q(\alpha,\alpha_v)+k\underbrace{\chi_Q(\alpha_v,\alpha_v)}_{=0}\\&=-\sum_{t\in(Q_v)_0}\chi_Q(\alpha,\varepsilon_t)\alpha_v(t)\geq 2\end{aligned}$$

where the last inequality holds because (Q,α) is reduced. A situation with one loop and 2 more incoming arrows in a vertex with dimension greater than k can never be cofree by Theorem 6.

We find that all vertices of the local quiver setting must have dimension 1. The only possibilities left are, according to Theorem 6, connected sums of $\tilde{A}_i$

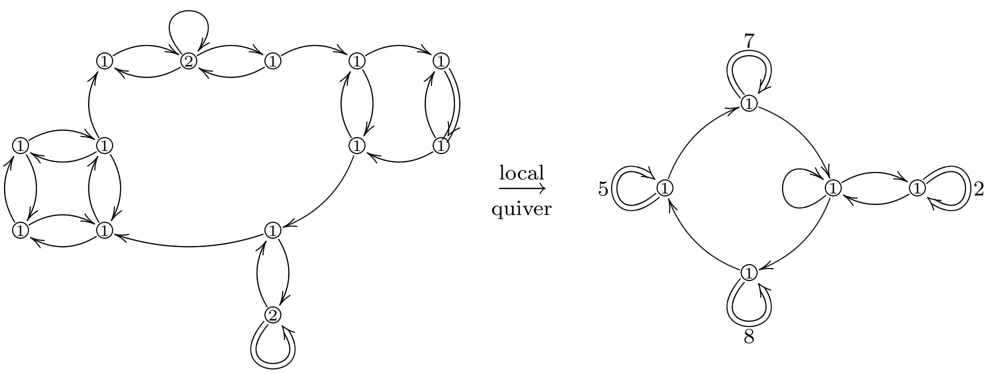
having dimension 1 on each vertex. $\square$

Now that we have found all possible local quiver settings, let us look at all possible reduced quiver settings that have a connected sum of $\tilde{A}_i$, with dimension vector $\mathbf{1}$ as local quiver.

Theorem 8 *Let (Q, α) be a reduced quiver setting with a local quiver (Q_p, α_p) given by a connected sum of $\tilde{A}_i$ with dimension vector $\mathbf{1}$ and n_v loops on each vertex v.*

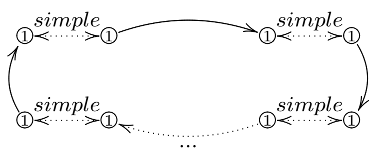
Then (Q, α) is of the following shape: take (Q_p, α_p) and replace every vertex v and its loops by a simple quiver setting (Q_v, α_v) with $1 - \chi_{Q_v}(\alpha_v, \alpha_v) = n_v$, in such a way that the $\tilde{A}_i$-arrows only start and end in a vertex with dimension 1.

We illustrate this theorem with an example:



Proof. We first show that the quivers described in the theorem have a local quiver that is a connected sum of $\tilde{A}_i$ with dimension 1 on every vertex.

Assume (Q, α) of the form



with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ where α_j is the dimension vector of the simple subquiver placed at vertex j of $\tilde{A}_n$.

The Euler form of this quiver setting is given by the block matrix

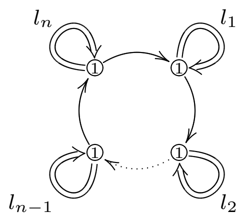
$$\begin{pmatrix} \boxed{A_1} & \boxed{\epsilon_{21}} & 0 & 0 & \cdots & 0 \\ 0 & \boxed{A_2} & \boxed{\epsilon_{32}} & 0 & \cdots & 0 \\ 0 & 0 & \boxed{A_3} & \boxed{\epsilon_{43}} & \cdots & 0 \\ 0 & 0 & 0 & \boxed{A_4} & \ddots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \boxed{\epsilon_{n,n-1}} \\ \boxed{\epsilon_{1n}} & 0 & 0 & 0 & 0 & \boxed{A_n} \end{pmatrix}$$

where A_j consists of the Euler matrix of the simple quiver at vertex j of $\tilde{A}_n$ and ϵ_{ij} is the matrix with -1 as entry (i,j) and zeroes elsewhere.

If we construct the local quiver by splitting the dimension vector α in n components

$$(\alpha_1, 0, \ldots, 0) \oplus (0, \alpha_2, 0, \ldots, 0) \oplus \ldots \oplus (0, \ldots, 0, \alpha_n).$$

we get a local quiver that has n vertices and, with exception of loops, the only arrows occurring are exactly the same as in the $\tilde{A}_n$. This results in



which is the first case of the cofree simple quivers of Theorem 6.

More generally, we start with a connected sum of $\tilde{A}_i$, where every vertex is replaced by a simple subquiver and $\tilde{A}_i$-arrows always start and end in a vertex with dimension 1. We construct a local quiver in the same way as above. The simple subquivers will be packed in 1 vertex and for the same reason as above, we get a local quiver that looks exactly the same as the original, replacing the simple subquivers by vertices with dimension 1, with a number of loops. We find a connected sum of cyclic quivers.

Remains to show that these quiver settings are the only possible reduced quiver settings that have a cofree local quiver. This result is an immediate consequence of the fact that it is impossible for 2 simple components of the local quiver to have a common vertex with nonzero dimension.

To see that it is impossible to have a common vertex, we use, as before, only reduced quiver settings (Q, α) with at least 2 vertices. Suppose (Q, α) has a local quiver with at least 2 components that overlap in a vertex $t \in Q_0$ (there exist components u and v with $\alpha_u(t), \alpha_v(t) \geq 1$). We will show that the local quiver (Q_p, α_p) always has a subquiver of the form

(i) $$\begin{tikzcd} 1 \arrow[r, Rightarrow] & 1 \end{tikzcd}$$ (ii) $$\begin{tikzcd} & 1 \arrow[d] \\ 1 \arrow[ur] \arrow[dr] & \\ & 1 \end{tikzcd}$$ (iii) $$\begin{tikzcd} 1 \arrow[r] \arrow[dr] & 1 \\ 1 \arrow[r] \arrow[ur] & 1 \end{tikzcd}$$

(where in (iii) the two left vertices may coincide) and therefore the local quiver cannot be a connected sum of $\tilde{A}_i$'s.

For a set of vertices of the local quiver, say Ω, we define the function

$$\kappa_t(\Omega) := \sum_{u \in \Omega} -\chi_Q(\alpha_u, \varepsilon_t).$$

If we take $\Omega = (Q_p)_0$, by Definition 6, $\kappa_t(\Omega) \geq 1$ and $\kappa_t(\Omega) \geq 2$ if there is at least 1 loop in t. If we remove vertices u from Ω with $-\chi_Q(\alpha_u, \varepsilon_t) \leq 0$, $\kappa_t(\Omega)$ still has the same lower bound.

Our next objective is to remove vertices from Ω in such a way that Ω has exactly 2 vertices u_i with $\alpha_{u_i}(t) > 0$.

We start with $\Omega = (Q_p)_0$ and remove vertices u with $-\chi_Q(\alpha_u, \varepsilon_t) \leq 0$ and $\alpha_u(t) > 0$ from Ω. We stop removing vertices if there are only 2 components u_i with $\alpha_{u_i}(t) > 0$ left in Ω. If it is not possible to reach an Ω with 2 components u_i with $\alpha_{u_i}(t) > 0$ by this procedure, we just remove vertices until no vertex w with $\chi_Q(\alpha_w, \varepsilon_t) \leq 0$ and $\alpha_w(t) > 0$ exists in Ω. We call the resulting set Ω_t.

If Ω_t has at least 3 components u_i with $\alpha_{u_i}(t) > 0$, each of these u_i has $-\chi_Q(\alpha_{u_i}, \varepsilon_t) \geq 1$, and the number of arrows between u_i and u_j is given by

$$-\chi_Q(\alpha_{u_i}, \alpha_{u_j}) = \sum_{s \in Q_0} -\chi_Q(\alpha_{u_i}, \varepsilon_s)\alpha_{u_j}(s) \geq -\chi_Q(\alpha_{u_i}, \varepsilon_t)\alpha_{u_j}(t) \geq 1$$

where the first inequality is obtained by simplicity of the components of the local quiver. This result in a subquiver of Q_p looking like (ii).

If Ω_t has 2 such components u and v, three situations may occur, but they all will lead to a situation (i), (ii) or (iii):

A. $(Q_u)_0 \not\subset (Q_v)_0$ and $(Q_v)_0 \not\subset (Q_u)_0$. The number of arrows from u to v is given by

$$-\sum_{s \in (Q_u)_0 \cap (Q_v)_0} \chi_Q(\alpha_u, \varepsilon_s)\alpha_v(s) - \sum_{s \in (Q_v)_0 \setminus (Q_u)_0} \chi_Q(\alpha_u, \varepsilon_s)\alpha_v(s) \qquad (4)$$

By simplicity of (Q_u, α_u), the first sum is always ≥ 0, and because there exist vertices in $(Q_v)_0 \setminus (Q_u)_0$, the second sum is always at least 1. So we have at least 1 arrow from u to v and, by the same argument, we have an arrow in the other direction. Since $\kappa_t(\Omega_t) \geq 1$, we have

$$-\chi_Q(\alpha_u, \varepsilon_t) - \chi_Q(\alpha_v, \varepsilon_t) + \sum_{w \in \Omega_t \setminus \{u,v\}} \chi_Q(\alpha_w, \varepsilon_t) \geq 1 \tag{5}$$

If the first term of (5) is at least one we get an extra arrow from u to v due to the first sum of (4), so we are left with $u \Rightarrow v$ in our local quiver. If the second term is at least one we get $v \Rightarrow u$ in our local quiver. If the third term is at least one, we get the situation (ii).

B. $(Q_u)_0 \subsetneq (Q_v)_0$. If Q_u has exactly one vertex, the number of vertices between v and u is given by $-\chi_Q(\alpha_v, \varepsilon_t)\alpha_u(t)$ and the condition on $\kappa(\Omega_t)$ translates to

$$-\chi_Q(\alpha_v, \varepsilon_t) + \sum_{w \in \Omega_t \setminus \{u,v\}} \chi_Q(\alpha_w, \varepsilon_t) \geq 2$$

(this is independent of the number of loops in t) and we see that all possible situations lead to a subquiver of (Q_p) of the form (i), (ii) or (iii).

If Q_u has at least 2 vertices, we can look at equation (4) from situation A, for arrows from u to v. We also use (5), to see we get another arrow from u tot v or vice versa, or an arrow to u and v from another vertex w. Let us now look at another vertex t_1 of $(Q_u)_0 \cap (Q_v)_0$. If here too we have $\kappa_{t_1}(\Omega_t) \geq 1$ this leads to another arrow from v to u. If $\kappa_{t_1}(\Omega_t) = 0$, there exists one vertex w' of the local quiver which we have removed while constructing Ω_t. This vertex has $-\chi_Q(\alpha_{w'}, \varepsilon_{t_1})$ and this leads to an arrow from w' to both u and v. Again we have a subquiver of (Q_p) of the form (i), (ii) or (iii).

C. $(Q_u)_0 = (Q_v)_0$. If $(Q_u)_0$ has 1 vertex, the same argument as in B holds. Suppose $(Q_u)_0$ has at least 2 vertices. Again, relation (5) gives us another arrow from u tot v or the other way round, or an arrow to u and v from another vertex w. Using the same strategy as in B, we get, for another vertex t_1 in the intersection, another arrow. If the intersection consist of 3 or more vertices, we get situation (i) - (iii). The last possibility to consider is the case where we have 2 vertices t, t_1 in the intersection. By the same arguments as before t and t_1 each give one arrow u to v or a reverse arrow, or an arrow from w to u and v. All cases gives (i) to (iii), except when we have exactly one arrow from u to v and vice versa. However, this is impossible because it yields $-\chi_Q(\alpha_u, \varepsilon_t) \geq 1$, $-\chi_Q(\alpha_v, \varepsilon_t) = 0$, $-\chi_Q(\alpha_v, \varepsilon_{t_1}) \geq 1$ and $-\chi_Q(\alpha_v, \varepsilon_{t_1}) = 0$, and the only way to obtain this is that (Q_u, α_u) or (Q_v, α_v) are not simple quiver settings. □

5 The Brauer-Severi Fibration over the Flat Locus

From the previous section, we know the only flat, non-Azumaya settings that can occur near a central singularity are cyclic quiver settings with dimension vector $\mathbf{1}$. In this section, we give a description of the fibers of the Brauer-Severi fibration over these points in the flat locus.

Lemma 4 *Let (Q, α) be a quiver setting that is a connected sum of k cyclic quivers $\tilde{A}_{n_i}$, $1 \leq i \leq k$ with dimension vector $\alpha = \mathbf{1}$,*

$$Q = \tilde{A}_{n_1} \#_{v_1} \dots \#_{v_{k-1}} \tilde{A}_{n_k}$$

then $\mathtt{Null}(Q, \alpha)$ has $(n_1 + 1). \dots .(n_k + 1)$ irreducible components T_i, each of which is a tree that is a connected sum of k quivers of type A_{n_i+1}.

Proof. We know that a representation V lies in the nullcone if and only if all traces along oriented cycles in Q become zero. This condition is equivalent to choosing at least one arrow in each component that gets assigned to 0 by V. But then choosing one arrow in each component gives a closed irreducible subset of the nullcone, and permuting the arrow chosen to be zero yields a covering of the nullcone by irreducible subsets, none of which lies in the union of the others. □

Now let $\xi \in \mathtt{flat}(Q, \alpha)$ with a local quiver setting $(Q_\xi, \mathbf{1})$ that is a connected sum of cyclic quivers and with $\gamma_\xi = (d_1, \dots, d_X)$. By Theorem 2 we know that each irreducible component C of $\pi^{-1}(\xi)$ is described by the moduli space $\mathtt{moss}_\theta(\tilde{T}, \mathbf{1})$ where T is an irreducible component of $\mathtt{Null}(Q_\xi, \mathbf{1})$ and hence by the previous lemma a tree. In the remainder of this section we will describe these components. In order to do so, we need some additional definitions.

Definition 7 *Let $(T, \mathbf{1})$ be a quiver setting of which the underlying quiver is a tree.*

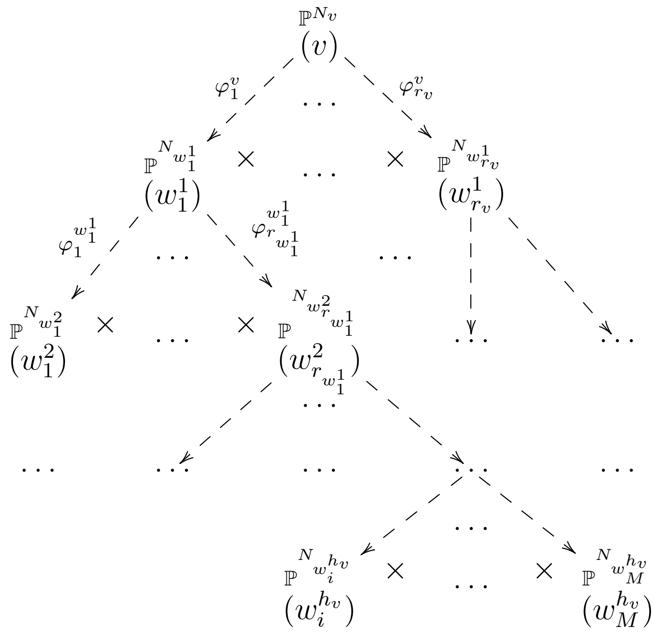
(1) A vertex $v \in T_0$ is called a root vertex *if it is a sink, that is, there are no arrows $a \in T_1$ such that $t(a) = v$.*
(2) A rooted tree *is a quiver T that is a tree and for which there exists a unique root vertex.*
(3) For a root vertex v, we denote by $T(v)$ the maximal rooted subtree of T with root vertex v.
(4) For a vertex w in $T(v)$, we let the root distance to v *be the length of the path connecting w to v, and denote this by $\mathcal{D}_v(w)$.*
(5) For a rooted subtree $T(v)$ we let $h_v := \max\{\mathcal{D}_v(w) \mid w \in T(v)_0\}$ and call this the height *of the subtree $T(v)$.*

Now let v be a root vertex in the irreducible component T of $\mathtt{Null}(Q_\xi, \mathbf{1})$, then we will assign a graph to $T(v)$ as follows. Assign to each vertex $w \in T(v)_0$ the projective space $\mathbb{P}^{N_w}$ with

$$N_w = \sum_{\text{(w)} \leftsquigarrow \text{(u)}} d_u - 1.$$

For each vertex w in $T(v)$ fix an ordering on the arrows entering w, denoting them $a_1^w, \dots a_{r_w}^w$. Denote the vertices at root distance s by w_i^s with i numbering the vertices at the fixed root distance s, grouping tails of arrows with the same head together. We now construct a series of projection maps

$$\begin{tikzcd}
\mathbb{P}^{N_v} \arrow[d, dashed, "\pi_1"] \\
\prod_{w \in Q_p(v)_0, \mathcal{D}_v(w)=1} \mathbb{P}^{N_w} \arrow[d, dashed, "\pi_2"] \\
\prod_{w \in Q_p(v)_0, \mathcal{D}_v(w)=2} \mathbb{P}^{N_w} \arrow[d, dashed, "\pi_3"] \\
\dots \arrow[d, dashed, "\pi_{h_v}"] \\
\prod_{w \in Q_p(v)_0, \mathcal{D}_v(w)=h_v} \mathbb{P}^{N_w}
\end{tikzcd}$$



where

$$\pi_i = \prod_{w \in Q_p(v)_0, \mathcal{D}_v(w)=i-1} \pi_i^w$$

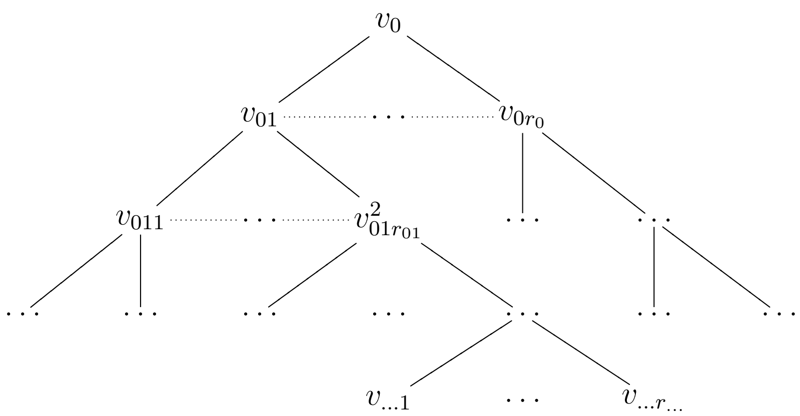
with

$$\pi_i^w = \prod_{j=1}^{r_w} \varphi_j^w$$

where φ_j^w is the projection on the projective coordinates numbered from $1 + \sum_{k=1}^{j-1}(N_{a_k^w} + 1)$ to $\sum_{k=1}^{j}(N_{a_k^w} + 1)$. We will denote the graph of this collection of rational maps by $\Gamma(v)$.

Let v and w be two root vertices, then their rooted subtrees must have a common subquiver, denoted by $T(v) \cap T(w)$ which is again a rooted tree. Denote the root vertex of this tree by $v \cap w$. We then have

Theorem 9 *Let $\xi \in \mathtt{flat}(Q, \alpha)$ with local quiver data $(Q, \mathbf{1}, \gamma)$ where Q is a connected sum of cyclic quivers. Let $(T, \mathbf{1})$ be the quiver setting of an irreducible component of $\mathtt{Null}(Q, \mathbf{1})$. Now let the rooted subtrees of T be connected* *as follows (we list the roots of the rooted subtrees):*



That is, $T(v_0)$ *has common subgraphs with* $T(v_{0i})$ *for* $1 \leq i \leq r_0$ *but not with any other* $T(w)$ *for* $w \notin \{v_{01}, \dots, v_{0r_0}\}$*, likewise for* $T(v_{01})$*, and so on. Now let*

$$\mathcal{F}_0 = (\dots((\Gamma(v_0) \times_{\Gamma(v_0 \cap v_{01})} \Gamma(v_{01})) \times_{\Gamma(v_0 \cap v_{02})} \Gamma(v_{02}) \dots) \times_{\Gamma(v_0 \cap v_{0r_0})} \Gamma(v_{0r_0})),$$

$$\mathcal{F}_1 = (\dots((\mathcal{F}_0 \times_{\Gamma(v_{01} \cap v_{011})} \Gamma(v_{011})) \times_{\Gamma(v_{01} \cap v_{012})} \Gamma(v_{012}) \dots) \times_{\Gamma(v_{0r_0} \cap v_{0r_0r_{0r_0}})} \Gamma(v_{0r_0r_{0r_0}})),$$

$$\dots$$

$$\mathcal{F}_n = (\dots((\mathcal{F}_{n-1} \times_{\Gamma(v_{01\dots1} \cap v_{01\dots11})} \Gamma(v_{01\dots11})) \times_{\Gamma(v_{01\dots1} \cap v_{01\dots12})} \Gamma(v_{01\dots12}) \dots)$$
$$\times_{\Gamma(v_{0r_0} \cap v_{0r_0r_{0r_0}\dots r_{0r_0r_{0r_0}\dots}})} \Gamma(v_{0r_0r_{0r_0}\dots r_{0r_0r_{0r_0}\dots}})),$$

then the irreducible component of $\pi^{-1}(p)$ *corresponding to* T *is equal to* $\mathcal{F}_n$*.*

Proof. We must describe $\texttt{moss}_\theta(\tilde{T}, \tilde{\mathbf{1}})$. We will first describe $\texttt{ress}_\theta(\tilde{T}, \tilde{\mathbf{1}})$ and then see what the action of $\mathrm{GL}(\tilde{\mathbf{1}})$ on this subspace of semistable representations does. First of all, let v be a root vertex in T and let $\tilde{T}(v)$ be the subquiver in $\tilde{T}$ corresponding to the rooted subtree of v. It is obvious that a representation V is semistable for this setting if and only if each vertex in T is reached by V. For a given vertex $w \in T(v)$ this means that the representation must be non-zero along at least one of the paths from v_0 to w, and this for each w. For any vertex w, denote the arrows from v_0 to w by $x_1^w, \dots, x_{\gamma_w}^w$. By abuse of notation, we will use the same notation for both the arrow a and the value assigned to it by V. Now define for a top vertex t

$$\mathbf{t} := (x_1^t, \dots, x_{\gamma_t}^t) \in \mathbb{C}^{\gamma_t}$$

and define inductively for any vertex w with incoming arrows $a_1, \dots, a_{i_w}$

$$\mathbf{w} := (x_1^w, \dots, x_{\gamma_w}^w, a_1 \mathbf{t}(\mathbf{a_1}), \dots, a_{i_w} \mathbf{t}(\mathbf{a_{i_w}})) \in \mathbb{C}^{N_w + 1}.$$

Then the semistability condition for V yields that $\mathbf{w} \neq 0$ for all w. Moreover, we may identify V with its image in $\prod_{w \in TT_0} \mathbb{C}^{N_w+1}$ under the map $V \mapsto (\mathbf{w})_{w \in T_0}$. The action of $\mathrm{GL}(\tilde{\mathbf{1}})$ on V translates in the natural action of $\prod_{w \in T_0} \mathbb{C}^*$

on $\prod_{w\in T_0} \mathbb{C}^{D_w+1}$ by left multiplication. This means the orbit of $\mathbf{w}$ corresponds to a point in $\prod_{w\in T_0} (\mathbb{P})^{N_w}$. Denote by

$$\overline{\mathbf{w}} \in \mathbb{P}^{N_w}$$

the projective coordinates obtained from $\mathbf{w}$, then the orbit of V is a M-tuple of projective points (with M the number of vertices in T):

$$\mathcal{O}_V = (\dots, \overline{\mathbf{v}}, \overline{\mathbf{t(a_1^v)}}, \dots, \overline{\mathbf{t(a_{n_v}^v)}}, \dots, \overline{\mathbf{w}}, \overline{\mathbf{t_1}}, \dots, \overline{\mathbf{t_{n_w}}}, \dots).$$

For the rooted subtree with root vertex v, the points corresponding to the rooted subtree may be depicted as

$$\begin{tikzcd}
\overline{\mathbf{t_1}} \arrow[dr, "a_1^w"'] & \cdots & \overline{\mathbf{t_{n_w}}} \arrow[dl, "a_{n_w}^w"] & \cdots & & \\
& \overline{\mathbf{w}} = (x_1^w : \cdots : x_{d_w}^w : a_1^w \overline{\mathbf{t_1}} : \cdots : a_{n_w}^w \overline{\mathbf{t_{n_w}}}) \arrow[d] & & & & \\
& \cdots \arrow[dr] & \cdots & \cdots \arrow[dl] & \cdots \arrow[d] & \cdots \arrow[dl] \\
& & \overline{\mathbf{t(a_1^v)}} \arrow[dr, "a_1^v"'] & \cdots & \overline{\mathbf{t(a_{n_v}^v)}} \arrow[dl, "a_{n_v}^v"] & \\
& & & \overline{\mathbf{v}} = (x_1^v : \cdots : x_{d_v}^v : a_1^v \overline{\mathbf{t(a_1^v)}} : \cdots : a_{n_v}^v \overline{\mathbf{t(a_{n_v}^v)}}) & &
\end{tikzcd}$$

This corresponds exactly to a point in $\Gamma(v)$, so for any root vertex v in T the restriction of (the orbit of) V to $T(v)$ yields a point in $\Gamma(v)$. Now let v_1 and v_2 be two root vertices with connected rooted subtrees $T(v_1)$ and $T(v_2)$. Assume these rooted subtrees coincide on a subquiver S, then this subquiver again is a tree with root vertex w. The points in $\mathcal{O}_V$ corresponding to v_1 and v_2 coincide on all vertices of S, thus yielding a point in

$$\Gamma(v_1) \times_{\Gamma(w)} \Gamma(v_2).$$

Repeating this argument until all root vertices are accounted for then precisely yields a point in $\mathcal{F}_n$. $\square$

Remark. These fibers are examples of framed quiver moduli as described by Markus Reineke in [11]. The results here however were obtained independently and through other methods than the results presented in [11].

The result above is a natural extension of [1]. In this paper M. Artin describes the fibers of a Brauer Severi variety of a maximal order over a smooth curve.

Theorem 10 (Artin) *Let A be a maximal order in a central simple algebra of dimension n^2 over a Dedekind domain R. If $p \in \mathtt{spec}R$ is a ramification point with ramification index m then the fiber of the Brauer Severi Fibration at p consists of m copies of the graph of the rational maps*

$$\begin{array}{c} \mathbb{P}_{n-1} \to \mathbb{P}_{\frac{m-1}{m}n-1} \to \dots \mathbb{P}_{\frac{1}{m}n-1} : \\ (x_1, \dots, x_n) \mapsto (x_1, \dots, x_{\frac{m-1}{m}n}, 0, \dots, 0) \mapsto \cdots (x_1, \dots, x_{\frac{n}{m}}, 0, \dots, 0). \end{array}$$

intersecting transversally.

Proof. The only strongly connected local quiver data for which this hold are $(\tilde{A}_k, \mathbf{1}, \frac{n}{k+1}\mathbf{1})$. The structure of the quiver and the dimension vector follow from the fact that $\mathtt{iss}_{\alpha_p} Q_p$ must be of dimension 1, the fact that γ_p must be a multiple of $\mathbf{1}$ is a consequence of the maximality of the order.The number $k+1$ can be identified with the ramification index m of the point.

Lemma 4 gives us he correct number of components. Every component corresponds to a tree which is in this case an ordinary dynkin quiver A_k, having a unique maximal rooted subtree $T(v)$ being the quiver itself. The component is thus just the graph of the sequence of maps indicated above because all the d_w are equal to $\frac{n}{m}$. □

6 Toric Varieties and the Brauer-Severi Fibration

We can also get a nice description of the fibers using toric geometry. This discussion closely follows [5].

In the set of semistable representations of the quiver we can embed the big torus $T_Q = (\mathbb{C}^*)^{\#Q_1}$ as an open subset. On this torus there is an action of $T_\alpha = \mathrm{GL}_\alpha = (\mathbb{C}^*)^{\#Q_0}$, so that the moduli space contains a torus $T = T_Q/T_\alpha = (\mathbb{C}^*)^{\#Q_1 - \#Q_0 + 1}$ (the extra 1 comes from the fact that the action is not free).

Let $v = v_0$ be the special source vertex and choose for every other vertex w an arrow a_w from v to w. Denote the set of all other arrows by P. We can identify T with $(\mathbb{C}^*)^{\#P}$ by choosing for each point in T the unique representative whose values at the a_w is 1.

The space in which we are going to construct our fan is then $\mathrm{Hom}(\mathbb{C}^*, T) \otimes \mathbb{R} \cong \mathbb{R}^P$. A well known fact in toric geometry is that one can reconstruct the cones from the variety using one parameter subgroups (1PSG's).

The space of one-parameter subgroups $\mathrm{Hom}(\mathbb{C}^*, T) = \mathbb{Z}^P$ can be identified with $\mathrm{Hom}(\mathbb{C}^*, T_Q)/\mathrm{Im}(\mathrm{Hom}(\mathbb{C}^*, T_\alpha) \to \mathrm{Hom}(\mathbb{C}^*, T_Q)) = \mathbb{Z}^{Q_1}/\mathbb{Z}^{Q_0-1}$. So every 1PSG $\bar{\lambda}$ of T corresponds to an equivalence class of vectors $\lambda : Q_1 \to \mathbb{Z}$ that differ by a vector of the form

$$\xi : Q_1 \to \mathbb{Z} : a \mapsto \zeta(h(a)) - \zeta(t(a))$$

where $\zeta : Q_0 \to \mathbb{Z}$ is a character of T_α.

Lemma 5 *In every equivalence class of $\mathbb{Z}^{Q_1}/\mathbb{Z}^{Q_0-1}$ there is a representant with only non-negative coefficients.*

Proof. Suppose that $\lambda \in \mathbb{Z}^{Q_1}$ is a vector with some negative coefficients. Choose an arrow a such that $\lambda(a) < 0$ and let V_1 be the set of vertices that are targets of paths containing a. Let V_0 be the complement of this set. Note that $s(a) \in V_0$ because the quiver has no cycles.

Consider the character $\zeta : Q_0 \to \mathbb{Z}$ mapping the vertices in V_i to i. This character gives a vector ξ_a that maps every arrow to something non-negative because there are no arrows from V_1 to V_0. Moreover $\xi_a(a) = 1$ so the vector

$$\lambda' = \lambda - \sum_{a, \lambda(a) < 0} \lambda(a)\xi_a$$

has no negative entries. □

The previous lemma implies that $\lim_{z \to 0} \bar{\lambda}(z)$ contains a representation that assigns to every arrow either a 1 (for the zeroes in the vector) or a 0 (for the strictly positive values in the vector). Also this limit representation must be semistable, so there exists a path from v to every vertex w containing only arrows with value 1.

Two such different limit points cannot belong to the same GL_α-orbit, because the GL_α-action can never change a zero into a one or vice versa.

Moreover, every semistable representation with values 0 or 1 can be seen as a limit point of a 1PSG (nl. one coming from a vector with zero's on the arrows with value 1, and positive values on the arrows with value 0). We can conclude that

Lemma 6 *There is a one to one correspondence between the limit points of 1PSGs and subquivers of Q which connect v to all other vertices in Q.*

Proof. We identify such subquivers with the representation of Q that maps the arrows of the subquiver to 1 and the others to 0. □

From toric geometry (see e.g. [4]) we know that the poset of cones in the fan of a toric variety is isomorphic to the poset of limit points under degeneration. The identification goes as follows: to every limit point p we assign the semigroup

$$\{\bar{\lambda} \in \mathrm{Hom}(\mathbb{C}^*, T) : \forall a \in Q_1 : p_a = 0 \implies (\lim_{z \to 0} \bar{\lambda}(z))_a = 0\}$$

This semigroup comes from the cone

$$\sigma_p := \{\eta \in \mathbb{R}^P : \forall a \in Q_1 : p_a = 0 \implies \eta(a) - \sum_{a, \eta(a) < 0} \eta(a)\xi_a = 0\}$$

We know the fiber is smooth so this poset comes from a simplicial set. Therefore we can conclude

Theorem 11 *The set of subquivers of Q such that v can be connected to all other vertices forms a poset under the reverse inclusion. This poset is simplicial and the set of cones*

$$\sigma_{Q'} := \{\eta \in \mathbb{R}^P : \forall a \in Q'_1 : \eta(a) - \sum_{a, \eta(a) < 0} \eta(a)\xi_a = 0\}$$

form a fan. Its corresponding smooth toric variety is isomorphic to the component of the Brauer-Severi fiber.

We will now use this identification to compute the cohomology of the fibers. This cohomology can in some ways be used as a shortcut to check certain properties of the fibers without having to go through the rather lengthy explicit description of Theorem 9.

For this we use the theorem by Fulton which allows one to compute the cohomology ring from the one dimensional cones in the fan:

Theorem 12 *For a smooth toric variety with fan Δ the cohomology is given by the quotient of the polynomial ring generated by the one-dimensional cones $D_i = \mathbb{N}v_i$ by the relations*

- $D_{i_1} \cdot \dots \cdot D_{i_k}$ *if* $\mathbb{N}v_{i_1} + \dots + \mathbb{N}v_{i_k}$ *is not contained in a cone of* Δ.
- $\sum_i \langle u, v_i \rangle D_i$ *for all possible* $u \in \mathit{Hom}(T, \mathbb{C}^*)$.

In order to translate this statement to the setting of this paper, we first of all note that the one-dimensional cones correspond to the subquivers that lack one arrow of the original. However, not all arrows correspond necessarily to a cone because it might be that the representations that map this arrow to zero is not semistable. This happens only if $a = a_w$ for some vertex and no arrow in P terminates in w. Let us postpone this case for a moment.

So suppose that there are at least two paths from v to every other vertex. For every arrow a let D_a be the corresponding cone. The vector corresponding to

D_a has the following form: if $a \in P$ then $v_a : P \to \mathbb{Z} : b \mapsto \delta_{ab}$, if $a = a_w$ for some vertex a_w then

$$v_a : P \to \mathbb{Z} : b \mapsto \begin{cases} -1 & h(b) = w \\ 1 & t(b) = w \\ 0 & \text{otherwise} \end{cases}$$

The second set of relations now becomes

$$\sum_a v_a(b) D_a \text{ for all possible } b \in P.$$

These relations imply that for two arrows a, b starting in v and ending in the same vertex w, $D_a = D_b$ inside the homology ring. Let us denote this generator by D_w. For arrows that do not start in v we have that $D_a = D_{h(a)} - D_{t(a)}$, so the D_w are the generators of the homology ring.

Now let us determine what happens with the first set of relations. A representation is not semistable if there is a vertex that is not the target of a path from v. So the product of some D_a is zero as long a the corresponding set of arrows meets all paths from v to a certain vertex w. As there are arrows from v to every other vertex this also implies that such a set of arrows must meet all the arrows terminating in a given vertex. Therefore the relations can be rewritten as

$$\prod_{a, h(a)=w} D_a \text{ or } \prod_{a, h(a)=w} (D_{h(a)} - D_{t(a)}) \text{ for all vertices } w \text{ and } D_v := 0$$

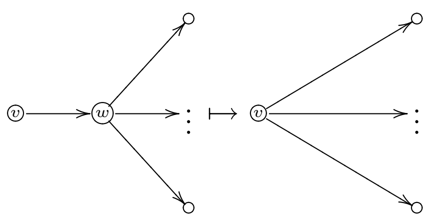
So we can conclude that

Theorem 13 *The cohomology ring of the component of the fiber $\pi^{-1}(\xi)$ associated to Q is isomorphic to the ring*

$$\mathbb{Z}[D_w : w \in Q_0]/(\prod_{a, h(a)=w} (D_{h(a)} - D_{t(a)}) : w \in Q_0 \setminus \{v\}, D_v)$$

Proof. For the case where there are at least two paths from v to every other vertex this follows immediately from the discussion above.

If there is a vertex w with a unique arrow a that terminates in it, we can construct a new quiver setting by removing this vertex and arrow like this



The moduli space of this new quiver is the same as the old one because the value of a is invertible and can be set to 1 using the action in w. The homology ring of the new quiver can be calculated as above, but if one calculates the relations for the old quiver one sees that these relations are the same because $D_w = 0$ by the relation in w. $\square$

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