COFREE QUIVER REPRESENTATIONS

RAF BOCKLANDT AND GEERT VAN DE WEYER

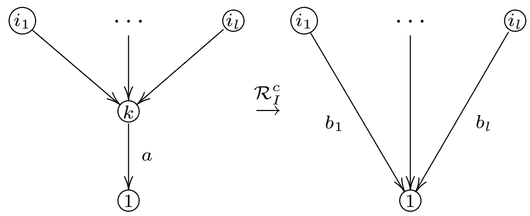
Abstract. We give a complete classification of all quivers Q and dimension vectors α for which the representation space $\mathsf{Rep}(Q,\alpha)$ is cofree, that is, for which $\mathbb{C}[\mathsf{Rep}(Q,\alpha)]$ is a graded free $\mathbb{C}[\mathsf{Rep}(Q,\alpha)]^{\mathsf{GL}_\alpha}$-module.

1. Introduction

Consider a linear reductive complex algebraic group G and a representation $\phi : G \to GL(V)$. Such a representation is called *cofree* if its coordinate ring $\mathbb{C}[V]$ is a graded free module over the ring of invariants $\mathbb{C}[V]^G$. Cofree representations were studied amongst others by Popov in [1] and Schwarz in [2] and were classified by Schwarz for G a connected simple complex algebraic group.

A representation of a quiver Q of dimension vector α is a natural example of the situation described in the previous paragraph through the natural action of the linear reductive complex algebraic group $\mathsf{GL}(\alpha)$ on such a representation by conjugation (base change). Recall that a quiver Q is a directed graph and that a dimension vector for such a quiver assigns to each vertex of the graph a positive integer. A representation of a given quiver with a given dimension vector then assigns to each vertex a complex vector space of dimension equal to the integer assigned to this vertex and to each arrow a linear map between the vector spaces on the vertices connected by this arrow. The representation theory of quivers has by now been shown to be very useful in such diverse fields as geometric invariant theory (through e.g. the construction of moduli spaces as in [3]), representation theory of (Lie) algebras (starting with the work of Gabriel [4]), quantum group theory (e.g. through the use of Hall algebras [5]), etc.

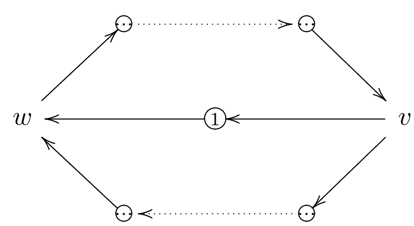
In the current paper we give a complete classification of all quivers Q and dimension vectors α for which the corresponding representation space $\mathsf{Rep}(Q,\alpha)$ is cofree. This classification is presented in Theorem 1.1. Recall that a quiver setting is a connected sum of two subquivers if the two subquivers have exactly one vertex in common. A quiver setting is called a prime quiver setting if it is not a connected sum of two subquivers. The prime components of a quiver setting are subquiver settings that are prime and are not contained in a larger subquiver that is also prime. By reduction step $\mathcal{R}_I^c$ we mean the construction of a new quiver from a given quiver by removing a vertex (and connecting all arrows) in the situation illustrated below, where k is not smaller than the number of cycles through Ⓚ.



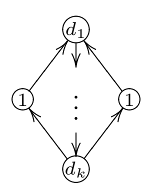
We have

Theorem 1.1. A quiver setting (Q, α) is cofree if and only if its prime components can be reduced using $\mathcal{R}_I^c$ to one of the following forms

- strongly connected quiver settings (P, ρ) for which
  (1) $\exists v \in P_0 : \rho(v) = 1$ and through which all cycles run
  (2) $\forall w \neq v \in P_0 : \rho(w) \geq \#\{ⓥ \rightsquigarrow ⓦ\} + \#\{ⓥ \leftsquigarrow ⓦ\} - 1$
- quiver settings (P, ρ) of the form

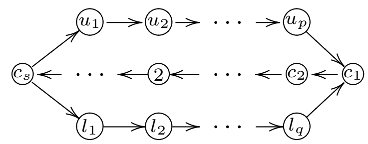


  for which at exactly vertex $x \in P_0$ in the path $v \rightsquigarrow w$ has minimal dimension.
- quiver settings (P, ρ) of the form



  with $d_i > 2$ for all $1 \leq i \leq k$.
- quiver settings of extended Dynkin type $\tilde{A}_n$
- quiver settings (P, ρ) consisting of two cyclic quivers, $\tilde{A}_{p+s}$ and $\tilde{A}_{q+s}$, coinciding on $s+1$ subsequent vertices, one of these, which we denote by f,

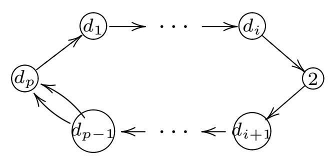
with dimension $\rho(f) = 2$:



with

(1) either $s = 0$ and $u_i, l_j, c_k \geq 2$ for all $1 \leq i \leq p$, $1 \leq j \leq q$ and $1 \leq k \leq s$ or
(2) $\rho(v) \geq 4$ for all vertices $v \neq f$.

- a cyclic quiver with exactly one vertex of dimension 2, extended with one extra arrow between two subsequent vertices



with $d_i \geq 4$ for all $1 \leq i \leq p$.

The classification is obtained starting from a classical result by Schwarz [6] which states that a representation ϕ is cofree if and only if it is coregular (that is, $\mathbb{C}[V]$ is isomorphic to a polynomial ring) and the codimension in $\mathbb{C}[V]$ of the zero set $N_G(V)$ of elements of positive degree in $\mathbb{C}[V]^G$ is equal to $\dim \mathbb{C}[V]^G$. This result, in combination with the classification of coregular quiver representations by the first author in [7] and the study of the nullcone of quiver representations by the second author in [8] yields the complete classification presented.

The paper is organized in the following manner. In Section 2 we collect most of the definitions and background material needed for the rest of the paper. In Sections 3 and 4 the methods to obtain the classification are discussed.

2. Preliminaries

In this section we gather together all necessary material for the rest of the paper.

2.1. **Definitions and Notations.**

Definition 2.1.

- A *quiver* is a fourtuple $Q = (Q_0, Q_1, s, t)$ consisting of a set of *vertices* Q_0, a set of *arrows* Q_1 and two maps $s : Q_1 \to Q_0$ and $t : Q_1 \to Q_0$ assigning to each arrow its source resp. its target vector:

$$\underset{t(a)}{\circ} \xleftarrow{\;a\;} \underset{s(a)}{\circ} \quad .$$

- A *dimension vector* of a quiver Q is a map

$$\alpha : Q_0 \to \mathbb{N} : v \mapsto \alpha(v) := \alpha_v$$

  and a *quiver setting* is a couple (Q, α) of a quiver and an associated dimension vector.
- Fix an ordering of the vertices of Q. The *Euler form of a quiver* Q is the bilinear form

$$\chi_Q : \mathbb{N}^{\#Q_1} \times \mathbb{N}\mathbb{Z}^{\#Q_1} \to \mathbb{Z}$$

  defined by the matrix having $\delta_{ij} - \#\{a \in Q_2 \mid h(a) = j, t(a) = i\}$ as element at location (i, j).
- A quiver is called *strongly connected* if and only if each pair of vertices in its vertex set belongs to an oriented cycle.

A quiver setting is graphically depicted by drawing the quiver and either listing in each vertex v the dimension $\alpha(v)$: $\textcircled{\scriptsize\alpha(v)}$ or by listing the name of the vertex, in which case the vertex is not encircled.

Definition 2.2.

- An α-dimensional representation V of a quiver Q assigns to each vertex $v \in Q_0$ a linear space $\mathbb{C}^{\alpha(v)}$ and to each arrow $a \in Q_1$ a matrix $V(a) \in M_{\alpha(t(a)) \times \alpha(s(a))}(\mathbb{C})$. We denote by $\mathsf{Rep}(Q, \alpha)$ the space of all α-dimensional representations of Q. That is,

$$\mathsf{Rep}(Q, \alpha) = \bigoplus_{a \in Q_1} M_{\alpha(t(a)) \times \alpha(s(a))}(\mathbb{C}).$$

- We have a natural action of the reductive group

$$\mathsf{GL}_\alpha := \prod_{v \in Q_0} \mathsf{GL}_{\alpha(v)}(\mathbb{C})$$

  on a representation V defined by basechange in the vectorspaces. That is

$$(g_v)_{v \in Q_0} . (V(a))_{a \in Q_1} = (g_{t(a)} V(a) g_{s(a)}^{-1})_{a \in Q_1}.$$

- The quotient space with respect to this action classifies all isomorphism classes of semisimple representations and is denoted by $\mathsf{iss}(Q,\alpha)$. The quotient map with respect to this action will be denoted by

$$\pi : \mathtt{rep}(Q,\alpha) \twoheadrightarrow \mathsf{iss}(Q,\alpha).$$

- The fibre of π in $\pi(0)$ is called the *nullcone* of the quiver setting and is denoted by $\mathsf{Null}(Q,\alpha)$.

For the study of the equidimensionality of the quotient map we introduce

Definition 2.3. For a given quiver setting (Q,α), we define the *defect of the equidimensionality*

$$\operatorname{def}(Q,\alpha) := \dim Null(Q,\alpha) - \dim \mathtt{rep}(Q,\alpha) + \dim \mathsf{iss}(Q,\alpha).$$

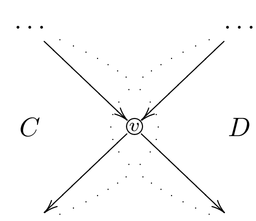
We then have

Proposition 1. [9, II.4.2, Folgerung 1] *For a given quiver setting (Q,α) as above, the quotient map π is equidimensional if and only if def$(Q,\alpha) = 0$. We will say that the quiver setting (Q,α) is equidimensional.*

In order to formulate our main result we need one last definition.

Definition 2.4.

- A quiver Q is said to be the *connected sum* of 2 subquivers C and D at vertex v, if $Q_0 = C_0 \cup D_0$, $Q_1 = C_1 \cup D_1$, $C_0 \cap D_0 = \{v\}$ and $C_1 \cap D_1 = \emptyset$. We denote this as $Q = C \underset{v}{\#} D$.

$C \underset{v}{\#} D :=$ 

- When three or more quivers are connected we write $R \underset{v}{\#} S \underset{w}{\#} T$ instead of $(R \underset{v}{\#} S) \underset{w}{\#} T$.
- We call a quiver setting (Q,α) *prime* if Q can not be written as $R\#_v S$ where v is a vertex with dimension 1.
- Prime subsettings that are not contained in any larger prime subsetting are called the prime components.

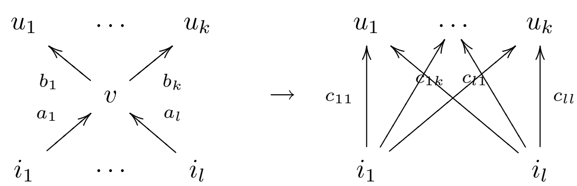
Note that a quiver setting is cofree if and only if all its prime components are cofree.

2.2. **Reducing Quiver Settings.** In [7], Raf Bocklandt introduced three different types of reduction moves on a quiver setting (Q, α). These are

$\mathcal{R}_I^v$: let v be a vertex without loops such that

$$\chi_Q(\alpha, \epsilon_v) \geq 0 \text{ or } \chi_Q(\epsilon_v, \alpha) \geq 0.$$

Construct a new quiver setting $(\mathcal{R}_I^v(Q), \mathcal{R}_I^v(\alpha))$ by removing v and connecting all arrows running through v:



For this step we have $\mathsf{iss}(Q, \alpha) \cong \mathsf{iss}(R_I^v(Q), R_I^v(\alpha))$.

$\mathcal{R}_{II}^v$: let v be a vertex with $\alpha(v) = 1$ and n loops. Let $(\mathcal{R}_{II}^v(Q), \alpha)$ be the quiver setting obtained by removing all these loops. We then have

$$\mathsf{iss}(Q, \alpha) \cong \mathsf{iss}(\mathcal{R}_{II}^v(Q), \alpha) \times \mathbb{A}^n.$$

$\mathcal{R}_{III}^v$: let v be a vertex with one loop and $\alpha(v) = n$ such that

$$\chi_Q(\alpha, \epsilon_v) = -1 \text{ or } \chi_Q(\epsilon_v, \alpha) = -1.$$

Let $(\mathcal{R}_{III}^v(Q), \alpha)$ be the quiver setting obtained by removing the loop in v and adding $n-1$ additional arrows between v and its neighbouring vertex with dimension 1 (all having the same orientation as the original arrow). For this step we have

$$\mathsf{iss}(Q, \alpha) \cong \mathsf{iss}(\mathcal{R}_{III}^v(Q), \alpha) \times \mathbb{A}^n.$$

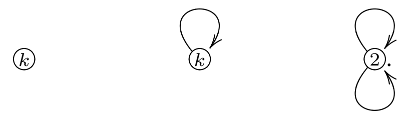
Definition 2.5. Let (Q, α) be a quiver setting.

(1) If none of the above reduction steps can be applied to (Q, α) then this setting is called *reduced*.

(2) By $(\mathcal{R}(Q), \mathcal{R}(\alpha))$ we denote the quiver setting obtained after repeatedly applying all of the above reduction steps until no longer possible. This setting is called the *reduced quiver setting of* (Q, α).

We now have the following two results.

Theorem 2.1 (Bocklandt, [7])**.** Let (Q, α) be a strongly connected quiver setting, then (Q, α) is coregular if and only if $(\mathcal{R}(Q), \mathcal{R}(\alpha))$ is one of the following three settings:



We will denote these settings by $\mathcal{Q}_0(k)$, $\mathcal{Q}_1(k)$ and $\mathcal{Q}_2$.

Theorem 2.2 (Van de Weyer, [8])**.** Let (Q, α) be a quiver setting, then

$$\mathrm{def}((Q, \alpha) \geq \mathrm{def}(\mathcal{R}(Q), \mathcal{R}(\alpha)).$$

These two results will be our main tools in classifying all cofree quiver settings.

Finally, we will use the Luna Slice Theorem, formulated for quiver representations (see [10]).

Theorem 2.3 (Le Bruyn-Procesi)**.** Let (Q, α) be a quiver setting. Let

$$\pi : \mathtt{rep}(Q, \alpha) \twoheadrightarrow \mathsf{iss}(Q, \alpha)$$

be the quotient with respect to the natural GL_α-action. Let $S \in \mathsf{iss}(Q, \alpha)$ correspond to the following decomposition in simples

$$S = S_1^{\oplus e_1} \oplus \cdots \oplus S_k^{\oplus e_k},$$

with S_i a simple representation of dimension vector α_i (for $1 \leq i \leq k$).

Define the quiver Q_S as the quiver with k vertices and $\delta_{ij} - \chi_Q(\alpha_i, \alpha_j)$ arrows from vertex i to vertex j. Define α_S as the dimension vector that assigns e_i to vertex i (for $1 \leq i \leq k$). Then

(1) there exists an étale isomorphism between an open neighbourhood of S in $\mathsf{iss}(Q, \alpha)$ and an open neighbourhood of the zero representation in $\mathsf{iss}(Q_S, \alpha_S)$.
(2) there is an isomorphism as GL_α-varieties

$$\pi^{-1}(S) \cong \mathsf{GL}_\alpha \times^{\mathsf{GL}_{\alpha_S}} \mathsf{Null}(Q_S, \alpha_S).$$

We define

Definition 2.6.

- An element $S = S_1^{\oplus e_1} \oplus \cdots \oplus S_k^{\oplus e_k}$ is said to be of *representation type* $(\alpha_1, e_1; \alpha_2, e_2, \ldots, \alpha_k, e_k)$ where α_i is the dimension vector of S_i for $1 \leq i, leqk$.

- The quiver setting (Q_S, α_S) is called the *local quiver* of S.

3. Quiver Settings Reducing to $\mathcal{Q}_0(1)$

Lemma 3.1. Suppose (Q, α) is cofree. If (Q', α') is

(1) a subquiver of (Q, α),

(2) a local quiver of (Q, α) or

(3) a quiver obtained by applying reduction moves to (Q, α),

then (Q', α') is also cofree.

We will call a path *quasiprimitive* if it does not run $n+1$ times trough a vertex of dimension n. We will denote a quasiprimitive path between v and w by $(v) \rightsquigarrow (w)$. Throughout the paper, we assume every path quasiprimitive unless stated otherwise. Cycles will also be assumed quasiprimitive. The advantage of working with quasiprimitive cycles is that there are only a finite number of them.

Reduction move $\mathcal{R}_I$ does not change the number of quasiprimitive cycles, the other moves respect the number of quasiprimitive cycles through the vertices where they are not applied.

To every point $V \in \mathsf{Rep}(Q, \alpha)$ and a vertex v, we can assign a new dimension vector σ^v where σ^v_w is the dimension of the vector space

$$\mathsf{Span}\Big(\bigcup_{(v) \overset{p}{\rightsquigarrow} (w)} \mathsf{Im}\, V_p \Big).$$

We will call σ^v the *relevant dimension vector with base* v. In this formula the trivial path through v is not counted. When we do count the trivial path then we will denote this by a $\bar{\sigma}^v$ (i.e. $\bar{\sigma}^v_v = \alpha_v$ while σ^v_v can be smaller). When the base vertex is obvious we will omit the superscript.

Theorem 3.2. Suppose (Q, α) a strongly connected quiver setting without loops. If Q has a vertex v with dimension 1 which all cycles run through then (Q, α) is cofree if and only if for every vertex w the dimension

$$\alpha_w \geq \#\{\, v \rightsquigarrow w \,\} + \#\{(v) \leftsquigarrow (w)\} - 1 \tag{1}$$

Proof. We will denote the set of representations with a given relevant dimension σ with base v by $\mathsf{Rel}_\sigma(Q, \alpha)$.

We now have that $V \in \mathsf{Null}(Q,\alpha)$ if and only if $\sigma_v = 0$, so

$$\mathsf{Null}(Q,\alpha) = \bigcup_{\sigma_v=0} \mathsf{Rel}_\sigma(Q,\alpha) \text{ and } \dim \mathsf{Null}(Q,\alpha) = \max_{\sigma_v=0} \dim \mathsf{Rel}_\sigma(Q,\alpha).$$

We can calculate the dimension of $\mathsf{Rel}_\sigma(Q,\alpha) \subset \mathsf{Null}(Q,\alpha)$ as follows. If there exists a vertex w such that

$$\delta_w := \sum_{t(a)=w} \bar{\sigma}_{s(a)} - \sigma_w < 0,$$

$\mathsf{Rel}_\sigma(Q,\alpha)$ will be empty (recall that $\bar{\sigma}$ is the same dimension vector as σ except that $\sigma_v = 1$). This also implies that $\sigma_w \le \#\{v \rightsquigarrow w\}$. If this is not the case

$$\begin{aligned}\dim \mathsf{Rel}_\sigma(Q,\alpha) &= \sum_{w\in Q_0} \sigma_w(\alpha_w - \sigma_w) + \sum_{a\in Q_1} (\alpha_{s(a)}\alpha_{t(a)} - \bar{\sigma}_{s(a)}(\alpha_{t(a)} - \sigma_{t(a)}))\\ &= \dim \mathsf{Rep}(Q,\alpha) - \sum_{w\neq v}\left(\sum_{t(a)=w} \bar{\sigma}_{s(a)} - \sigma_w\right)(\alpha_w - \sigma_w) - \sum_{t(a)=v} \sigma_{s(a)}\\ &= \dim \mathsf{Rep}(Q,\alpha) - \sum_{w\neq v}\sum_{t(a)=w} \delta_w(\alpha_w - \sigma_w) - \sum_{t(a)=v} \sigma_{s(a)}\end{aligned}$$

The first term on the first line calculates the dimension of all possible choices of a σ_w-dimensional subspaces in an α_w-dimensional subspace for every w. The second term gives the dimension of the space of all possible maps $R_a, a \in Q_1$ mapping the correct subspaces onto each other.

Now we calculate the last term of the third line. For a given n we have that

$$\sum_{t(a)=v} \sigma_{s(a)} = \#\{\text{cycles of length} \le n\} + \sum_{\substack{w \overset{p}{\rightsquigarrow} v\\ |p|=n, w\neq v}} \sigma_w - \sum_{w\neq v} \delta_w \cdot \#\{\, w \overset{p}{\rightsquigarrow} v \,, |p| < n\}.$$

We will prove this statement by induction. For $n = 1$ only the middle term of the right hand side is non zero and it is equal to the right side. Suppose that the formula is proven for n and we want to prove it for $n+1$. We split up every σ_w as $\sum_{t(a)=w} \sigma_{s(a)} - \delta_w$. If $s(a) = v$ then ap is a cycle of length $n+1$ and in this case

$\sigma_{s(a)} = 1$. We will put these terms apart.

$$\begin{aligned}\sum_{\substack{w \overset{p}{\rightsquigarrow} v\\|p|=n, w\neq v}} \sigma_w &= \sum_{\substack{w \overset{p}{\rightsquigarrow} v\\|p|=n, w\neq v}} \left(\sum_{t(a)=s(p)} \sigma_{s(a)} - \delta_w\right)\\ &= \sum_{\substack{v \overset{ap}{\rightsquigarrow} v\\|ap|=n+1}} 1 + \sum_{\substack{w \overset{ap}{\rightsquigarrow} v\\|ap|=n+1, w\neq v}} \sigma_w - \sum_{\substack{w \overset{p}{\rightsquigarrow} v\\|p|=n, w\neq v}} \delta_w\\ &= \#\{\text{cycles of length } n+1\} + \sum_{\substack{w \overset{p}{\rightsquigarrow} v\\|p|=n+1, w\neq v}} \sigma_{s(p)} - \sum_{w\neq v} \delta_w \cdot \#\{\ w \overset{p}{\rightsquigarrow} v\ , |p| = n\}.\end{aligned}$$

We can substitute this formule in $(*)$ and add the left term respectively the right terms to obtain the equation for $n+1$. Because every cycle contains v and the length of the paths is bounded, we get for $n \gg 1$ that middle term becomes zero and hence

$$\begin{aligned}\sum_{t(a)=v} \sigma_{s(a)} &= \#\{\text{QP cycles}\} - \sum_{w\neq v} \delta_w \cdot \#\{ⓦ \overset{p}{\rightsquigarrow} ⓥ\}\\ &= \dim \mathsf{iss}(Q,\alpha) - \sum_{w\neq v} \delta_w \cdot \#\{ⓦ \overset{p}{\rightsquigarrow} ⓥ\}\end{aligned}$$

The last equality holds because (Q,α) reduces to a quiver with one vertex of dimension 1 and k loops where k is the number of cycles.

The formula for the dimension of $\mathsf{Rel}_\sigma(Q,\alpha)$ now becomes

$$\dim \mathsf{Rel}_\sigma(Q,\alpha) = \dim \mathsf{Rep}(Q,\alpha) - \mathsf{iss}(Q,\alpha) - \sum_{w\neq v} \delta_w\Big(\sum_{t(a)=w} (\alpha_w - \sigma_w) - \#\{\ w \overset{p}{\rightsquigarrow} v\ \}\Big)$$

Note that if $\delta_w > 0$ then $\sigma_w \leq \#\{\ v \rightsquigarrow w\ \} - 1$, so if we suppose that

$$\alpha_w \geq \#\{\ v \rightsquigarrow w\ \} + \#\{\ v \leftsquigarrow ⓦ\} - 1$$

then

$$(\alpha_w - \sigma_w) - \#\{p : ⓥ \rightsquigarrow ⓦ\}) \geq 0.$$

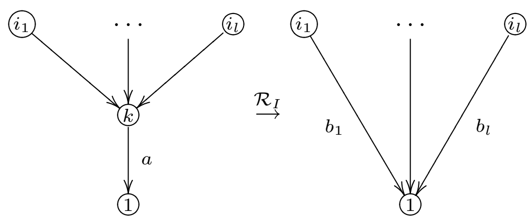
We now have that $\dim \mathsf{Rel}_\sigma(Q,\alpha) \leq \dim \mathsf{Rep}(Q,\alpha) - \mathsf{iss}(Q,\alpha)$ and therefore (Q,α) is cofree.

On the other hand if there exists a vertex w such that

$$\alpha_w < \#\{\ v \rightsquigarrow w\ \} + \#\{\ v \leftsquigarrow w\ \}$$

we can construct a relevant dimension vector σ such that the corresponding δ is only nonzero for such w (this is allways possible because the dimensions of the other vertices are always big enough). The dimension of $\mathsf{Rel}_\sigma(Q,\alpha)$ is then bigger than $\dim \mathsf{Rep}(Q,\alpha) - \mathsf{iss}(Q,\alpha)$ so (Q,α) is not cofree. $\square$

Lemma 3.3. Let (Q,α) be a quiver setting containing a subsetting like below and let (Q',α') be the setting obtained by reducing the central vertex.



(1) If k is not smaller than the number of cycles through ⓚ then (Q,α) is cofree if and only if (Q',α') is cofree.
(2) If k is smaller than the number of cycles through ⓚ then (Q,α) is not cofree.

Proof.

(1) We only have to proof that the condition is sufficient. The necessity follows from [8]. Let π denote the projection $\mathsf{Null}(Q,\alpha) \to \mathsf{Null}(Q',\alpha')$. The dimension of the generic fiber of $V \in \mathsf{Null}(Q',\alpha')$ is

$$\dim \mathsf{Rep}(Q,\alpha) - \dim \mathsf{Rep}(Q',\alpha') = \sum_r i_r(k-1) + k$$

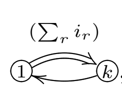
and this occurs when at least one of the linear maps V_{b_i} is nonzero. If they are all zero the dimension is

$$\max(\sum_r i_r(k-1) + k, \sum_r i_r k)$$

This is because the fiber is identical to the zero fiber of the map

$$\mathsf{Hom}(X,Y) \times \mathsf{Hom}(Y,Z) \mapsto \mathsf{Hom}(X,Z) : (x,y) \mapsto y \circ x$$

with X a vector space of dimension $\sum_r i_r$, Y a vector space of dimension k and Z a vector space of dimension 1. This zero fiber is easily seen to be identical to the nullcone of the quiver setting



,

and by Theorem 3.2 we know this dimension to be $\max(\sum_r i_r(k-1)+k, \sum_r i_r k)$. The second argument is the larger one if $\sum_r i_r > k$, and in that case the dimension of the subset

$$X := \{V \in \mathsf{Null}(Q', \alpha') | \forall r \leq l : V_{b_r} = 0\}$$

is at most $\dim \mathsf{Null}(Q', \alpha') - \sum_r i_r - \#\{\text{cycles through } ⓚ\}$ Indeed, through every $V \in X$ we can draw an affine space

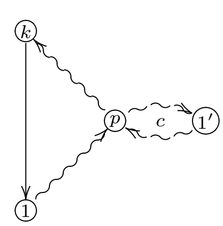
$$V + \{W \mid \forall 1 \leq j \leq l : \mathsf{Span}(\mathsf{Im}\, V_{① \rightsquigarrow (i_j)}) \subset \ker W_{b_j}, \forall v \neq b_1, \dots, b_l \in Q'_0 : W_v = 0\}$$

with dimension at least $\sum_r i_r - \#\{\text{cycles through } ⓚ)\}$ and all these spaces are disjunct. This yields

$$\begin{aligned}
\dim \pi^{-1}(X) &\leq \dim X + \sum_r i_r k \\
&\leq \dim \mathsf{Null}(Q', \alpha') - \sum_r i_r + \#\{\text{cycles through } ⓚ + \sum_r i_r k \\
&\leq \dim \mathsf{Null}(Q', \alpha') + \sum_r i_r(k-1) + k \\
&\leq \dim \mathsf{Rep}(Q', \alpha') - \dim \mathsf{iss}(Q', \alpha') + \sum_r i_r(k-1) + k \\
&\leq \dim \mathsf{Rep}(Q, \alpha) - \dim \mathsf{iss}(Q, \alpha)
\end{aligned}$$

(2) Let $k < \#\{① \rightsquigarrow ⓚ\}$ and let Q_k be the subquiver containing all cycles through ⓚ. If (Q, α) is cofree then (Q_k, α) must also be cofree. We will show that this cannot be the case.

If every cycle in Q_k runs through ① then by Theorem 3.2 we are done. For the other cases we will reduce the setting to this case. Suppose that there is a cycle not through ①. If the minimal dimension of the vertices in this cycle is at least 2 we can get rid of this cycle using $\mathcal{R}_I$ and $\mathcal{R}_{III}$ while keeping the number of cycles through ⓚ constant. If the minimal dimension is 1 then we have a situation like

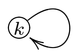
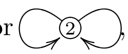


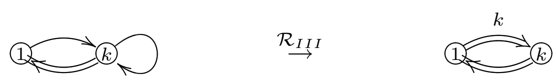
There are no other paths going from vertices in c to ⓚ because there can only be one cycle through ① and (1′). This means that we can remove c using $\mathcal{R}_I$ and $\mathcal{R}_{II}$.

□

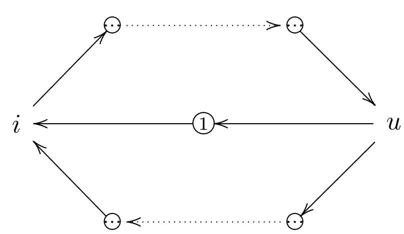
By $\mathcal{R}_I^c$ we denote the reduction move described in the previous lemma; that is, the reduction of the vertex with dimension k when this dimension is not smaller than the number of cycles running through it. In order to find the cofree quiver settings not satisfying the conditions of Theorem 3.2 we now have to classify all prime quiver settings that cannot be reduced using $\mathcal{R}_I^c$. We will consider different cases

I. There exists a cycle c of higher dimension.

If (Q, α) does not reduce to  or , there is a unique arrow from the one to this cycle and a unique arrow back because otherwise we could reduce the setting to something containing a subsetting like



or its dual. Those two are not cofree by Theorem 3.2. Hence, (Q, α) must be of the form



Not all of these settings are cofree.

Theorem 3.4. A quiver setting of the form above is cofree if and only if at most one vertex in the path $i \rightsquigarrow u$ has minimal dimension.

Proof. Let (Q, α) be a setting of the previous form. For every positive integer n we denote by p_n (resp. p_{-n}) the path of length n starting (ending) in i consisting only of arrows in c. The notation $i + n$ $(i - n)$ will stand for the target of p_n (source of p_{-n}).

$u - i$ will be the smallest nonnegative integer such that $i + (u - i) = u$. Let m be the first nonnegative integer such that α_{i+m} is minimal and let k be the length of c.

To every nilpotent representation $V \in \mathsf{Rep}(Q,\alpha)$ we can assign a sequence

$$\sigma_n := \begin{cases} \alpha_{i-n} & n \leq 0 \\ \dim \mathsf{Im}\, V_{p_n} & n > 0 \end{cases}$$

Let δ be the sequence of the differences between consecutive $\sigma's$.

$$\delta_n := \sigma_n - \sigma_{n+1}$$

These two sequences satisfy the following properties

(1) $\sigma_j - \sigma_{j+k} = \delta_j + \dots + \delta_{j+k-1} \geq 1$ if $j \geq 0$ and $\sigma_j > 0$. Otherwise the map from the vertex $i+j$ to itself is not nilpotent.
(2) $\sum_j \delta_j = \alpha_i$.

We define s to be the largest number such that image of the vertex ① is contained within the image of p_{sk}.

Now let $\mathsf{Rel}_{\sigma,s}Q$ be the set of all representations with a given σ and s then the codimension of this set in $\mathsf{Rep}(Q,\alpha)$ can be written as

$$\begin{aligned} \operatorname{codim} \mathsf{Rel}_{\sigma,s}Q &= \sum_{j\geq 0}(\sigma_{j-k}-\sigma_j)\delta_j + \alpha_i - \sigma_{ks} + \sigma_{sk+u} \\ &= \sum_{j\geq 0}(\sigma_{j-k}-\sigma_j)\delta_j + \alpha_i - (\delta_{sk+1} + \dots \delta_{sk+u}) \\ &= \sum_{j\geq 0} f_{js}\delta_j + \alpha_i \end{aligned}$$

where

$$f_{js} = \begin{cases} \sigma_{j-k} - \sigma_j - 1 & j \in [sk+i, sk+u] \\ \sigma_{j-k} - \sigma_j & j \notin [sk+i, sk+u] \end{cases}$$

The codimension of the nullcone is the minimum of all possible $\operatorname{codim} \mathsf{Rel}_{\sigma,s}Q$.

We will now prove that if there is at most one vertex in ⓘ$\rightsquigarrow$ⓤ which has the dimension of c then (Q,α) is cofree i.e. $\operatorname{codim} \mathsf{Null}(Q,\alpha) = 2\alpha_{i+m}$.

If $s > 0$ we have that all $f_{js} \geq 0$ and if $j > k$ then $f_{js} \geq \delta_{j-k+1} + \dots + \delta_j - 1$. This implies that there is at most one $j \in [i+sk, u+sk]$ such that $\delta_j \neq 0$ and $f_{js} = 0$. And for this j we have that $\delta_j = 1$.

The other j such that $\delta_j \neq 0$ and $f_{js} = 0$ are elements of $[1,m]$ and because $f_{j0} = 0$ if $\sigma_{j-1} - \alpha_j = \delta_j$ we also have that $\sum_{f_{js}=0, j\in[1,m]} \delta_j \leq \alpha_i - \alpha_{i+m}$, so

$$\operatorname{codim} \mathsf{Rel}_{\sigma,s}Q \geq \sum_j \delta_j - \alpha_i + \alpha_{i+m} - 1 + \alpha_i = \alpha_i + \alpha_{i+m} - 1 \geq 2\alpha_{i+m} - 1.$$

If codim $\mathsf{Rel}_{\sigma,s}Q$ were equal to $2\alpha_{i+m}-1$ we must have that $m=0$ and all $f_{js}\leq 1$. The equality also implies that if $\delta_j=1$ then $\delta'_j=0 \forall j<j'<j+k$ and $\delta_{j+k}=1$ if $\sigma_{j+k-1}>0$. Let j be the first nonnegative integer such that $\delta_j=1$ then $i+j<u$ because there should exist a $ks<j'\leq ks+(u-i)$ such that $\delta_{j'}=1$. The dimension of the target of the j^{th} arrow is bigger than $\alpha_{i+m}=\alpha_i$ by the condition in the theorem, so $\delta_{js}>1$.

If $s=0$ we can have that some of the f_{sj} are negative. For this to happen we should have that $\delta_j=\sigma_j-\alpha_{i+j}$ and $m\neq 0$. Again we have that the sum of the δ_j for which this happens is not bigger than $\alpha_i-\alpha_{i+m}$. We can be even more specific: if there are $f_sj=0$ with $j<\min(m,u-i)$ and $\delta_j\neq 0$ we have that they decrease the sum $\sum_{f_{js}=0,j\in[1,m]}\delta_j$ so we have the inequality

$$\sum_{f_{js}=-1,j\in[1,m]} + \sum_{f_{js}=0,j\in[1,m]\cap[1,u-i]} \leq \alpha_i-\alpha_{i+m}$$

An analoguous reasoning as in the case for $s>0$ leads to the conclusion that for the other f_{sj} that are zero between m and $u-i$ there can be at most one that has $\delta_j=1$ and the rest has $\delta_j=0$.

The j for which $\delta_j\neq 0$ and $f_{sj}=0$

$$\begin{aligned}\text{codim}\,\mathsf{Rel}_{\sigma,s}Q &\geq \sum_j \delta_j - 2\sum_{f_{sj}=-1}\delta_j - \sum_{f_{sj}=0}\delta_j + \alpha_i = \alpha_i+\alpha_{i+m}-1\\ &\geq \sum_j \delta_j - 2(\alpha_i-\alpha_{i+m}) - 1 + \alpha_i = 2\alpha_{i+m}-1.\end{aligned}$$

The equality can only occur if all $f_{sj}\leq 1$ but for the unique $j\in[m,u-i]$ such that $\delta_j=1$, we have $f_{sj}=\alpha_{\bar{j}}-\sigma_j+1\geq 2$. □

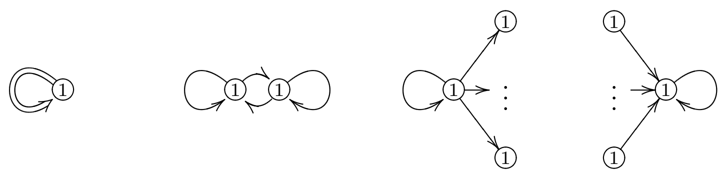
II. There exists no cycle c of higher dimension.

If $\alpha=1$ every arrow must be contained in only one cycle because the codimension of the nullcone is then equal to the minimal number of arrows that is needed to block all cycles. In combination with the condition that (Q,α) must be prime we can conclude that Q is of extended dynkin type $\tilde{A}_n$.

In the case that there are vertices of higher dimension, there is allway such a vertex that can be reduced. This is because an $\mathcal{R}_I$-move on a vertex with dimension 1 does not change the reducibility conditions on the other vertices. Therfore we can construct the setting (Q^I,α^I) where all vertices of higher dimension have been reduced, this new setting needs not to be prime so it is a connected sum of $\tilde{A}_n$'s.

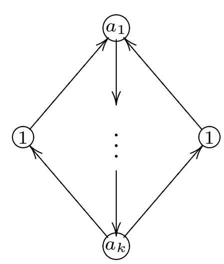
Every arrow in (Q^I, α^I) corresponds to a unique path in (Q, α). If v is a vertex of Q with $\alpha_v > 1$ then we can look at the subquiver P_v^I of Q^I whose arrows correspond to paths through v. If S is the set of source vertices for these arrows and T the set of target vertices then there is an arrow between every vertex of S and every vertex of T. This is because we can mix two paths $p = p_1 v p_2$ and $q = q_1 v q_2$ to $p_1 v q_2$ and $q_1 v p_2$ which connect the source of p with the target of q and vice versa.

Because between every two vertices in Q^I there can be only one cycle, the only possibilities for P_v^I are



In the last 2 cases the vertex can be reduced using a sequence of $\mathcal{R}_I^c$-reductions

Because (Q, α) is prime we have then that Q^I must be equal to P_v^I. The first case has been dealt with in Theorem 3.2. For the second case (Q, α) must look like



with all the $a_i > 2$.

Lemma 3.5. All settings of the previous form are cofree

Proof. We have to prove that the codimension of the nullcone is not smaller than 3. For a given representation V we can define the subspaces $H^1_{a_i}$ and $H^2_{a_i}$, as the images of the two maps that go from the ones to the vertex a_i.

For every vertex we have 5 possible states

A: $0 \neq H^1_{a_i} \neq H^2_{a_i} \neq 0$,

B1: $0 = H^1_{a_i} \neq H^2_{a_i} \neq 0$,

B2: $0 \neq H^1_{a_i} \neq H^2_{a_i} = 0$,

C: $H^1_{a_i} = H^2_{a_i} \neq 0$,

0: $H^1_{a_i} = H^2_{a_i} = 0$.

The states can be ordered as follows: $A > Bj > 0$, $A > C > 0$. To every representation we can assign a sequence of decreasing states S_{a_i}. We will now calculate the codimension of the set of nilpotent representations with a given sequence.

Every transition to a lower state gives a nonzero contribution to the codimension:

A $\to Bj:\ +a_{i+1}, \to C:\ +a_{i+1}-1, \to 0:\ +2a_{i+1}$

Bj $\to 0:\ +a_{i+1},$

C $\to 0:\ +a_{i+1}.$

Apart from that the final state also gives a contribution: on of the two arrows leaving a_k must contain both $H^i_{a_k}$ the other one must contain one of the $H^i_{a_k}$ so we get as contributions:

A	3
B1	1
B2	1
C	2
0	0

As all the $a_i \geq 2$ we have that the codimension is at least 3 for every sequence. □

4. Quivers Settings reducing to $\mathcal{Q}_1(k)$, $k \geq 2$, and $\mathcal{Q}_2$

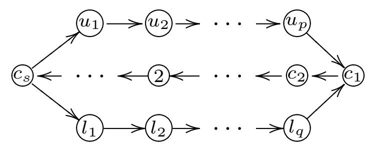
In this section we will classify all cofree quiver settings reducing to $\mathcal{Q}_1(k)$ and $\mathcal{Q}_2$. First of all note that $\mathcal{Q}_1(k)$ is a reduced quiver setting only if its dimension vector k is at least 2. But then the only quiver setting reducing to $\mathcal{Q}_1(k)$ is the cyclic quiver with smallest dimension in its dimension vector equal to k. The nilpotent representations of the cyclic quiver were studied extensively in a.o. [11] and it is known that there are only finitely many orbits of nilpotent representations of the cyclic quiver. But a classical result (e.g. [9, II.4.2, Satz 1]) then yields that the quotient map must be equidimensional, so we have a first result

Proposition 2. *Let $\tilde{A}_n$ be the cyclic quiver with n vertices and let α be a dimension vector for $\tilde{A}_n$ such that $\alpha(v) \geq 2$ for some vertex v of $\tilde{A}_n$, then $(\tilde{A}_n, \alpha)$ is a cofree quiver setting.*

Next, we turn our attention towards all quivers reducing to $\mathcal{Q}_2$. Recall the following lemma from [12]

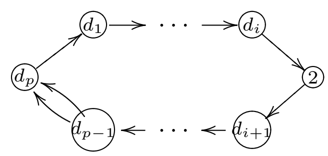
Lemma 4.1. Let (Q, α) be a quiver setting reducing to $\mathcal{Q}_2$, then (Q, α) is of one of the following forms

- two cyclic quivers, $\tilde{A}_{p+s}$ and $\tilde{A}_{q+s}$, coinciding on $s+1$ subsequent vertices, one of these of dimension 2:



with $u_i, l_j, c_k \geq 2$ for all $1 \leq i \leq p$, $1 \leq j \leq q$ and $1 \leq k \leq s$. Such a setting will be denoted by $\mathcal{Q}_2(p,q,s)$;

- a cyclic quiver with at least one vertex of dimension 2, extended with one extra arrow between two subsequent vertices



with $d_i \geq 2$ for all $1 \leq i \leq p$. Such a setting will be denoted by $\mathcal{Q}_2^{double}(p)$.

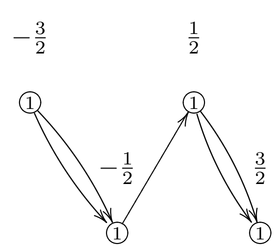
When considering the situations of the lemma above, the following result will prove useful.

Lemma 4.2. The quiver settings

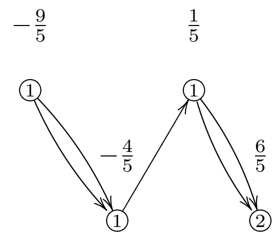
with $3 \geq d \geq 2$ are not equidimensional.

Proof. We will prove this for the first setting, which we will denote by (Q,α). The proof for the second setting is completely analogous. We will construct a stratum in the Hesselink stratification of $\mathsf{Null}(Q,\alpha)$ of dimension strictly greater than $\dim \mathsf{Rep}(Q,\alpha) - \dim \mathsf{iss}(Q,\alpha)$, so $\mathrm{def}(Q,\alpha) > 0$. Using the notations and conventions from [13], consider the following level quivers with corresponding coweight:

if $d = 2$:



if $d = 3$:



For $d = 2$ this level quiver with corresponding coweight determines a Hesselink stratum of dimension 9, whereas $\dim \mathsf{Rep}(Q, \alpha) - \dim \mathsf{iss}(Q, \alpha) = 7$. For $d = 3$ the level quiver determines a Hesselink stratum of dimension 14 whereas $\dim \mathsf{Rep}(Q, \alpha) - \dim \mathsf{iss}(Q, \alpha) = 13$.

$\square$

An immediate consequence of this lemma is

Lemma 4.3. With notations as above

(1) $\mathcal{Q}_2(p, q, s)$ is not cofree if $s \geq 1$ and at least vertices have dimension 2 or 3;
(2) $\mathcal{Q}_2^{double}(p)$ is not cofree if at least one two vertices have at least 2 or 3.

Proof. Consistently applying reduction step $\mathcal{R}_I$ to the vertex with greatest dimension in either of these settings reduces the quiver setting to one of the two settings from Lemma 4.2. Now because these settings are not cofree, the original quiver setting cannot be cofree either. $\square$

We will now show that these are the only situations reducing to $\mathcal{Q}_2$ that are not cofree. For this, we need another two lemmas.

Lemma 4.4. Consider the map

$$\pi : \mathsf{Hom}(U, V) \times \mathsf{Hom}(V, W) \twoheadrightarrow \mathsf{Hom}(U, W) : (X, Y) \mapsto Y \circ X$$

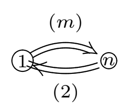
where U and V are vector spaces of dimension at least 2, W is a vector space of dimension exactly 2 and $\dim U - \dim V + 1 \geq 0$, then

(1) $\dim \pi^{-1}(0) = \dim U \dim V + \dim V - \dim U + 1$;
(2) for $Z \neq 0 \in \mathsf{Hom}(U, W)$ with $rk(Z) <= 1$ we have that $\dim \pi^{-1}(Z) = \dim U \dim V + \dim V - \dim U$;
(3) for $Z \in \mathsf{Hom}(U, W)$ with $rk(Z) = 2$ we have that $\dim \pi^{-1}(Z) = \dim U \dim V + 2 \dim V - 2 \dim U$.

If $\dim U - \dim V + 1 < 0$ we have for all $Z \in \mathsf{Hom}(U, W)$ that

$$\dim \pi^{-1}(Z) = \dim U \dim V + 2 \dim V - 2 \dim U.$$

Proof. We let $\dim U = m$ and $\dim V = n$ so we may identify the above situation with the quotient map of the representation space of the following quiver setting (Q, α):

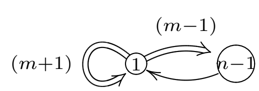


In order to prove (1), we have to compute the dimension of the nullcone of this quiver setting. This was done in Theorem 3.2 and from this we obtain

$$\dim \mathsf{Null}(Q, \alpha) = mn + 2n - 2m + m + 2 - n - 1$$

if $m + 2 - n - 1 \geq 0$.

To show that (2) holds, we first note that if $Z \neq 0$ has rank at most one, it has representation type $((1, 1), 1; (0, 1), n - 1)$. Its local quiver setting (Q_Z, α_Z) then corresponds to the quiver setting

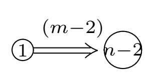


and the dimension of its fiber has to be

$$\begin{aligned}
\dim \pi^{-1}(Z) &= \dim \mathsf{Null}(Q_Z, \alpha_Z) + \dim GL_\alpha - \dim GL_{\alpha_Z} \\
&= (m-1)(n-1) + n - 1 - (m-1) + (m-n) + 1 + n^2 - 1 - (n-1)^2 \\
&= mn - m + n
\end{aligned}$$

where the dimension of the nullcone is again due to Theorem 3.2.

Finally, (3) holds by a similar computation. If Z has rank 2 it has to be of representation type $((1, 2), 1; (0, 1); n - 2)$. Then its local quiver (Q_Z, α_Z) becomes



and the dimension of its fiber becomes

$$\begin{aligned}
\dim \pi^{-1}(Z) &= \dim \mathsf{Null}(Q_Z, \alpha_Z) + \dim GL_\alpha - \dim GL_{\alpha_Z} \\
&= (m-2)(n-2) + 1 + n^2 - 1 - (n-2)^2 \\
&= mn - 2m + 2n
\end{aligned}$$

If $\dim U - \dim V + 1 < 0$ in each of the situations the local quiver is cofree, proving the last claim of the lemma. □

Lemma 4.5. Let $(\tilde{A}_n, \alpha)$ be a cyclic quiver setting with $\min_{v \in (\tilde{A}_n)_0} \alpha(v) = 2$. Call the arrows of this cyclic quiver a_0 through a_n, with $s(a_0)$ a vertex v_0 with $\alpha(v_0) = 2$ and $t(a_i) = s(a_{i+1 \bmod n})$. Then $\mathsf{Null}(\tilde{A}_n, \alpha)$ has no irreducible component C contained in

$$N_n = \{V \in \mathsf{Null}(\tilde{A}_n, \alpha) \mid V(a_n)V(a_{n-1}). \dots .V(a_0) = 0\}.$$

Proof. We will prove this by induction on n. For $n = 0$ the claim is trivial as $\mathsf{Null}(\tilde{A}_0, 2)$ is the irreducible variety of all nilpotent 2×2 matrices. Assume the claim holds for $1, \dots, n-1$. Let $v_1 = t(a_0)$ and $v_2 = t(a_1)$. Applying reduction step $\mathcal{R}_I$ to v_1 maps N_n onto N_{n-1}. Denote the map corresponding to this reduction step by π. Let C be an irreducible component in N_n and let C' be its image in N_{n-1}. If C' contains a representation W such that $rk(W(a_0)) = 2$ then computations similar to the ones in the previous lemma show that

$$\dim \pi^{-1}(W) = 2\alpha(v_1) + \alpha(v_1)\alpha(v_2) - 2\alpha(v_2).$$

This yields

$$\begin{aligned} \dim C &\leq \dim C' + \dim \pi^{-1}(W) \\ &= \dim C' + 2\alpha(v_1) + \alpha(v_1)\alpha(v_2) - 2\alpha(v_2) + 2\alpha(v_1) \\ &< \dim \mathsf{Null}(\tilde{A}_{n-1}, \alpha') + 2\alpha(v_1) + \alpha(v_1)\alpha(v_2) - 2\alpha(v_2) \\ &= \dim \mathsf{Rep}(\tilde{A}_n, \alpha) - 2\alpha(v_1) - \alpha(v_1)\alpha(v_2) + 2\alpha(v_2) - 2 \\ &\quad + \alpha(v_1)\alpha(v_2) - 2\alpha(v_2) + 2\alpha(v_1) \\ &= \dim \mathsf{Rep}(\tilde{A}_n, \alpha) - 2. \end{aligned}$$

So C cannot be an irreducible component of $\mathsf{Null}(\tilde{A}_n, \alpha)$.

If C' contains only representations W such that $W(a_0) = 0$, we obtain that

$$\dim \pi^{-1}(W) = 2\alpha(v_1) + \alpha(v_1)\alpha(v_2) - 2\alpha(v_2) + \max(0, \alpha(v_2) - \alpha(v_1) + 1).$$

And as $W(a_0) = 0$ we have

$$\dim C' \leq \dim \mathsf{Rep}(\tilde{A}_{n-1}, \alpha') - 2\alpha(v_2).$$

This gives

$$\begin{aligned}
\dim C &\leq \dim C' + \dim \pi^{-1}(W) \\
&= \dim C' + 2\alpha(v_1) + \alpha(v_1)\alpha(v_2) - 2\alpha(v_2) + 2\alpha(v_1) + \max(0, \alpha(v_2) - \alpha(v_1) + 1) \\
&\leq \dim \mathsf{Rep}(\tilde{A}_{n-1}, \alpha') - 2\alpha(v_2) + 2\alpha(v_1) + \alpha(v_1)\alpha(v_2) - 2\alpha(v_2) \\
&\quad + \max(0, \alpha(v_2) - \alpha(v_1) + 1) \\
&= \dim \mathsf{Rep}(\tilde{A}_n, \alpha) - 2\alpha(v_1) - \alpha(v_1)\alpha(v_2) + 2\alpha(v_2) - 2\alpha(v_2) \\
&\quad + \alpha(v_1)\alpha(v_2) - 2\alpha(v_2) + 2\alpha(v_1) + \max(0, \alpha(v_2) - \alpha(v_1)) + 1 \\
&= \dim \mathsf{Rep}(\tilde{A}_n, \alpha) - 2\alpha(v_2) + \max(0, \alpha(v_2) - \alpha(v_1) + 1) \\
&\leq \dim \mathsf{Rep}(\tilde{A}_n, \alpha) - 3.
\end{aligned}$$

So again, C cannot be an irreducible component of $\mathsf{Null}(\tilde{A}_n, \alpha)$.

If C' contains only representations W such that $rk(W(a_0)) \leq 1$, let W be a representation such that $rk(W(a_0)) = 1$. We have that

$$\dim \pi^{-1}(W) = 2\alpha(v_1) + \alpha(v_1)\alpha(v_2) - 2\alpha(v_2) + \max(0, \alpha(v_2) - \alpha(v_1)).$$

We also have that C' is strictly contained within the irreducible set

$$R = \{W \in \mathsf{Rep}(\tilde{A}_{n-1}, \alpha') \mid rk(W(a_0)) \leq 1\}$$

where

$$\dim R = \dim \mathsf{Rep}(\tilde{A}_{n-1}, \alpha') - \alpha(v_2) + 1.$$

First of all assume $\alpha(v_2) \leq \alpha(v_1)$. In this case we again have

$$\begin{aligned}
\dim C &\leq \dim C' + \dim \pi^{-1}(W) \\
&= \dim C' + 2\alpha(v_1) + \alpha(v_1)\alpha(v_2) - 2\alpha(v_2) + 2\alpha(v_1) \\
&< \dim \mathsf{Null}(\tilde{A}_{n-1}, \alpha') + 2\alpha(v_1) + \alpha(v_1)\alpha(v_2) - 2\alpha(v_2) \\
&= \dim \mathsf{Rep}(\tilde{A}_n, \alpha) - 2\alpha(v_1) - \alpha(v_1)\alpha(v_2) + 2\alpha(v_2) - 2 \\
&\quad + \alpha(v_1)\alpha(v_2) - 2\alpha(v_2) + 2\alpha(v_1) \\
&= \dim \mathsf{Rep}(\tilde{A}_n, \alpha) - 2.
\end{aligned}$$

So C cannot be an irreducible component of $\mathsf{Null}(\tilde{A}_n, \alpha)$.

Now assume $\alpha(v_2) > \alpha(v_1) \geq 2$, then

$$\dim C \leq \dim C' + \dim \pi^{-1}(W)$$

$$
\begin{aligned}
&= \dim C' + 2\alpha(v_1) + \alpha(v_1)\alpha(v_2) - 2\alpha(v_2) + \alpha(v_2) - \alpha(v_1) \\
&< \dim R + 2\alpha(v_1) + \alpha(v_1)\alpha(v_2) - 2\alpha(v_2) + \alpha(v_2) - \alpha(v_1) \\
&= \dim \mathsf{Rep}(\tilde{A}_n, \alpha) - 2\alpha(v_1) - \alpha(v_1)\alpha(v_2) + 2\alpha(v_2) - \alpha(v_2) + 1. \\
&\quad +2\alpha(v_1) + \alpha(v_1)\alpha(v_2) - 2\alpha(v_2) + \alpha(v_2) - \alpha(v_1) \\
&= \dim \mathsf{Rep}(\tilde{A}_n, \alpha) - \alpha(v_1) + 1.
\end{aligned}
$$

If $\alpha(v_1) \geq 3$ this means $\dim C < \dim \mathsf{Null}(\tilde{A}_n, \alpha)$ so C cannot be an irreducible component. If $\alpha(v_1) = 2$ we reorder the vertices so that v_1 becomes the first vertex and apply the reduction step on v_2. Repeating the different possible cases from the previous paragraphs then yields that C cannot be an irreducible component as in the last case we have $\alpha(v_2) \geq 3$. □

These last lemmas now allow us to prove

Theorem 4.6. A quiver setting (Q, α) reducing to $\mathcal{Q}_2$ is cofree if and only if either

(1) it is of the form $\mathcal{Q}_2(p, q, s)$ with either $s = 0$ or $\alpha(v) \geq 4$ for all v but one;
(2) it is of the form $\mathcal{Q}_2^{double}(p)$ for any p and $\alpha(v) \geq 4$ for all v different from the final vertex.

Proof. In order to prove (1), first assume $s = 0$. In this case the quiver (Q, α) is a connected sum of two cyclic quiver settings $(\tilde{A}_p, \alpha_p)$ and $(\tilde{A}_q, \alpha_q)$ in a vertex with dimension 2 which we denote by v. We will denote the arrows of the first cyclic quiver by $a_0, \ldots, a_p$ and of the second cyclic quiver by $b_0, \ldots, b_q$ with $s(a_0) = s(b_0) = v$ and $t(a_i) = s(a_{i+1 \bmod p+1})$ resp. $t(b_j) = s(a_{j+1 \bmod q+1})$. We have an embedding

$$\mathsf{Null}(Q, \alpha) \subset \mathsf{Null}(\tilde{A}_p, \alpha_p) \times \mathsf{Null}(\tilde{A}_q, \alpha_q).$$

Any maximal irreducible component of $\mathsf{Null}(Q, \alpha)$ has to be a real closed subset of an irreducible component of $\mathsf{Null}(\tilde{A}_p, \alpha_p) \times \mathsf{Null}(\tilde{A}_q, \alpha_q)$. Indeed, if we denote a representation in such an irreducible component of $\mathsf{Null}(Q, \alpha)$ as (V, W), we know that

$$tr(V(a_p).\ldots.V(a_0)W(b_q).\ldots.W(b_0)) = 0$$

Now by Lemma 4.5 we know that in all irreducible components of $\mathsf{Null}(\tilde{A}_p, \alpha_p)$ there are elements V satisfying

$$V(a_p).\ldots.V(a_0) \neq 0,$$

and in all irreducible components of $\mathsf{Null}(\tilde{A}_q, \alpha_q)$ there are elements W satisfying

$$W(b_q)\ldots.V(b_0) \neq 0.$$

This means in any irreducible component of $\mathsf{Null}(\tilde{A}_p, \alpha_p)$ we get a representation V such that

$$V(a_p)\ldots.V(a_0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and in any irreducible component of $\mathsf{Null}(\tilde{A}_q, \alpha_q)$ we get a representation W such that

$$W(b_q)\ldots.V(b_0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

This yields

$$tr(V(a_p)\ldots.V(a_0)W(b_q)\ldots.W(b_0)) = 1.$$

But then we have

$$\dim \mathsf{Null}(Q, \alpha) <= \dim \mathsf{Null}(\tilde{A}_p, \alpha_p) + \dim \mathsf{Null}(\tilde{A}_q, \alpha_q) - 1$$

and hence

$$\dim \mathsf{Null}(Q, \alpha) = \dim \mathsf{Rep}(Q, \alpha) - \dim \mathsf{iss}(Q, \alpha).$$

Now assume $s > 0$ and $\alpha(v) \geq 4$ for all but one vertex f which has $\alpha(f) = 2$. Let us first consider the situation where $\#\{a \in Q_0 \mid h(a) = f\} = 1$ and let $v = s(a)$ for a the unique arrow with $t(a) = f$. We apply reduction step $\mathcal{R}_I$ to v and consider the corresponding map

$$\pi : \mathsf{Null}(Q, \alpha) \twoheadrightarrow \mathsf{Null}(\mathcal{R}_I(Q), \mathcal{R}_I(\alpha)).$$

If $\sum_{x \in Q_1, t(x)=v} \alpha(s(x)) - \alpha(v) + 1 \leq 0$ then the dimension d of any fiber of π equals

$$d = \alpha(v) \sum_{x \in Q_1, t(x)=v} \alpha(s(x)) + 2\alpha(v) - 2 \sum_{x \in Q_1, t(x)=v} \alpha(t(x)).$$

But then, for any maximal irreducible component C of $\mathsf{Null}(Q, \alpha)$ and any element $Z \in \pi(C)$ we have by the dimension formula for morphisms in combination with Lemma 4.4

$$\begin{aligned} \dim C &\leq \dim \pi(C) + \dim \pi^{-1}(Z) \\ &= \dim \mathsf{Null}(\mathcal{R}_I(Q), \mathcal{R}_I(\alpha)) + \alpha(v) \sum_{x \in Q_1, h(x)=v} \\ &\quad \alpha(t(x)) + 2\alpha(v) - 2 \sum_{x \in Q_1, h(x)=v} \alpha(t(x)) \end{aligned}$$

$$\begin{aligned}
&= \dim \mathsf{Rep}(\mathcal{R}_I(Q),\mathcal{R}_I(\alpha)) - \dim \mathsf{iss}(\mathcal{R}_I(Q),\mathcal{R}_I(\alpha)) \\
&\quad +\alpha(v)\sum_{x\in Q_1,t(x)=v}\alpha(s(x)) + 2\alpha(v) - 2\sum_{x\in Q_1,t(x)=v}\alpha(s(x)) \\
&= \dim \mathsf{Rep}(Q,\alpha) - \alpha(v)\sum_{x\in Q_1,t(x)=v}\alpha(s(x)) - 2\alpha(v) \\
&\quad +2\sum_{x\in Q_1,t(x)=v}\alpha(s(x)) - \dim \mathsf{iss}(Q,\alpha) \\
&\quad +\alpha(v)\sum_{x\in Q_1,t(x)=v}\alpha(s(x)) + 2\alpha(v) - 2\sum_{x\in Q_1,t(x)=v}\alpha(s(x)) \\
&= \dim \mathsf{Rep}(Q,\alpha) - \dim \mathsf{iss}(Q,\alpha)
\end{aligned}$$

so $\dim \mathsf{Null}(Q,\alpha) = \dim \mathsf{Rep}(Q,\alpha) - \dim \mathsf{iss}(Q,\alpha)$.

Now assume $\sum_{x\in Q_1,t(x)=v}\alpha(s(x)) - \alpha(v) + 1 > 0$ and let C be a maximal irreducible component of $\mathsf{Null}(Q,\alpha)$. By the dimension formula for morphisms we have for any element in $Z \in \pi(C)$ that

$$\dim(C) \le \dim \pi(C) + \dim \pi^{-1}(Z).$$

If $\pi(C)$ contains an element Z such that

$$rk((Z(x))_{x\in Q_1,h(x)=f}) = 2$$

then by lemma 4.4 we have

$$\begin{aligned}
\dim C &\le \dim \pi(C) + \dim \pi^{-1}(Z) \\
&= \dim \mathsf{Null}(\mathcal{R}_I(Q),\mathcal{R}_I(\alpha)) \\
&\quad +\alpha(v)\sum_{x\in Q_1,t(x)=v}\alpha(s(x)) + 2\alpha(v) - 2\sum_{x\in Q_1,t(x)=v}\alpha(s(x)) \\
&= \dim \mathsf{Rep}(Q,\alpha) - \dim \mathsf{iss}(Q,\alpha)
\end{aligned}$$

through the same computation as in the previous paragraph. If $\pi(C)$ does not contain such an element, we must have

$$rk((Z(x))_{x\in Q_1,t(x)=f}) \le 1.$$

Assume we have an element with $rk((Z(x))_{x\in Q_1,t(x)=f}) = 1$, then

$$\pi(C) \subset L_1 \times \mathsf{Rep}(\overline{Q},\overline{\alpha})$$

with $(\overline{Q},\overline{\alpha})$ the quiver setting with all arrows x with $h(x) = f$ removed and L_1 the set of all linear maps from a vectorspace of dimension $\sum_{x\in Q_1,t(x)=v}\alpha(s(x))$ to a vectorspace of dimension 2 that have rank at most 1. By [9, II.4.1, Lemma 1], we have that L_1 is irreducible of dimension $\sum_{x\in Q_1, t(x)=v}\alpha(s(x))+1$. Now any Z in $\pi(C)$ has to satisfy $tr(X)=tr(Y)=0$ for X the cycle along the first cyclic quiver and Y the cycle along the second cyclic quiver. This means

$$\dim \pi(C) \leq \dim L_1 + \dim \mathsf{Rep}(\overline{Q},\overline{\alpha}) - 2.$$

But then

$$\begin{aligned}
\dim C &\leq \dim \pi(C) + \dim \pi^{-1}(Z)\\
&= \dim L_1 + \dim \mathsf{Rep}(\overline{Q},\overline{\alpha}) - 2\\
&\quad +\alpha(v)\sum_{x\in Q_1, t(x)=v}\alpha(s(x)) + \alpha(v) - \sum_{x\in Q_1, t(x)=v}\alpha(s(x))\\
&= \dim \mathsf{Rep}(Q,\alpha) - \alpha(v) - 1\\
&\leq \dim \mathsf{Rep}(Q,\alpha) - 5\\
&= \dim \mathsf{Rep}(Q,\alpha) - \dim \mathsf{iss}(Q,\alpha)
\end{aligned}$$

Finally, assume we have no element Z in $\pi(C)$ such that $rk((Z(x))_{x\in Q_1, t(x)=f})\geq 1$, then $(Z(x))_{x\in Q_1, t(x)=f}=0$ and

$$\pi(C) \subset \{0\}\times \mathsf{Rep}(\overline{Q},\overline{\alpha}).$$

But then

$$\begin{aligned}
\dim C &\leq \dim \pi(C) + \dim \pi^{-1}(Z)\\
&\leq \dim \mathsf{Rep}(\overline{Q},\overline{\alpha})\\
&\quad +\alpha(v)\sum_{x\in Q_1, t(x)=v}\alpha(s(x)) + \alpha(v) - \sum_{x\in Q_1, t(x)=v}\alpha(s(x)) + 1\\
&\leq \dim \mathsf{Rep}(Q,\alpha) - \alpha(v) - \sum_{x\in Q_1, t(x)=v}\alpha(s(x)) + 1\\
&\leq \dim \mathsf{Rep}(Q,\alpha) - 5\\
&= \dim \mathsf{Rep}(Q,\alpha) - \dim \mathsf{iss}(Q,\alpha)
\end{aligned}$$

To show that the settings in (2) are cofree, the same computations as in the previous paragraph may be made. Finally, in order to show that these are all cofree settings reducing to $\mathcal{Q}_2$, note that all other possibilities were already shown to be not cofree in lemma 4.3. □

References

[1] V. L. Popov. Representations with a free module of coinvariants. *Func. Anal. Appl.*, 10(3):242–244, 1976.

[2] Gerald Schwarz. Representations of simple lie groups with a free module of coinvariants. *Invent. Math.*, 50:1–12, 1978.

[3] A. King. Moduli of representations of finite dimensional algebras. *Quat.J.Math. Oxford*, 45:515–530, 1994.

[4] Peter Gabriel and Andrei V. Roiter. *Representations of Finite-Dimensional Algebras*. Springer, 1992.

[5] Claus Michael Ringel. Hall algebras and quantum groups. *Invent. Math.*, 101:583–591, 199.

[6] Gerald Schwarz. Representations of simple lie groups with regular rings of invariants. *Invent. Math.*, 49:167–191, 1978.

[7] Raf Bocklandt. Smooth quiver representation spaces. *J. of Algebra*, 253:296–313, 2002.

[8] Geert Van de Weyer. A proof of the popov conjecture for quivers. *J. Algebra*, 274:373–386, 2004.

[9] Hanspeter Kraft. *Geometrische Methoden in der Invariantentheorie*. Aspects of Mathematics. Vieweg, 1986.

[10] Lieven Le Bruyn and Claudio Procesi. Semi simple representations of quivers. *Transactions Americ.Math.Society*, 317:585–598, 1990.

[11] George Lusztig. Canonical bases arising from quantized enveloping algebras. *J.Amer.Math.Soc.*, 3:447–498, 1990.

[12] Geert Van de Weyer. *Nullcones of Quiver Representations*. Ph.D. thesis, University of Antwerp, 2003.

[13] Lieven Le Bruyn. Optimal filtrations on representations of finite dimensional algebras. *Transactions Americ.Math.Society*, 353:411–426, 2000.

Universiteit Antwerpen (UA), B-2600 Antwerp (Belgium)

E-mail address: `rafael.bocklandt@ua.ac.be`

Universiteit Antwerpen (UA), B-2000 Antwerp (Belgium)

E-mail address: `geert.vandeweyer@ua.ac.be`