GRADED CALABI YAU ALGEBRAS OF DIMENSION 3

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ABSTRACT. In this paper we prove that Graded Calabi Yau Algebras of dimension 3 are isomorphic to path algebras of quivers with relations derived from a superpotential. We show that for a given quiver Q and a degree d, the set of good superpotentials of degree d, i.e. those that give rise to Calabi Yau algebras is either empty or almost everything (in the measure theoretic sense). We also give some constraints on the structure of quivers that allow good superpotentials, and for the simplest quivers we give a complete list of the degrees for which good superpotentials exist.

1. INTRODUCTION AND MOTIVATION

If one studies boundary conditions of the B-model in super string theory over an n-dimensional Calabi Yau manifold X, one obtains naturally the derived category of coherent sheaves $\mathcal{D}^b\mathtt{Coh}X$ [10]. This category is called a Calabi Yau category of dimension three, i.e. the third shift in the derived category is a Serre Functor:

$$\forall A, B \in \mathcal{D}^b\mathtt{Coh}X : \ \mathrm{Hom}_{\mathcal{D}^b\mathtt{Coh}X}(A, B) \cong \mathrm{Hom}_{\mathcal{D}^b\mathtt{Coh}X}(B, A[3])^*,$$

where the isomorphisms are natural in A and B. In general this category is too big to study its structure directly and therefore it is interesting to look at full triangulated subcategories of $\mathcal{D}^b\mathtt{Coh}X$ that can be modeled using derived categories of module categories of noncommutative algebras. In string theoretical papers this is often done using path algebras of quivers with relations coming form a superpotential: if Q is a quiver and $\mathbb{C}Q$ the corresponding path algebra, then a superpotential is an element of the vector space $\mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$. On this space we can define for every arrow a a 'derivation' ∂_a that cuts out a (for a precise definition see section 2.1). Given a superpotential W one can construct the *vacualgebra* [4]

$$A_W := \mathbb{C}Q/(\partial_a W : a \in Q_1).$$

In the exemplary cases worked out by physicists, the derived category of finite dimensional modules of the vacualgebra is indeed a Calabi Yau category, and hence these algebras are called Calabi Yau Algebras.

In this note we will show that in the case of graded algebras, every graded path algebra with relations that is Calabi Yau of dimension 3 must be isomorphic to a vacualgebra of some superpotential. The converse is not true but we will show that being a Calabi Yau algebra of dimension 3 corresponds to the exactness of a certain bimodule complex. Therefore, for a given quiver Q and a given degree d the subset of superpotentials of degree d that give rise to Calabi Yau vacualgebras is either empty or almost everything. Furthermore we will use Groebner basis techniques to explicitly determine the list of degree of good superpotentials of simple quivers.

The results in this paper build further on ideas introduced by M. Van den Bergh in [8]. Similar results on Calabi Yau algebras in different settings have been obtained by R. Rouquier and V. Ginzberg [1].

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2. Preliminaries

2.1. **Path Algebras with relations.** As usual a *quiver* Q is an oriented graph. We denote the set of vertices by Q_0, the set of arrows by Q_1 and the maps h, t assign to each arrow its head and tail. A *nontrivial path* p is a sequence of arrows $a_1 \cdots a_k$ such that $t(a_i) = h(a_{i+1})$, whereas a *trivial path* is just a vertex. We will denote the length of a path by $|p| := k$ and the head and tail by $h(p) = h(a_1),\ t(p) = t(a_k)$. A path is called a cycle if $h(p) = t(p)$. A quiver is called *connected* if it is not the disjoint union of two subquivers and it is *strongly connected* if there is a cycle through each pair of vertices.

The path algebra $\mathbb{C}Q$ is the complex vector space with as basis the paths in Q and the multiplication of two paths p, q is their concatenation pq if $t(p) = h(q)$ or else 0. We can put a gradation on $\mathbb{C}Q$ using the length of the paths. The space spanned by all paths of nonzero length is a graded ideal of $\mathbb{C}Q$ and we will denote it by $\mathcal{J}$.

The vector space $\mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$ has as basis the set of cycles up to cyclic permutation of the arrows. We can embed this space into $\mathbb{C}Q$ by mapping a cycle onto the sum of all its possible cyclic permutations:

$$\circlearrowleft : \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q] \to \mathbb{C}Q : a_1 \cdots a_n \mapsto \sum_i a_i \cdots a_n a_1 \cdots a_{i-1}.$$

Another convention we will use is the inverse of arrows: if $p := a_1 \cdots a_n$ is a path and b an arrow, then $pb^{-1} = a_1 \cdots a_{n-1}$ if $b = a_n$ and zero otherwise. Similarly one can define $b^{-1}p$. These new defined maps can be combined to obtain a 'derivation'

$$\partial_a : \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q] \to \mathbb{C}Q : p \mapsto \circlearrowleft(p)a^{-1} = a^{-1} \circlearrowleft(p).$$

From now on A will denote the quotient algebra $\mathbb{C}Q/\mathcal{I}$ by a finitely generated graded ideal $\mathcal{I} \subset \mathcal{J}^2$. The set $\mathcal{R} \subset \mathcal{I}$ will be a minimal set of homogeneous generators each sitting inside some $i\mathbb{C}Qj,\ i, j \in Q_0$.

We denote the semi-simple (left) A-module $A/A_{\geq 1} \cong \mathbb{C}Q/\mathcal{J}$ by S. S is the direct sum of $\#Q_0$ simple one-dimensional A-modules S_i, each corresponding to a vertex $i \in \mathbb{C}Q$. To each vertex we can also assign a projective module P_i which is the left ideal Ai and $S_i = P_i/(P_i)_{\geq 1}$. Although it is a little sloppy we will also use S to denote the subring $A_0 \cong \mathbb{C}Q_0$, generated by the vertices.

2.2. **Calabi Yau Categories.** Let $\mathcal{C}$ be an abelian $\mathbb{C}$-linear category and $\mathcal{D}^b\mathcal{C}$ its bounded derived category. Using the shift we can define a graded functor (s, η^s) in the sense of A.5.2 where s is the shift functor and the η^s gives natural isomorphisms

$$\eta^s_A : s(A[1]) \to (sA)[1] : x \mapsto -x.$$

As explained in the appendix, these maps are uniquely determined by the demand of compatibility with the triangulated structure of $\mathcal{D}^b\mathcal{C}$.

Definition 2.1. The category $\mathcal{D}^b\mathcal{C}$ is called *Calabi Yau of dimension* n if there are natural isomorphisms

$$\nu_{A,B} : \mathrm{Hom}_{\mathcal{D}^b\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}^b\mathcal{C}}(B, s^n A)^*,\ (^* \text{ is the complex dual})$$

or, in other words, the n^{th} shift is a Serre Functor.

Starting with a graded path algebra with relations A, we can construct the category of finite dimensional left A-modules: $\mathtt{Rep}A$. This is an abelian category so we can construct its bounded derived category $\mathcal{D}^b\mathtt{Rep}A$. We will call A a graded Calabi Yau Algebra of dimension n if $\mathcal{D}^b\mathtt{Rep}A$ is a Calabi Yau category of dimension n.

Although the definition is asymmetric in the sense that one only uses left modules, it is easy to see that if A is Calabi Yau, the derived category of finite dimensional *right* modules $\mathcal{D}^b\mathtt{RRep}A$ is also a Calabi Yau Category. This can be proved using the complex dual as an anti-equivalence between $\mathcal{D}^b\mathtt{Rep}A$ and $\mathcal{D}^b\mathtt{RRep}A$: let M, N be complexes of right modules and define

$$\nu^{\mathtt{RRep}A}_{M,N} : \mathrm{Hom}_{\mathcal{D}^b\mathtt{RRep}A}(M, N) \to \mathrm{Hom}_{\mathcal{D}^b\mathtt{RRep}A}(N, s^n M)^*$$

by the equality

$$\nu_{M,N}^{\mathsf{RRep}A}(f)(g) = \nu_{N^*,M^*}^{\mathsf{Rep}A}(s^n(f^T))(s^n(g^T)).$$

The Calabi Yau property of the derived category can be tracked back to the original category to give us properties that we will often use

Property 2.2. *If A is Calabi Yau of dimension n then*

C1 *The global dimension of A is also n.*

C2 *If $X, Y \in \mathsf{Rep}A$ then*

$$\mathsf{Ext}_A^k(X,Y) \cong \mathsf{Ext}_A^{n-k}(Y,X)^*.$$

C3 *The identifications above gives us a pairings $\langle,\rangle_{XY}^k : \mathsf{Ext}_A^k(X,Y) \times \mathsf{Ext}_A^{n-k}(Y,X) \to \mathbb{C}$ which satisfy*

$$\langle f,g\rangle_{XY}^k = \langle 1_X, g*f\rangle_{XX}^0 = (-1)^{k(n-k)}\langle 1_Y, f*g\rangle_{YY}^0,$$

where $*$ denotes the standard composition of extensions.

Proof. (1) : if $i > n$ then $\mathsf{Ext}_A^i(M,N) = \mathsf{Ext}^{n-i}(M,N) = 0$ so $\mathrm{gldimA} \leq \mathrm{n}$ and $\mathsf{Ext}_A^n(A/A_+, A/A_+) = \mathsf{Hom}_A(A/A_+, A/A_+) = A/A_+ \neq 0$ so $\mathrm{gldimA} \geq \mathrm{n}$. For $(2-3)$ see the appendix. □

3. Graded Calabi Yau algebra's of dimension $n \leq 3$

In this section we will give descriptions of the types of quivers and relations that appear in graded Calabi Yau algebras of dimension 3.

From now on we will also assume that the quiver Q is connected. This is not a severe restriction because A is the direct sum of subalgebras defined over its connected components. Many properties like the Calabi Yau property transfer from the algebra to its direct summands: $A_1 \oplus A_2$ is Calabi Yau of dimension n if both A_1 and A_2 are Calabi Yau of dimension n. This follows from the fact that the representation category (and hence the derived category) of A decomposes as the direct sum of $\mathsf{Rep}A_1$ and $\mathsf{Rep}A_2$.

Theorem 3.1. *If A is Calabi Yau of dimension 3 then*

(1) *there is a homogeneous superpotential $W \in \mathbb{C}Q/[\mathbb{C}Q,\mathbb{C}Q]$ such that*

$$A \cong \mathbb{C}Q/(\partial_a W : a \in Q_1),$$

(2) *every arrow in Q is contained in a cycle of $\circlearrowright W$.*

(3) *every vertex in Q is the source of two arrows and the target of two arrows.*

Proof. As the global dimension of A must be 3, there is a projective graded resolution

$$\bigoplus_{j\in Q_0} P_j^{m_{ij}} \xrightarrow{(f_r)} \bigoplus_{t(r)=i} P_{h(r)} \xrightarrow{(rb^{-1})} \bigoplus_{t(b)=i} P_{h(b)} \xrightarrow{(\cdot b)} P_i \twoheadrightarrow S_i.$$

In the diagram above the $r's$ are the relations in $\mathcal{R}$ and the $b's$ are arrows, the f_r are maps that are not further specified. Using the Calabi Yau property and comparing dimensions we can conclude that

(1) $m_{ij} = \mathsf{Dim}\,\mathsf{Ext}^3(S_i,S_j) = \mathsf{Dim}\,\mathsf{Hom}(S_j,S_i) = \delta_{ij}$,

(2) $\#\{r \in \mathcal{R} : h(r) = j, t(r) = i\} = \mathsf{Dim}\,\mathsf{Ext}^2(S_i,S_j) = \mathsf{Dim}\,\mathsf{Ext}^1(S_i,S_j) = \#\{a \in Q_1 : i \xleftarrow{a} j\}$.

Because of (1) we can identify each f_r with an element in $iAh(r)$. Consider the finite dimensional quotient algebra

$$M = A/(f_r : r \in \mathcal{R}, A_n : n \geq N) \text{ where } \forall r : N > \deg f_r.$$

The Calabi Yau property allows us to calculate the dimension of iMj:

$$\mathsf{Dim}\, iMj = \mathsf{Dim}\,\mathsf{Hom}(P_i, Mj) = \mathsf{Dim}\,\mathsf{Ext}^3(S_i, Mj) \overset{CY}{=} \mathsf{Dim}\,\mathsf{Hom}(Mj, S_i) = \delta_{ij},$$

and conclude that M must be isomorphic to the degree zero part of A. As (2) implies there are only as many f_r as there are arrows, we can conclude that the f_r are linear and form a basis for A_1. Hence, by linearly combining our original relations, we can assume that the f_r can be identified with the arrows. Let r_a be the (nonzero) relation for which $f_{r_a} = a$. This relation occurs only in the resolution of $S_{t(r_a)} = S_{h(a)}$ and therefore $h(a) = t(r_a)$ and $t(a) = h(r_a)$.

Every arrow a is contained in a cycle: ar_a, so if there is a path between two vertices there is also a path in the opposite direction. This means we that because Q is assumed to be connected, Q is also strongly connected. We will now prove that all the r_a have the same degree.

Let a be the arrow for which r_a has minimal degree. First of all note that if two arrows a, b share their heads then $\deg r_a = \deg r_b$ because they occur in the same resolution. Denote by $r_{ab} := r_a b^{-1}$ the terms that appear in the middle map of the resolution. These terms are only nonzero if $t(b) = h(a)$. The fact that the maps in the resolutions form a complex implies that $\sum_{h(a)=i} ar_{ab}$ is zero in A. If $\deg r_a = \deg ar_{ab}$ is minimal then there exist scalars (g_{bc}) such that

$$\sum_{h(a)=i} ar_{ab} = \sum_{\substack{h(c)=h(b)\\ t(c)=h(a)}} g_{bc} r_c = \sum_{\substack{h(c)=h(b)\\ t(c)=t(b)}} g_{bc} \sum_{t(d)=h(b)} r_{cd} d \text{ evaluated in } \mathbb{C}Q.$$

The $\deg r_c$ (which is the same for all c with $h(c) = h(b)$ including b itself) must also be minimal. All arrows following an arrow of minimal r_a-degree are also minimal, so by induction all arrows in Q have the same degree.

We will now prove that (g_{ab}) can be seen as a diagonal matrix. First note that

$$\mathsf{Ext}^1(S_i, S_j) = \mathrm{Hom}(\bigoplus_{t(a)=i} P_{h(a)}, S_j) \cong \mathbb{C}^{\{i \to j\}}$$

and on the other hand

$$\mathsf{Ext}^2(S_j, S_i) = \mathrm{Hom}(\bigoplus_{t(r_a)=j} P_{h(r_a)}, S_i) = \mathrm{Hom}(\bigoplus_{h(a)=j} P_{t(a)}, S_i) \cong \mathbb{C}^{\{i \to j\}}.$$

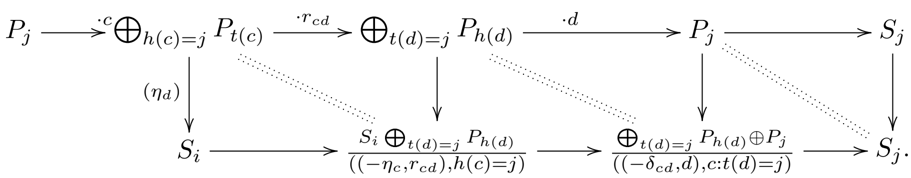
We can compose the spaces in two different ways:

$$\mathsf{Ext}^1(S_i, S_j) \times \mathsf{Ext}^2(S_j, S_i) \to \mathsf{Ext}^3(S_j, S_j) \cong \mathbb{C} : (\xi_a) * (\eta_b) = \sum_a \xi_a \eta_a$$

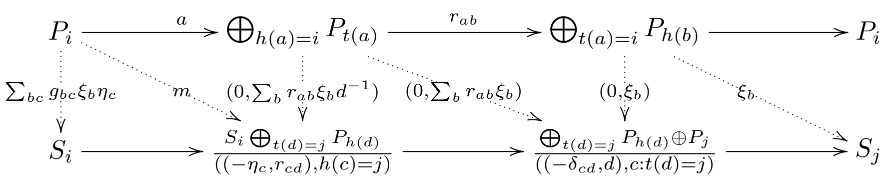
and

$$\mathsf{Ext}^2(S_j, S_i) \times \mathsf{Ext}^1(S_i, S_j) \to \mathsf{Ext}^3(S_i, S_i) \cong \mathbb{C} : (\eta_b) * (\xi_a) = \sum_{a,b} g_{ab} \xi_a \eta_b$$

We only work out the last composition since the other one is similar. We extend the sequence (η_b) to a sequence running over all arrows by adding zeros. We push out (dotted lines) the map η forward along the resolution to obtain an exact sequence $S_i \to \cdots \to S_j$:



We use this sequence to pull back (dotted arrows) the map (ξ_b)



where

$$m = (0, \sum_{ab} a r_{ab} \xi_b d^{-1}) = (0, \sum_{bce} g_{bc} r_{ce} e \xi_b d^{-1}) = (0, \sum_{bc} g_{bc} r_{cd} \xi_b) = (\sum_{bc} g_{bc} \eta_c \xi_b, 0)$$

Because of the Calabi Yau property there exist traces $\mathrm{Tr}_{S_j} : \mathsf{Ext}^3(S_j, S_j) \to \mathbb{C}$. As these Ext-spaces are one-dimensional we can represent these traces by scalars α_j. Property A.5.2 can be rewritten as

$$\mathrm{Tr}_{S_j}((\xi_a) * (\eta_b)) = \mathrm{Tr}_{S_i}((\eta_b) * (\xi_a))$$
$$\alpha_j \sum_a \xi_a \eta_a = \alpha_i \sum_{a,b} g_{ab} \xi_a \eta_b.$$

As this holds for arbitrary (ξ_a) and (η_b) we can conclude that

$$g_{ab} = \frac{\alpha_{h(a)}}{\alpha_{t(a)}} \delta_{ab}.$$

Now we construct the element

$$\sum_{a,b \in Q_1} \alpha_{h(a)} a r_{ab} b,$$

Which is a sum of cycles. It is also a homogeneous element that is invariant under cyclic permutation:

$$\sum_{a,b} \alpha_{h(a)} r_{ab} ab = \sum_{a,b} \alpha_{t(b)} r_{ab} ab = \sum_{a,b} \alpha_{t(b)} \frac{\alpha_{h(b)}}{\alpha_{t(b)}} b r_{ba} a = \sum_{a,b} \alpha_{h(b)} b r_{ba} a.$$

This implies that we can identify it with $\circlearrowleft (W)$ where $W \in \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$ such that r_a is a scalar multiple of $\partial_a W$.

To prove the last condition on the structure of the quiver, assume first that v is the tail of a unique arrow a and let the b_i be the vertices whose head is $t(a)$. As $r_a = \sum_i b_i r_{b_i a}$ and $r_a \neq 0$ in $\mathbb{C}Q$, there must be at least one $r_{b_i a} \neq 0$ in $\mathbb{C}Q$ and because of its degree it is also nonzero in A. Now $r_{b_i a}$ sits inside the kernel of $P_{h(a)} \xrightarrow{\cdot a} P_{t(a)}$ because $\partial_{b_i} W = r_{b_i a} a$. This would imply that the resolution for $S_{h(a)}$ is not exact. Using right modules instead of left one proves that every vertex is also the tail of at least two arrows. □

For reasons of completeness we also include the descriptions of Calabi Yau algebras of smaller dimension because the techniques to do this are similar.

The zero-dimensional case is trivial and consists of the semi-simple algebras i.e. quivers without arrows. The one-dimensional case consists a direct sums of $\mathbb{C}[X]$ (disjunct unions of one-vertex-one-loop quivers). This is a consequence of property $C2$: $\#\{i \leftarrow j\} = \mathsf{Dim}\,\mathsf{Ext}^1(S_i, S_j)$=$\mathsf{Dim}\,\mathsf{Hom}(S_j, S_i) = \delta_{ij}$.

Theorem 3.2. *If A is Calabi Yau of dimension 2 then A is the preprojective algebra of a non-Dynkin quiver (for a definition of a preprojective algebra see* [7]*).*

Proof. As the global dimension of $A = \mathbb{C}Q/\mathcal{I}$ is now 2, the projective graded resolutions look like

$$\bigoplus_{t(r)=i} P_{h(r)} \xhookrightarrow{\cdot ra^{-1}} \bigoplus_{t(a)=i} P_{h(a)} \xrightarrow{\cdot a} P_i \twoheadrightarrow S_i$$

From the Calabi Yau property C2, we deduce that

$$\#\{r \in \mathcal{R} | h(r) = i, t(r) = j\} = \mathsf{Dim}\,\mathsf{Ext}^2(S_i, S_j) \stackrel{CY}{=} \mathsf{Dim}\,\mathsf{Hom}(S_j, S_i) = \delta_{ij},$$

i.e. for every vertex there is a unique relation and vice versa.

Now, similarly to the three dimensional case, we consider the finite dimensional quotient algebra $M = A/(ra^{-1} : r \in \mathcal{R}, a \in Q_1, A_n : n \geq N)$ where $\forall r : N > \deg r$. The Calabi Yau property allows us to calculate the dimension of iMj:

$$\mathsf{Dim}\, iMj = \mathsf{Dim}\,\mathsf{Hom}(P_i, Mj) = \mathsf{Dim}\,\mathsf{Ext}^2(S_i, Mj) \stackrel{CY}{=} \mathsf{Dim}\,\mathsf{Hom}(Mj, S_i) = \delta_{ij}$$

and conclude that M must be isomorphic to the degree zero part of A. This implies that the ra^{-1} are all linear and span A_1. For every a there is also at most one r such that ra^{-1} is nonzero: the unique r with $t(r) = t(a)$. If we group the relations together in $R = \sum_{r \in \mathcal{R}} r$ then there exists an invertible complex matrix g_{ab} such that

$$Ra^{-1} = \sum_{a,b} g_{ab} b.$$

We can use this g to explicitly calculate the pairing (the calculation is analogous to the thee-dimensional case).

$$\mathsf{Ext}^1(S_i, S_j) \times \mathsf{Ext}^1(S_j, S_i) \to \mathsf{Ext}^2(S_i, S_i)(\xi_a) * (\eta_b) = \sum_{ab} g_{ab} \xi_a \eta_b.$$

Property $C3$ now implies that g_{ab} is antisymmetric and non-degenerate so using a base transformation on the arrows we can put g_{ab} in its standard symplectic form. The fact that $g_{ab} \neq 0 \implies h(a) = t(b) \wedge t(a) = h(b)$ indicates that this base transformation only mixes arrows with identical head and tail. In this new basis the arrows can be partitioned in couples (a, a^*) with $g_{aa^*} = 1$ and $g_{ab} = 0$ if $b \neq a^*$. The relation R assumes the form of the standard preprojective relations:

$$\sum_a aa^* - a^* a$$

where a runs over the unstarred half of the arrows. Also Q cannot be the double of a Dynkin quiver because A must have global dimension 2, see [7]. □

4. Selfdual Resolutions

In this section we use the notion of selfdual resolutions to give a criterium to check whether a vacualgebra A_W is indeed Calabi Yau.

4.1. **Projective A-modules.** Let A be a finitely generated graded algebra that is the quotient of a path algebra $\mathbb{C}Q$ and let $S = A_0$.

For every finite dimensional S-bimodule T we can define a projective A-bimodule

$$F_T := A \otimes_S T \otimes_S A.$$

We denote the full subcategory of $\mathsf{Mod} A - A$ containing these projective modules as $\mathcal{P}$. The basic objects of this category are of the form $F_{ij} := F_{Si \otimes jS} = Ai \otimes jA$ with $i, j \in Q_0$.

The bimodule homomorphisms between $F_T \in \mathcal{P}$ and a bimodule $M \in \mathsf{Mod} A - A$ can be identified with

$$\mathrm{Hom}_{A-A}(F_T, M) \cong T^* \underset{S\text{–}S}{\otimes} M.$$

The tensor product in this formula tensors over both the left and the right S action. The identification can be expressed explicitly as

$$\theta \underset{S\text{–}S}{\otimes} m : b_1 \otimes_S t \otimes_S b_2 \mapsto \sum_{i,j \in Q_0} \theta(itj) b_1 imj b_2$$

A special role is played by $F_{S \otimes S} \cong A \otimes A$. We will denote this space by F. On this vector space we can define two commuting A-bimodule structures

$$F_{\text{Outer}} : (a_1(b_1 \otimes b_2)a_2) = a_1 b_1 \otimes b_2 a_2,$$
$$F_{\text{Inner}} : (a_1(b_1 \otimes b_2)a_2) = b_1 a_2 \otimes a_1 b_2.$$

If we use no subscript, we automatically assume the outer structure. These structures are both isomorphic as bimodules to the free bimodule of rank one and the isomorphism between them is given by the twist

$$\tau : F_{\text{Outer}} \to F_{\text{Inner}} : (b_1 \otimes b_2) \mapsto (b_2 \otimes b_1).$$

The existence of these two commuting structures implies that for any A-bimodule M the object $\mathrm{Hom}_{A-A}(M, F_{\text{Outer}})$ is again an A-bimodule using the inner structure. This bimodule will be denoted by $M^\vee$. Maps can also be dualized in the standard way to turn $-^\vee$ into a functor:

$$\forall f \in \mathrm{Hom}_{A-A}(M,N) : \forall m \in M : \forall \nu \in N^\vee :\ f^\vee(\nu)(m) := \nu(f(m)).$$

For the standard projective bimodules we have the following natural identities

- $F_T^\vee = \mathrm{Hom}_{A-A}(F_T, F) \cong (T^* \underset{S\text{–}S}{\otimes} F)_{\text{Inner}} \cong A \otimes_S T^* \otimes_S A = F_{T^*}$,
- $\mathrm{Hom}_{A-A}(F_T, M) \cong T^* \underset{S\text{–}S}{\otimes} M \cong F_{T^*} \underset{A\text{–}A}{\otimes} M \cong F_T^\vee \underset{A\text{–}A}{\otimes} M$.

We can also write out the duality for the morphisms:

$$(\theta \underset{S\text{–}S}{\otimes} a_1 \otimes_S t \otimes_S a_2)^\vee = t \underset{S\text{–}S}{\otimes} a_2 \otimes_S \theta \otimes_S a_1$$

These formulas imply that there is a natural equivalence between

$$(- \underset{A\text{–}A}{\otimes} -)^* \text{ and } (-^\vee \underset{A\text{–}A}{\otimes} -^*) : \mathcal{P} \times \mathsf{Rep}A - A \to \mathsf{Mod}\mathbb{C}$$

and between $-^{\vee\vee}|_{\mathcal{P}}$ and $\mathcal{P} \hookrightarrow \mathsf{Mod}A - A$. These functors and identities can be transferred to complexes if we assume that

$$(M^\bullet)^* = (M^*_{-i}, -(d^M_{-i+1})^*) \text{ and } (P^\bullet)^\vee = (P^\vee_{-i}, -(d^P_{-i+1})^\vee)$$

Keeping all this in mind we can propose the following definition:

Definition 4.1. A projective resolution $P^\bullet$ of left A-bimodules is selfdual with shift n if and only if there exists a commutative diagram

$$\begin{array}{ccccccccc}
P_n & \xrightarrow{d_n} & P_{n-1} & \xrightarrow{d_{n-1}} & \cdots & \longrightarrow & P_1 & \xrightarrow{d_1} & P_0 \\
\downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \alpha_{n-1}} & & & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} \\
P_0^\vee & \xrightarrow{-d_1^\vee} & P_1^\vee & \xrightarrow{-d_2^\vee} & \cdots & \longrightarrow & P_{n-1}^\vee & \xrightarrow{-d_n^\vee} & P_n^\vee
\end{array}$$

for which the α_i are isomorphisms of A-bimodules. In short hand we can write $P^\bullet \cong (P^\bullet)^\vee[n]$.

Theorem 4.2. *If an algebra A has a selfdual resolution of length n with entries in $\mathcal{P}$ then A is Calabi Yau of dimension n.*

Proof. Let $M^\bullet$ and $N^\bullet$ be two complexes in $\mathsf{Rep}A$. Standard homological algebra allows us to identify naturally

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}^b\mathsf{Rep}A}(M^\bullet, N^\bullet) &\cong \mathrm{Hom}_{\mathcal{D}^b\mathsf{Mod}A-A}(A, N^\bullet \otimes (M^\bullet)^*) \\
&\cong \mathrm{Hom}_{\mathcal{D}^b\mathsf{Mod}A-A}(P^\bullet, N^\bullet \otimes (M^\bullet)^*) \\
&\cong H^0\mathrm{RHom}_{\mathcal{D}^b\mathsf{Mod}A-A}(P^\bullet, N^\bullet \otimes (M^\bullet)^*) \\
&\cong H^0\mathrm{Hom}^\bullet_{\mathcal{K}\mathsf{Mod}A-A}(P^\bullet, N^\bullet \otimes (M^\bullet)^*)
\end{aligned}$$

So if we can prove that there is a natural equivalence between

$$\mathrm{Hom}^\bullet_{\mathcal{K}\mathsf{Mod}A-A}(P^\bullet, N^\bullet \otimes (M^\bullet)^*) \text{ and } \mathrm{Hom}^\bullet_{\mathcal{K}\mathsf{Mod}A-A}(P^\bullet, M^\bullet \otimes (N^\bullet)^*[n])$$

we are done.

Now using the fact that the resolution is composed of projectives in $\mathcal{P}$ we can make the following identifications

$$\begin{aligned}
&(\mathrm{Hom}^\bullet_{\mathcal{K}\mathsf{Mod}A-A}(P^\bullet, N^\bullet \otimes (M^\bullet)^*))^* \\
&\cong ((P^\bullet)^\vee \underset{A\text{–}A}{\otimes} N^\bullet \otimes (M^\bullet)^*)^* \\
&\cong (P^\bullet)^{\vee\vee} \underset{A\text{–}A}{\otimes} (N^\bullet \otimes (M^\bullet)^*)^* \\
&\cong (P^\bullet) \underset{A\text{–}A}{\otimes} M^\bullet \otimes (N^\bullet)^* \\
&\overset{\alpha}{\cong} (P^\bullet)^\vee[n] \underset{A\text{–}A}{\otimes} M^\bullet \otimes (N^\bullet)^* \\
&\cong (P^\bullet)^\vee \underset{A\text{–}A}{\otimes} M^\bullet \otimes (N^\bullet)^*[n] \\
&\cong \mathrm{Hom}^\bullet_{\mathcal{D}^b\mathsf{Mod}A-A}(P^\bullet, M^\bullet \otimes (N^\bullet)^*[n])
\end{aligned}$$

which are natural in the $M^\bullet$ an $N^\bullet$. $\square$

For an explicit write-out of the corresponding pairing between $\mathrm{Hom}_{\mathcal{K}\mathsf{Mod}A-A}(P^\bullet, N^\bullet \otimes (M^\bullet)^*)$ and $\mathrm{Hom}_{\mathcal{K}\mathsf{Mod}A-A}(P^\bullet, M^\bullet \otimes (N^\bullet)^*[n])$ we first need some notation: for simplicity we will work with elements that are pure tensors:

$$f \in \mathrm{Hom}_{\mathcal{K}\mathsf{Mod}A-A}(P^\bullet, N^\bullet \otimes (M^\bullet)^*) : f^{ij} = \phi^{ij} \underset{A\text{–}A}{\otimes} \mu^{ij} \in P^{i\vee} \underset{A\text{–}A}{\otimes} N^{i+j} \otimes M^{j*}$$

$$g \in \mathrm{Hom}_{\mathcal{K}\mathsf{Mod}A-A}(P^\bullet, M^\bullet \otimes (N^\bullet)^*[n]) : g^{ij} = \gamma^{ij} \underset{A\text{–}A}{\otimes} m^{ij} \in P^{i\vee} \underset{A\text{–}A}{\otimes} M^j \otimes (N^{n-i+j})^*.$$

With these expressions for f and g we can track back the pairing in the previous identifications:

$$\langle f, g\rangle_{M^\bullet N^\bullet} = \sum_{ij} \mathrm{Tr}\mu^{ij} \circ \phi^{ij} \alpha^{-1}_{n-i}(\gamma^{n-i,j}) m^{n-i,j}.$$

4.2. **Superpotentials and Selfduality.** In the case of a graded algebra $A := \mathbb{C}Q/\mathcal{I}$, $\mathcal{I} \subset \mathcal{J}^2$ one can construct its minimal resolution using standard presentations of $\mathcal{I}^n/\mathcal{I}^{n+1}$. These objects, introduced in [6], consist of quintuples $(U, V, r, l, \Delta)_n$ where

(1) $U, V \subset \mathcal{I}^n$ are S-bimodule complements such that
$$\mathcal{I}^n = U \oplus \mathcal{J}\mathcal{I}^n + \mathcal{I}^n\mathcal{J} \text{ and } \mathcal{J}\mathcal{I}^n \cap \mathcal{I}^n\mathcal{J} = V \oplus \mathcal{J}\mathcal{I}^n\mathcal{J},$$

(2) $r, l : \mathcal{I}^n \to A \otimes_S U \otimes_S A$ are a $\mathbb{C}Q - S$ and a $S - \mathbb{C}Q$-bimodule section of the $\mathbb{C}Q$-bimodule morphism
$$e : A \otimes_S U \otimes_S A \to \mathcal{I}^n : 1 \otimes_S u \otimes_S 1 \mapsto u.$$
and use these to define a map
$$d : A \otimes_S V \otimes_S A \to A \otimes_S U \otimes_S A : 1 \otimes_S v \otimes_S 1 \mapsto l(v) - r(v)$$

(3) $\Delta : \mathcal{I}^n \to A \otimes_S V \otimes_S A$ is a $\mathbb{C}Q$-bimodule derivation (i.e. a S-bimodule morphism satisfying $\Delta(azb) = \Delta(az)b + a\Delta(zb) - a\Delta(z)b$) such that $d\Delta = l - r$ and $\forall v \in V : \Delta(v) = 1 \otimes_S v \otimes_S 1$.

Although the map d is a morphism as $\mathbb{C}Q$-bimodules it can also be considered as a morphism of A-modules $d_A : F_V \to F_U$ because the $\mathbb{C}Q$-action factors over A. The same can be done with e provided we factor out $\mathcal{I}^{n+1}$ in the target: $e_A : F_U \to \mathcal{I}^n/\mathcal{I}^{n+1}$. To turn Δ into a A-bimodule morphism we have to do two things: look at the subspace $\mathcal{I}^{n+1} \subset \mathcal{I}^n$ (this turns the derivation law into a morphism law) and mod out $\mathcal{I}^{n+2}$ (this turns the domain into a A-bimodule):

$$c_A : \frac{\mathcal{I}^{n+1}}{\mathcal{I}^{n+2}} \to F_V : x + \mathcal{I}^{n+2} \mapsto \Delta(x).$$

These maps can be packed together in sequences of $A - A$ bimodules

$$0 \longrightarrow \frac{\mathcal{I}^{n+1}}{\mathcal{I}^{n+2}} \xrightarrow{c_A} F_{\frac{\mathcal{J}\mathcal{I}^n \cap \mathcal{I}^n\mathcal{J}}{\mathcal{J}\mathcal{I}^n\mathcal{J}}} \xrightarrow{d_A} F_{\frac{\mathcal{I}^n}{\mathcal{J}\mathcal{I}^n + \mathcal{I}^n\mathcal{J}}} \xrightarrow{e_A} \frac{\mathcal{I}^n}{\mathcal{I}^{n+1}} \longrightarrow 0.$$

In [6] it is proved that these sequences are exact and they can be spliced together to get a projective bimodule resolution of $\mathcal{I}^0/\mathcal{I}^1 = A$. This resolution is not minimal but it can be made minimal if one cuts out the excess summands that occur at the splicing boundaries. These terms are of the form

$$A \otimes_S \frac{\mathcal{I}^{n+1}}{\mathcal{I}^{n+1} \cap \mathcal{J}\mathcal{I}^n\mathcal{J}} \otimes_S A \cong A \otimes_S \frac{\mathcal{I}^{n+1} + \mathcal{J}\mathcal{I}^n\mathcal{J}}{\mathcal{J}\mathcal{I}^n\mathcal{J}} \otimes_S A \subset F_{\frac{\mathcal{J}\mathcal{I}^n \cap \mathcal{I}^n\mathcal{J}}{\mathcal{J}\mathcal{I}^n\mathcal{J}}}$$

We will now apply this to the case of Calabi Yau algebras of dimension 3. As we already know from section 3.1 the ideal is generated by

$$\partial_a W, a \in Q_1$$

where $W \in \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$ is a superpotential and as the global dimension is 3 we only need to look at the standard presentations for $n = 0, 1$.

The case $n = 0$ has the same form for every algebra

- $U_0 = S, V_0 = \mathbb{C}Q_1$
- $l_0 : a \mapsto a \otimes_S t(a) \otimes_S 1$
  $r_0 : a \mapsto 1 \otimes_S h(a) \otimes_S a$,
- $\Delta : a_1 \cdots a_k \mapsto \sum_{1 \le j \le k} a_1 \cdots a_{j-1} \otimes_S a_j \otimes_S a_{j+1} \cdots a_k)$.

For $n = 1$ we do not need to bother about the Δ_1 because it does not affect the minimal resolution:

- $\mathcal{I} = \mathbb{C}\{\partial_a W, a \in Q_1\} \oplus \mathcal{J}\mathcal{I} + \mathcal{I}\mathcal{J},\ U_1 = \mathbb{C}\{\partial_a W, a \in Q_1\} \cong \mathbb{C}Q_1^{op}$.
  For V_1 we chose a complement that contains the subspace $\mathbb{C}\{i \circlearrowright W, i \in Q_0\}$.
- $l_1 : x\partial_a W y \mapsto x \otimes_S \partial_a W \otimes_S y$ if $y \notin \mathcal{I}$,
  $r_1 : x\partial_a W y \mapsto x \otimes_S \partial_a W \otimes_S y$ if $x \notin \mathcal{I}$.

Because the $i \circlearrowright W$ are not contained in $\mathcal{I}^2$ they are not cut out by restricting to the minimal resolution. Moreover, because

$$\begin{aligned}
\mathsf{Ext}^3_A(S, S) &= \mathrm{Hom}_S(\frac{V_1}{\mathcal{I}^2 \cap V_1}, S) \\
&\overset{CY}{\cong} \mathrm{Hom}_A(S, S)^* \\
&\cong \mathrm{Hom}_S(\mathbb{C}\{i, i \in Q_0\}, S)^* \\
&\cong \mathrm{Hom}_S(\mathbb{C}\{i \circlearrowright W, i \in Q_0\}, S)^*
\end{aligned}$$

We have that the third term in the minimal resolution must be $F_{\frac{V_1}{\mathcal{I}^2 \cap V_1}} = F_{\mathbb{C}\{i \circlearrowright W, i \in Q_0\}} \cong F_S$. Putting everything together we get

$$F_{\mathbb{C}\{i \circlearrowright W, i \in Q_0\}} \xrightarrow{\delta_3} F_{\mathbb{C}\{\partial_a W, a \in Q_1\}} \xrightarrow{\delta_2} F_{\mathbb{C}Q_1} \xrightarrow{\delta_1} F_S \longrightarrow 0$$

with maps

$$\begin{aligned}
\delta_1(1 \otimes_S a \otimes_S 1) &= a \otimes_S t(a) \otimes_S 1 - 1 \otimes_S h(a) \otimes_S a \\
\delta_2(1 \otimes_S \partial_a W \otimes_S 1) &= \Delta(\partial_a W) \\
\delta_3(1 \otimes_S W \otimes_S 1) &= \sum_{a \in Q_1} a \otimes_S \partial_a W \otimes_S 1 - 1 \otimes_S \partial_a W \otimes_S a
\end{aligned}$$

A more explicit write-out of the complex C_W, whose 0^{th} homology is equal to A, in terms of the basic projective F_{ij} looks like

$$C_W : \quad \bigoplus_{i \in Q_0} F_{ii} \xrightarrow{(\cdot \tau da \cdot)} \bigoplus_{a \in Q_1} F_{t(a)h(a)} \xrightarrow{(\cdot \partial^2_{ba} W \cdot)} \bigoplus_{b \in Q_1} F_{h(b)t(b)} \xrightarrow{(\cdot db \cdot)} \bigoplus_{i \in Q_0} F_{ii} \xrightarrow{m} A$$

where the differential is $da := a \otimes t(a) - h(a) \otimes a$ and the second derivatives are $\partial^2_{ba} W = \pi_{F_{t(b)h(b)}} \Delta \partial_a W$. More explicitly, if c is a cycle then

$$\partial^2_{ba} c = \sum_{p_1, p_2 : \circlearrowright a p_1 b p_2 = \circlearrowright c} p_1 \otimes p_2.$$

Note that because $\circlearrowright W$ is invariant under cyclic permutation, $\partial^2_{ab}W = \tau\partial^2_{ba}W$

$$\left(\bigoplus_{i\in Q_0} F_{ii} \xrightarrow{(\cdot\tau da\cdot)} \bigoplus_{a\in Q_1} F_{t(a)h(a)} \xrightarrow{(\cdot\partial^2_{ba}W\cdot)} \bigoplus_{b\in Q_1} F_{h(b)t(b)} \xrightarrow{(\cdot db\cdot)} \bigoplus_{j\in Q_0} F_{jj}\right)^{\vee}$$

$$\begin{array}{ccccccccc} = \bigoplus_{j\in Q_0} F^{\vee}_{jj} & \xrightarrow{(\cdot db\cdot)^{\vee}} & \bigoplus_{b\in Q_1} F^{\vee}_{h(b)t(b)} & \xrightarrow{(\cdot\partial^2_{ba}W\cdot)^{\vee}} & \bigoplus_{a\in Q_1} F^{\vee}_{t(a)h(a)} & \xrightarrow{(\cdot\tau da\cdot)^{\vee}} & \bigoplus_{i\in Q_0} F^{\vee}_{ii} \\ \downarrow\tau & & \downarrow\tau & & \downarrow\tau & & \downarrow\tau \\ = \bigoplus_{j\in Q_0} F_{jj} & \xrightarrow{(\cdot\tau db\cdot)} & \bigoplus_{b\in Q_1} F_{t(b)h(b)} & \xrightarrow{(\cdot\partial^2_{ab}W\cdot)} & \bigoplus_{a\in Q_1} F_{h(a)t(a)} & \xrightarrow{(\cdot da\cdot)} & \bigoplus_{i\in Q_0} F_{ii} \end{array}$$

This complex is selfdual and the isomorphism connecting the complex with its dual is composed of the standard identifications we used in the previous paragraph.

So the sufficient condition of selfduality is also necessary for Calabi Yau algebras of dimension 2.

Theorem 4.3. *A vacualgebra A_W is Calabi Yau of dimension 3 if and only if the complex C_W is a projective resolution of A_W as an A_W-bimodule.*

This fact has a nice interpretation for the classification of *good superpotentials* i.e. superpotentials with a vacualgebra that is indeed Calabi Yau.

Corollary 4.4. *For a given quiver Q and a given dimension d, the subset of $\mathsf{Sup}_d Q$ of good superpotentials of degree d is either the empty set or almost everything (in the measure theoretic sense).*

Proof. The condition we must check is that the standard complex is indeed a resolution. Because the resolution is graded we can check this separately for every degree so the subspace of good superpotentials is an intersection of a countable number of Zariski open sets. If one of these sets is empty we're in the first case and otherwise the complement of this set is a countable union of hypersurfaces, which has measure zero for the standard measure on $\mathbb{C}^n$. □

Remark 4.5. For global dimension two we can do a similar thing. Recall that if A is Calabi Yau of dimension two, then the set of arrows partitions in pairs (a, a^*) with opposite head and tail.

The selfdual resolution now looks like

$$\bigoplus_{i\in Q_0} F_{ii} \xrightarrow{\begin{pmatrix}\cdot\tau da^*\cdot\\ \cdot-\tau da\cdot\end{pmatrix}} \bigoplus_{(a,a^*)} F_{t(a)h(a)} \oplus F_{t(a^*)h(a^*)} \xrightarrow{\begin{pmatrix}\cdot da\cdot & \cdot da^*\cdot\end{pmatrix}} \bigoplus_{i\in Q_0} F_{ii} \xrightarrow{m} A$$

This is indeed the standard resolution for preprojective algebras of non-Dynkin quivers (see [7]).

4.3. **The matrix valued Hilbert polynomial.** For a graded algebra $A = \mathbb{C}Q/(\mathcal{R})$ one can define the *matrix valued Hilbert series*

$$H_A(t) := h_0 + h_1 t + h_2 t^2 + \cdots$$

where the h_k are matrices in $\mathtt{Mat}_{\#Q_0\times\#Q_0}(\mathbb{C})$ and

$$(h_k)_{ij} = \dim iA_kj \text{ (A_k is the degree k part of A)}$$

The matrix valued Hilbert series of a Calabi Yau algebra can be computed from its bimodule resolution:

Theorem 4.6. *If a vacualgebra A_W with $\deg W = d \geq 3$ is Calabi Yau then*

$$H_{A_W}(t) = \frac{1}{1 - M_Q t + M_Q^T t^{d-1} - t^d}$$

where M_Q is the incidence matrix of Q. This equality must be evaluated in the ring of formal power series $\mathtt{Mat}_{\#Q_0\times\#Q_0}(\mathbb{C}[\![t]\!])$.

Proof. The Hilbert polynomial of F_{kl} is equal to

$$H_{F_{kl}}(t) = H_A(t)e_{kl}H_A(t)$$

where e_{kl} is the matrix with 1 on the entry k, l and zero elsewhere. So from the exactness of the resolution C_W and the fact that $H_0(P^\bullet) = A$ we get

$$\begin{aligned} H_A(t) &= H_{F_S} - t(H_{F_{\mathbb{C}Q_1}} - t^{d-2}(H_{F_{\mathbb{C}Q_1^{op}}} - tH_{F_S})) \\ &= H_A(t)1H_A(t) - tH_A(t)M_QH_A(t) + t^{d-1}H_A(t)M_Q^TH_A(t) - t^dH_A(t)1H_A(t) \end{aligned}$$

Note that $H_A(t)$ is invertible because $H_A(0) = \mathbf{1}$. Multiplying to the left and the right by $H_A(t)^{-1}$ and taking the inverse we obtain the equality. □

The bimodule resolution gives us also resolutions of the left modules S. Writing out the dimensions of these resolutions gives the equation

$$1 = H_A(t) - tM_QH_A(t) + t^{d-1}M_Q^TH_A(t) - t^dH_A(t).$$

This is nothing new, but as this equation corresponds to a real resolution we can derive certain inequalities that must be met:

- I1 $H_A(t) \geq 0$
- I2 $(M_Q^T - t)H_A(t) \geq 0$
- I3 $(M_Q - M_Q^Tt^{d-2} - t^{d-1})H_A(t) \geq 0$

Note that a matrix valued series $f(t)$ is positive if all its entries $(f_k)_{ij}$ are positive. These inequalities can be useful to check whether quivers have good superpotentials of a given degree.

5. APPLICATIONS

5.1. **Groebner Bases and Superpotentials.** To show that for a given quiver and a given degree there exist good superpotentials one has to check whether one can find a superpotential W such that C_W is exact. To do this we will use the technique of Groebner bases as outlined in [12], adapted to path algebras. Suppose Q is a quiver with n arrows and put an order on the arrows: $a_1 > \cdots > a_n$. One can extend this order to the set of paths with nonzero length using the *deglex ordering* method:

$$a_{i_1} \cdots a_{i_p} < a_{j_1} \cdots a_{j_q}$$

if and only if $p < q$ or $p = q$ and $\exists \nu \leq p : a_{i_\nu} < a_{j_\nu} \wedge \forall \mu < \nu : i_\mu = j_\mu$. We denote the leading monomial term (according to the deglex ordering) of $f \in \mathbb{C}Q$ by $\mathsf{lt}(f)$. Recall that a (not necessarily finite) set of elements $G \subset \mathcal{I} \lhd \mathbb{C}Q$ is a *Groebner basis* if all $\mathsf{lt}(g), g \in G$ are different and

$$\mathsf{lt}(\mathcal{I}) := (\mathsf{lt}(f) : f \in \mathcal{I}) = (\mathsf{lt}(g) : g \in G)$$

where the equality is taken as ideals in $\mathbb{C}Q$. Groebner bases are very useful in determining the structure of an algebra. They can be used to determine the Hilbert polynomial because

$$H_{\mathbb{C}Q/\mathcal{I}} = H_{\mathbb{C}Q/\mathsf{lt}\mathcal{I}}$$

and they can be used to check whether certain expressions in $\mathbb{C}Q$ are zero in $\mathbb{C}Q/\mathcal{I}$:

$$f \in \mathcal{I} \implies \mathsf{lt}(f) \in \mathsf{lt}(\mathcal{I}) = (\mathsf{lt}(g) : g \in G)$$

To check whether a given set of relations is indeed a Groebner basis one can use the method of Bergman diamonds [3]. For any f in $\mathbb{C}Q$, an elementary reduction of f by $g \in G$ is the new expression

$$\rho_g(f) := f - \zeta agb \text{ if } a, b \text{ are paths and } \zeta \in \mathbb{C} \text{ s.t.} \mathsf{lt}(f) = \zeta a \mathsf{lt}(g) b \text{ or } f \text{ otherwise.}$$

If G is a set of relations then a triple of monomial terms (a, b, c) is called an *ambiguity* if $ab = \mathsf{lt}(g_1), bc = \mathsf{lt}(g_2)$ with $g_1, g_2 \in G$. An ambiguity is called resolvable if there is a sequence of elementary reductions such that

$$\rho_1 \cdots \rho_m(g_1c - ag_2) = 0.$$

Now Bergman's diamond lemma states that if all leading terms are different and all ambiguities are resolvable then G is a Groebner basis.

We will now give a useful criterion to find good superpotentials.

Lemma 5.1. *Suppose every vertex in Q is the source and the target of at least two arrows and W is a superpotential such that*

- *The leading terms of the relations $\partial_a W$ are all different and the ambiguities are in 1 to 1 correspondence to the vertices and are of the form*

$$\mathrm{amb_v} = (a, \mathsf{lt}(\partial_a W)b^{-1}, b) = (a, a^{-1}\mathsf{lt}(\partial_b W), b) \text{ with } a\mathsf{lt}(\partial_a W) = \mathsf{lt}(\partial_b W)b = \mathsf{lt}(vWv),$$

- *for every vertex v there is at least one arrow $a, t(a) = v$ such that $\forall b \in Q_0$: $\mathsf{lt}(\partial_b W)a^{-1} = 0$.*

then A_W is Calabi Yau.

Proof. First note that the condition implies that $\{\partial_a W : a \in Q_1\}$ is a Groebner basis: an ambiguity of the form $\mathrm{amb}_v(a, \mathsf{lt}(\partial_a W)b^{-1}, b)$ is resolvable because

$$\sum_{h(c)=v} c\partial_c W = \sum_{t(c)=v} \partial_c W c$$

and hence

$$a\partial_a W - \partial_b W b = \sum_{t(c)=v, c\neq b} \partial_c W c - \sum_{h(c)=v, c\neq a} c\partial_c W.$$

Note that the leading terms of the summands in the right hand side are all different because the $\mathsf{lt}(\partial_c W)$ are and there is only one ambiguity corresponding to v. We can remove each term using an elementary reduction, starting with the one with the highest leading term. Therefore the ambiguity is resolvable.

To calculate the Hilbert series one must calculate $(h_k)_{vw}$, which is equal to the number of words between v and w of a given length k not containing $\mathsf{lt}(\partial_a W)$'s. This can be done using recursion:

$$(h_k)_{vw} = \underbrace{\sum_u h_{vu}^{k-1} \#\{u \leftarrow w\}}_{\text{add an arrow}} - \underbrace{\sum_u h_{vu}^{k-d+1} \#\{w \leftarrow u\}}_{\text{remove those ending in } \mathsf{lt}(\partial_b W)} + \underbrace{h_{vw}^{k-d}}_{\text{remove double counting}}.$$

There are no further terms needed: a word ending w can only be double counted once because of the form and number of the ambiguities. The Hilbert series of A_W is thus

$$H_{A_W}(t) = \frac{1}{1 - M_Q t + M_Q^T t^{d-1} - t^d}.$$

Using the exactness of the first 2 terms of C_W we can calculate the Hilbert series of the kernel of the third map

$$H_{A_W}(t) - ktH_{A_W}(t)^2 + kt^{d-1}H_{A_W}(t)^2 = t^d H_{A_W}(t)^2.$$

This is the same as the Hilbert series of the last term so if we can prove that the last map is an injection we are done. There is indeed no element $\sum_j f_j \otimes g_j \in A_W \otimes_S A_W$ such that

$$\forall i \geq k : \sum_j f_j b \otimes g_j - f_j \otimes b g_j = 0.$$

The deglex ordering on $\mathbb{C}Q$ can be transferred to and ordering on the monomials of $\mathbb{C}Q \otimes_S \mathbb{C}Q$:

$$v_1 \otimes v_2 > w_1 \otimes w_2 \iff v_1 > w_1 \text{ or } v_1 = w_1 \text{ and } v_2 > w_2.$$

This ordering is compatible with the multiplicative structure on $\mathbb{C}Q \otimes_S \mathbb{C}Q$. Let $f_1 \otimes g_1$ be the highest order term, then the highest order term of $\sum_j f_j b \otimes g_j - f_j \otimes b g_j$ is $f_1 b \otimes g_1$. Therefore $f_1 \notin (\mathsf{lt}(\partial_a W) : a \in Q_1)$ but $f_1 b \in (\mathsf{lt}(\partial_a W) : a \in Q_0)$ for every $b \in Q_0$. This would imply that for every b with $h(b) = t(f_1)$ there is a $c \in Q_1$ such that $f_1 b$ ends in $\mathsf{lt}(\partial_c W)$, contradicting the second condition on W. □

The conditions imposed on the superpotential are very strict and there are far more good superpotential that do not meet these conditions. In general the ideal generated by a good superpotential will not have a finite Groebner basis. However for many quivers and degrees we will be able to find superpotentials that satisfy the demands of the lemma.

5.2. **The one vertex situation.** First note that if Q has only one vertex and one loop, then none of the vacualgebras can be Calabi Yau of dimension 3 because these algebras are finite dimensional and hence $H_A(t)$ cannot be the inverse of the polynomial $1 - t + t^{d-1} - t^d$.

So, in this section, let Q be a quiver with one vertex and $k \geq 2$ loops and let $\mathsf{Sup}_d \subset \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$ be the subspace of all superpotentials of degree d with $d \geq 3$. We will show that the space of good superpotentials is non-empty if and only if $(k, d) \neq (2, 3)$.

If $(k, d) = (2, 3)$ then there are no good superpotentials because the inequality $(I2)$ does not hold:

$$(2-t)\frac{1}{1-2t+2t^2-t^d} = 2 + 3t + 2t^2 - t^3 + 2t^4 + \cdots \not\geq 0.$$

For every other couple (k, d) we can find at least one good superpotential.

Lemma 5.2. *Take $\mathbb{C}Q \cong \mathbb{C}\langle X_1, \ldots, X_n\rangle$ and $X_1 > X_2 > \cdots > X_n$, then the following superpotentials are good:*

(1) $W = X_1X_2X_3 + X_1X_3X_2 + \sum_{j>3} X_1X_j^2 + [\mathbb{C}Q, \mathbb{C}Q]$,
(2) $W = \sum_{k \geq l > 1} X_1^{d-2}X_lX_k + [\mathbb{C}Q, \mathbb{C}Q]$.

Proof. We calculate the leading terms of the relations

(1) $\mathsf{lt}(\partial_{X_1}W) = X_2X_3,\ \mathsf{lt}(\partial_{X_2}W) = X_1X_3,\ \mathsf{lt}(\partial_{X_3}W)X_1X_2,\ \mathsf{lt}(\partial_{X_k}W)X_1X_k, \ldots$
(2) $\mathsf{lt}(\partial_{X_1}W) = X_1^{d-3}X_2^2,\ \mathsf{lt}(\partial_{X_2}W) = X_1^{d-1}X_2,\ \ldots,\ \mathsf{lt}(\partial_{X_k}W) = X_1^{d-1}X_k.$

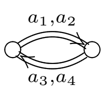
The only ambiguity we can construct is

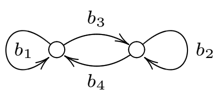
(1) (X_1, X_2, X_3) between $\partial_{X_1}W$ and $\partial_{X_2}W$,
(2) $(X_1, X_1^{d-3}X_2, X_2)$ between $\partial_{X_1}W$ and $\partial_{X_2}W$.

We also see that none of the leading terms ends in X_1. □

Remark 5.3. In the cases where (k, d) equals $(2, 4)$ or $(3, 3)$ one can obtain a complete classification of the good superpotentials because then we are in the case of Artin-Shelter regular algebras [2].

5.3. **Special Quivers.** The simplest quivers with more than one vertex that can have good potentials are

$Q_1 :=$ 

$Q_2 :=$ 

Theorem 5.4.

- $\mathbb{C}Q_1/[\mathbb{C}Q_1, \mathbb{C}Q_1]_d$ *contains good superpotentials if and only if* $d \geq 4$ *and* d *is even.*
- $\mathbb{C}Q_1/[\mathbb{C}Q_2, \mathbb{C}Q_2]_d$ *contains good superpotentials if and only if* $d \geq 4$.

Proof. For both quivers d must be bigger than or equal to 4 because otherwise the inequalities $I1 - I3$ are not satisfied. For Q_1, d must be even because every cycle has even length.

Assume the orders $a_1 > a_2 > a_3 > a_4$, $b_1 > b_2 > b_3 > b_4$ and define the following superpotentials

$$Q_1 : a_1a_3(a_2a_4)^{\frac{d}{2}-1} + a_3a_1(a_4a_2)^{\frac{d}{2}-1} + [\mathbb{C}Q, \mathbb{C}Q]$$
$$Q_2 : b_1^{d-2}b_3b_4 + b_2^{d-2}b_4b_3 + [\mathbb{C}Q, \mathbb{C}Q]$$

The leading terms of the relations are

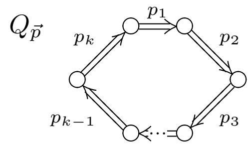
$$Q_1 : a_3(a_2a_4)^{\frac{d}{2}-1},\ a_3a_1a_4(a_2a_4)^{\frac{d}{2}-2},\ a_1(a_4a_2)^{\frac{d}{2}-1},\ a_1a_3a_2(a_4a_2)^{\frac{d}{2}-2},$$
$$Q_2 : b_1^{d-3}b_3b_4,\ b_2^{d-3}b_4b_3,\ b_2^{d-2}b_4,\ b_1^{d-2}b_3$$

For each of the quivers there are two ambiguities (one for each vertex)

$$Q_1 : (a_1, a_3a_2(a_4a_2)^{\frac{d}{2}-2}, a_4),\ (a_3, a_1a_4(a_2a_4)^{\frac{d}{2}-2}, a_2)$$
$$Q_2 : (b_1, b_1^{d-3}b_3, b_4),\ (b_2, b_2^{d-3}b_4, b_3)$$

Finally none of the relations end in a_1, a_3 and b_1, b_2. □

The method described above can be extended to lots of other quivers and degrees, especially quivers of the form



The number of arrows between consecutive vertices can differ (but is ≥ 2).

Theorem 5.5. *Let Q be a quiver of the form above with $k \geq 2$ vertices and let $p_i \geq 2$ be the number of arrows between the i^{th} and the $i+1^{th}$ vertex. If $d = \ell k$ with $\ell \geq 2$ then Q has good superpotentials of dimension d.*

Proof. For every vertex $v \in Q_0$, we will denote the consecutive vertex by $v+1$, so $\forall a \in Q_1 : h(a) = t(a)+1$. Fix an order on the arrows of Q and let a_i, b_i the highest and second highest arrow arriving in the vertex i.

Define the superpotential

$$W := \sum_{i \in Q_0} a_i a_{i-1} b_{i-2} \cdots b_{i-\ell k+1}$$
$$+ \sum_{c \neq a_i, b_i} c a_{h(c)-1} b_{h(c)-2} \cdots b_{h(c)-k+1} (c b_{h(c)-1} b_{h(c)-2} \cdots b_{h(c)-k+1})^{\ell-1} + [\mathbb{C}Q, \mathbb{C}Q]$$

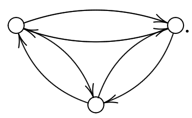
The leading terms of the relations now look like

$$\mathsf{lt}(\partial_{a_i} W) = a_{i-1} b_{i-2} \cdots b_{i-\ell k+1}$$
$$\mathsf{lt}(\partial_{b_i} W) = a_{i-1} a_{i-2} b_{i-3} \cdots b_{i-\ell k+1}$$
$$\mathsf{lt}(\partial_c W) = a_{h(c)-1} b_{h(c)-2} \cdots b_{h(c)-k+1} (c b_{h(c)-1} b_{h(c)-2} \cdots b_{h(c)-k+1})^{\ell-1}$$

It is easy to check that all ambiguities are of the form $(a_i, a_{i-1}b_{i-2}\cdots b_{i-\ell k+2}, b_{i-\ell k+1})$ and none of the relations ends in some a_i. □

Remark 5.6. If $\ell = 1$ the situation is more complicated because the solutions of the inequalities I1-I3 are not easy to determine. It is not the case that if they are satisfied for $Q_{\vec{p}}$ that they are also satisfied for a quiver $Q_{\vec{p}'}$ with $(p'_1, \cdots, p'_k) \geq (p_1, \cdots, p_k)$. F.i. a quiver with arrows $\vec{p} = (2,2,2,2)$ has good superpotentials but one with $\vec{p} = (6,2,2,2)$ has not.

The method of finding these very special superpotentials does not always work. As a counterexample consider the quiver



One can check that there are no superpotentials of dimension 4 satisfying the conditions from 5.1 although Groebner basis computations in GAP [9](up to a certain degree because the full Groebner basis could be infinite) seem to indicate that a generic superpotential is indeed good.

The general picture that arises from computations is that as soon as conditions I1-I3 are met by the Hilbert series then there do exist good superpotentials, but we have no proof for this statement.

6. AKNOWLEDGEMENTS

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APPENDIX A. THE SIGNS OF SERRE DUALITY

BY MICHEL VAN DEN BERGH

A.1. **Introduction.** In this self-contained appendix we determine the exact signs which occur in Serre duality (see for example Proposition A.5.2 for the Calabi-Yau case). Although the answer is the obvious, the verification turned out to be slightly more tricky than foreseen.

We thank Bernhard Keller for pointing out Example A.3.2 (see [11] and [14] for further information) and suggesting that, likewise, the correct signs in Serre duality should be determined by the requirement that the Serre functor be exact.

A.2. **Graded categories.**

Definition A.2.1. A graded (pre-additive) category is a pair $(\mathcal{C}, S)$ where $\mathcal{C}$ is a pre-additive category and S is an automorphism of $\mathcal{C}$.

Remark A.2.2. It is customary to only require S to be an autoequivalence. The stronger condition that S is an automorphism is usually satisfied in practice and up to an appropriate notion of equivalence we may always reduce to this case.

In a graded category $(\mathcal{C}, S)$ we may define the *graded Hom-sets* between objects by

$$\mathrm{Hom}^i_{\mathcal{C}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, S^i B)$$

and

$$\mathrm{Hom}^{\mathrm{gr}}_{\mathcal{C}}(A, B) = \bigoplus_i \mathrm{Hom}^i_{\mathcal{C}}(A, B)$$

There is an obvious graded composition

$$- * - : \mathrm{Hom}^j_{\mathcal{C}}(B, C) \times \mathrm{Hom}^i_{\mathcal{C}}(A, B) \to \mathrm{Hom}^{i+j}_{\mathcal{C}}(A, C) : (g, f) \mapsto: S^i(g) f$$

We denote by $\mathcal{C}^{\mathrm{gr}}$ the category $\mathcal{C}$ equipped with graded Hom-sets.

A *graded functor* between graded categories $(\mathcal{C}, S)$, $(\mathcal{D}, T)$ is an additive functor $U : \mathcal{C} \to \mathcal{D}$ together with a natural isomorphism $\eta^U : U \circ S \to T \circ U$. By a slight abuse of notation we will write the composition

$$U \circ S^i \to T \circ U \circ S^{i-1} \to \cdots \to T^i \circ U$$

as $(\eta^U)^i$.

Associated to (U, η^U) there is a functor $U^{\mathrm{gr}} : \mathcal{C}^{\mathrm{gr}} \to \mathcal{D}^{\mathrm{gr}}$ given by

$$U^{\mathrm{gr}}(f_i) = (\eta^U)^i_B \circ U(f_i) \tag{1}$$

for $f_i \in \mathrm{Hom}^i_{\mathcal{C}}(A, B)$. It is clear that the formation of $(-)^{\mathrm{gr}}$ is compatible with compositions.

A.3. **Triangulated categories.** We will assume that triangulated categories have a strictly invertible shift functor. Up to equivalence we may always reduce to this case.

Definition A.3.1. An exact functor $U : \mathcal{S} \to \mathcal{T}$ between triangulated categories is a graded functor $(U, \eta^U) : (\mathcal{S}, [1]) \to (\mathcal{T}, [1])$ such that for any distinguished triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

the following triangle

$$UA \xrightarrow{Uf} UB \xrightarrow{Ug} UC \xrightarrow{\eta^U_A \circ Uh} (UA)[1]$$

is distinguished.

Example A.3.2. Let $s : \mathcal{A} \to \mathcal{A}$ be the functor which coincides with $[1]$ on objects and maps but for which $\eta^s_A : s(A[1]) \to (sA)[1]$ is given by $-\mathrm{id}_{A[2]}$. Then (s, η^s) is an exact endofunctor on $\mathcal{A}$. Note in contrast that $[1]$ itself, while being a graded endofunctor, is *not* exact.

A.4. **Serre functors.** Let k be a field and assume that $\mathcal{C}$ is a Hom-finite k-linear category.

Definition A.4.1. $\mathcal{C}$ satisfies *Serre duality* if there is an auto-equivalence $F : \mathcal{C} \to \mathcal{C}$ together with isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{C}}(B,FA)^* \tag{2}$$

natural in A, B. Such an F is called a *Serre functor* for $\mathcal{C}$.

Putting $B = A$ in (2) yields a canonical element

$$\mathrm{Tr}_A : \mathrm{Hom}_{\mathcal{C}}(A,FA) \to k$$

corresponding to the identity in $\mathrm{Hom}_{\mathcal{C}}(A,A)$. It is easy to see that $\mathrm{Tr}_A(- \circ -)$ defines a non-degenerate pairing

$$\mathrm{Hom}_{\mathcal{C}}(B,FA) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \to k$$

and that the map (2) is given by $f \mapsto: \mathrm{Tr}_A(- \circ f)$. In addition we have the following fundamental identity [13]

$$\mathrm{Tr}_A(g \circ f) = \mathrm{Tr}_B(Ff \circ g) \tag{3}$$

Now assume that $(\mathcal{C}, S)$ is graded and assume that $\mathcal{C}$ has a Serre functor F. We may make F into a graded functor as follows: we have to give maps

$$\eta_A^F : (F \circ S)(A) \to (S \circ F)(A)$$

natural in A. Using non-degeneracy of the trace pairing we define these maps via the requirement

$$\mathrm{Tr}_A(S^{-1}(\eta_A^F \circ f)) = -\mathrm{Tr}_{SA}(f) \tag{4}$$

for any $f : SA \to (F \circ S)(A)$.

Remark A.4.2. The minus sign in this formula is an arbitrary choice in the graded context, but it is forced in the triangulated context. See the proof of Theorem A.4.4 below.

Proposition A.4.3. *(Graded Serre duality) For $f_i \in \mathrm{Hom}^i_{\mathcal{C}}(A,B)$, $g_{-i} \in \mathrm{Hom}^{-i}_{\mathcal{C}}(B,FA)$ we have*

$$\mathrm{Tr}_A(g_{-i} * f_i) = (-1)^i \mathrm{Tr}_B(F^{\mathrm{gr}} f_i * g_{-i})$$

Proof. We have

$$\begin{aligned}
\mathrm{Tr}_B(F^{\mathrm{gr}} f_i * g_{-i}) &= \mathrm{Tr}_B(S^{-i}(F^{\mathrm{gr}} f_i) \circ g_{-i}) && \text{(by (A.2))}\\
&= \mathrm{Tr}_B(S^{-i}((\eta^F)^i_B \circ F(f_i) \circ S^i g_{-i})) && \text{(by (1))}\\
&= (-1)^i \mathrm{Tr}_{S^i B}(F(f_i) \circ S^i g_{-i}) && \text{(by (4))}\\
&= (-1)^i \mathrm{Tr}_A(S^i g_{-i} \circ f_i) && \text{(by (3))}\\
&= (-1)^i \mathrm{Tr}_A(g_{-i} * f_i) && \text{(by (A.2))} \quad \square
\end{aligned}$$

Assume now that $\mathcal{A}$ is a Hom-finite k-linear triangulated category with a Serre functor F.

Theorem A.4.4. [5] *F is an exact functor when equipped with the graded structure obtained from* (4) *(with $S = [1]$).*

Proof. This is proved by Bondal and Kapranov in [5]. We give a somewhat more direct proof which makes the signs evident.

We start with a distinguished triangle.

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

We have to construct a map δ such that the following diagram is a morphism of distinguished triangles

$$\begin{array}{ccccccc} FA & \xrightarrow{Ff} & FB & \xrightarrow{Fg} & FC & \xrightarrow{\eta_A^F \circ Fh} & (FA)[1] \\ \| & & \| & & \uparrow{\scriptstyle \delta} & & \| \\ FA & \xrightarrow[Ff]{} & FB & \xrightarrow[\alpha]{} & X & \xrightarrow[\beta]{} & (FA)[1] \end{array}$$

where X is the cone of Ff.

In equations:

$$\eta_A^F \circ Fh \circ \delta = \beta \tag{5}$$

$$\delta \circ \alpha = Fg \tag{6}$$

For any $x : A \to X[-1]$ we deduce from (5)

$$(\eta_A^F \circ Fh \circ \delta)[-1] \circ x = \beta[-1] \circ x$$

Using (4) this is equivalent to

$$\mathrm{Tr}_{A[1]}(Fh \circ \delta \circ x[1]) = -\mathrm{Tr}_A(\beta[-1] \circ x)$$

which using (3) can be further rewritten as

$$\mathrm{Tr}_C(\delta \circ x[1] \circ h) = -\mathrm{Tr}_A(\beta[-1] \circ x) \tag{7}$$

Using the non-degeneracy of the trace pairing we see that (5) is equivalent to the validity of (7) for all $x : A \to X[-1]$. Similarly (6) is equivalent to

$$\mathrm{Tr}_C(\delta \circ \alpha \circ y) = \mathrm{Tr}_C(Fg \circ y) = \mathrm{Tr}_B(y \circ g)$$

for all $y : C \to FB$.

Summarizing: we have to find δ such that the following equations

$$\begin{aligned} \mathrm{Tr}_C(\delta \circ x[1] \circ h) &= -\mathrm{Tr}_A(\beta[-1] \circ x) \\ \mathrm{Tr}_C(\delta \circ \alpha \circ y) &= \mathrm{Tr}_B(y \circ g) \end{aligned} \tag{8}$$

hold for all $x \in \mathrm{Hom}_{\mathcal{A}}(A, X[-1])$ and $y \in \mathrm{Hom}_{\mathcal{A}}(C, FB)$.

We may view the equations (8) as fixing the value of the function $\mathrm{Tr}_C(\delta \circ -)$ on two sub vector spaces of $\mathrm{Hom}_{\mathcal{A}}(C, X)$. Since Tr_C is non-degenerate such a system can be solved provided we give the same value on the intersection. Thus we have to show

$$\alpha \circ y = x[1] \circ h \text{ then } \mathrm{Tr}_B(y \circ g) = -\mathrm{Tr}_A(\beta[-1] \circ x)$$

To prove this assume $\alpha \circ y = x[1] \circ h$ and consider the following commutative diagram

$$\begin{array}{ccccccc} FA & \xrightarrow{Ff} & FB & \xrightarrow{\alpha} & X & \xrightarrow{\beta} & (FA)[1] \\ \uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle y} & & \uparrow{\scriptstyle x[1]} & & \uparrow{\scriptstyle \psi[1]} \\ B & \xrightarrow[g]{} & C & \xrightarrow[h]{} & A[1] & \xrightarrow[-f[1]]{} & B[1] \end{array} \tag{9}$$

where ψ exists because of the axioms of triangulated categories.

We compute

$$\begin{aligned} \mathrm{Tr}_B(y \circ g) &= \mathrm{Tr}_B(Ff \circ \psi) \\ &= \mathrm{Tr}_A(\psi \circ f) \\ &= -\mathrm{Tr}_A(\beta[-1] \circ x) \end{aligned}$$

In the third line we have used the commutativity of the rightmost square in (9). □

A.5. **The Calabi-Yau case.**

Definition A.5.1. A triangulated category with Serre functor F is *Calabi-Yau of dimension* n if $F \cong s^n$ as graded functors, where s is as in Example A.3.2.

Proposition A.5.2. *Assume that $\mathcal{A}$ is Calabi-Yau of dimension n. Then for $f_i \in Hom^i_{\mathcal{A}}(A,B)$, $g_{n-i} \in Hom^{n-i}_{\mathcal{A}}(B,A)$ we have*

$$Tr_A(g_{n-i} * f_i) = (-1)^{i(n-i)} Tr_B(f_i * g_{n-i}) \tag{10}$$

Proof. We view g_{n-i} as an element of $\mathrm{Hom}^{-i}_{\mathcal{A}}(A, FB)$ by using the naive identification on objects $(FB)[-i] = B[n][-i] = B[n-i]$. To avoid confusion we put $h_{-i} = g_{n-i}$.

Graded Serre duality now reads as

$$\mathrm{Tr}_A(h_{-i} * f_i) = (-1)^i \mathrm{Tr}_B((s^{\mathrm{gr}})^n(f_i) * h_{-i})$$

Writing out everything explicitly we get

$$\begin{aligned}\mathrm{Tr}_A(h_{-i}[i] \circ f_i) &= (-1)^i \mathrm{Tr}_B\big(\big((\eta^s)^{ni}_A \circ f_i[n]\big)[-i] \circ h_{-i}\big) \\ &= (-1)^i \mathrm{Tr}_B\big((\eta^s)^{ni}_A[-i] \circ f_i[n-i] \circ h_{-i}\big)\end{aligned}$$

Now composing with $(\eta^s)^{ni}_A[-i]$ is just multiplying by $(-)^{ni}$. Thus we obtain

$$\mathrm{Tr}_A(h_{-i}[i] \circ f_i) = (-1)^{i+ni} \mathrm{Tr}_B(f_i[n-i] \circ h_{-i})$$

which translates into (10). □

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