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Smooth Order Singularities

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ABSTRACT: In [4] it was shown that the center of Cayley-Hamilton smooth orders is smooth whenever the central dimension is at most two and that there may be singularities in higher dimensions. In this paper, we give methods to classify central singularities of smooth orders up to smooth equivalence in arbitrary dimension and show that these methods are strong enough to complete the classification in dimension ≤ 6. In particular we show that there is exactly one possible singularity type in dimension three : the conifold singularity. In dimensions 4 (resp. 5,6) there are precisely 3 (resp. 10,53) types of singularities.

Contents

1. Introduction

One can define smoothness for a noncommutative algebra either by extending the homological (Serre) or the categorical (Grothendieck) characterization of commutative regular algebras to the noncommutative world. In this paper we follow the second approach, started off by W. Schelter [8] and C. Procesi [7], as we have an étale local description of these *Cayley-Hamilton smooth orders* by the results of [4]. This local structure then gives restrictions on the central simple algebras possessing a noncommutative smooth model.

An *algebra with trace map* (A, tr) is an associative $\mathbb{C}$-algebra having a linear trace map $tr : A \longrightarrow A$ satisfying $tr(ab) = tr(ba)$, $tr(a)b = btr(a)$ and $tr(tr(a)b) = tr(a)tr(b)$. Morphisms in the category of algebras with trace are trace preserving $\mathbb{C}$-algebra morphisms. One has the identity

$$\prod_{i=1}^{n}(t - x_i) = \sum_{i=0}^{n}(-1)^i \sigma_i t^{n-i}$$

where the σ_i are the elementary symmetric polynomials in the x_i. There is another generating set of the symmetric polynomials given by the power sums $\tau_k = \sum_i x_i^k$, so there are polynomials with rational coefficients $\sigma_k = p_k(\tau_1, \tau_2, \ldots, \tau_n)$ and we define the function σ_k formally on any algebra with trace (A, tr) to be

$$\sigma_k(a) = p_k(tr(a), tr(a^2), \ldots, tr(a^n))$$

This allows us to define a *formal n-th Cayley-Hamilton polynomial* for (A, tr) by

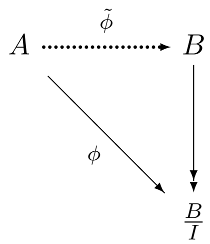
$$\chi_{n,a}(t) = \sum_{i=0}^{n}(-1)^i \sigma_i(a) t^{n-i}$$

and we say that (A, tr) is an n-th Cayley-Hamilton algebra (or that $A \in \mathtt{alg@n}$) if

$$tr(1) = n \qquad \text{and} \qquad \chi_{n,a}(a) = 0 \text{ in } A \text{ for all } a \in A$$

The archetypical example of an n-th Cayley-Hamilton algebra is an order over a normal domain in a central simple algebra of degree n.

A *Cayley-Hamilton smooth algebra* is an affine $\mathbb{C}$-algebra in $\mathtt{alg@n}$ satisfying Grothendieck's lifting characterization with respect to test-objects (B, I) in $\mathtt{alg@n}$, that is, any trace preserving algebra map ϕ



can be lifted to a trace preserving algebra map $\tilde{\phi}$ completing the diagram. C. Procesi proved in [7] that this categorical condition is equivalent to the geometric statement that the scheme $\mathtt{trep}_n\ A$ of trace preserving n-dimensional representations of A is a smooth affine variety (though it may have several connected components). Moreover, the algebraic quotient variety

$$\mathtt{tiss}_n\ A = \mathtt{trep}_n\ A//GL_n$$

with respect to the natural base-change action has as its coordinate ring the central subalgebra $tr(A)$ and its geometric points parametrize the trace preserving semi-simple n-dimensional representations of A. Of particular interest to us is the case of Cayley-Hamilton smooth *orders*, that is, when there is a Zariski open subset of $\mathtt{tiss}_n\ A$ corresponding to simple n-dimensional representations and (consequently) that $tr(A) = Z(A)$ the center of A.

If A is a Cayley-Hamilton smooth order and m is a maximal central ideal, then one can use the Luna slice theorem to determine the algebra structures of the m-adic completions (the étale local structure)

$$\hat{A}_m \qquad \text{and} \qquad \widehat{Z(A)}_m$$

in terms of a *marked* quiver setting $(Q^\bullet, \alpha)$, see [4]. To be precise, let m be the point of $\mathtt{tiss}_n\ A$ corresponding to the trace preserving semi-simple n-dimensional representation

$$M = S_1^{\oplus e_1} \oplus \ldots \oplus S_k^{\oplus e_k}$$

where the S_i are simple d_i-dimensional representations of A occurring with multiplicity e_i whence $n = \sum e_i d_i$. Consider the quiver Q on k vertices $\{v_1, \ldots, v_k\}$ (corresponding to the distinct simple components) such that the number of directed arrows from v_i to v_j is given by the dimension of the subspace of the extension space $Ext^1_A(S_i, S_j)$ consisting of trace preserving algebra maps, see [4]. Let α be the dimension vector $(e_1, \ldots, e_k)$ (given by the multiplicities), then the GL_n-étale structure of $\mathtt{trep}_n\ A$ in a neighborhood of the orbit $\mathcal{O}(M)$ is isomorphic to the associated fiber bundle

$$GL_n \times^{GL(\alpha)} \mathtt{rep}_\alpha\ Q^\bullet$$

where $GL(\alpha) \hookrightarrow GL_n$ is determined by the dimensions d_i and where $\texttt{rep}_\alpha\ Q^\bullet$ is the vectorspace of all α-dimensional representations of the marked quiver $Q^\bullet$ (this means that some of the loops in Q acquire a marking imposed by the trace preserving linear conditions, a representation of $Q^\bullet$ is a representation of Q such that the matrix corresponding to a marked loop has trace zero). In particular, this implies that $\hat{A}_m$ is Morita equivalent to the completion of the algebra of $GL(\alpha)$-equivariant maps $\texttt{rep}_\alpha\ Q^\bullet \longrightarrow M_n(\mathbb{C})$ at the maximal ideal corresponding to the zero representation and that $Z(\hat{A})_m$ is isomorphic to the completion

$$\mathbb{C}[[\texttt{rep}_\alpha\ Q^\bullet]]^{GL(\alpha)}$$

of the ring of polynomial quiver invariants at the maximal graded ideal. This fact allows us to study the central singularities of Cayley-Hamilton smooth orders. In [4] it was shown that the center is smooth whenever the Krull dimension of the smooth order is ≤ 2 and that there are central singularities possible in dimensions ≥ 3.

Recall that two commutative local rings C_m and D_n are said to be *smooth equivalent* if there are numbers k and l such that

$$\hat{C}_m[[x_1, \ldots, x_k]] \simeq \hat{D}_n[[y_1, \ldots, y_l]]$$

A classification of all commutative singularities up to smooth equivalence is a hopeless task. Still, because central singularities of Cayley-Hamilton smooth orders are determined by quiver invariants we will prove methods to attack this classification problem in principle and illustrate the methods by giving a full classification in dimensions ≤ 6. The main result of this paper is

Theorem 1 *Let d be the dimension of the central variety $\texttt{tiss}_n\ A$ of a Cayley-Hamilton smooth order A. Then, if $d \leq 2$, $\texttt{tiss}_n\ A$ is smooth. If $d = 3$ (resp. $4, 5, 6$) there are exactly one (resp. three, ten and fifty three) types of central singularities possible.*

In dimension three, the only possible central singularity is the so called *conifold singularity*

$$\mathbb{C}[[u, v, x, y]]/(uv - xy)$$

In section two we give a general strategy to classify smooth equivalence classes of central singularities in any dimension, based on the reduction steps of [1] in the classification of the smooth quiver settings. In section three we give the proofs of the claims made and in the final two sections we give the details of the remaining classification result in dimensions 5 and 6.

2. The strategy

By the étale classification it suffices to classify marked quiver settings up to smooth equivalence, that is, we want to determine when

$$\mathbb{C}[\texttt{rep}_{\alpha_1}\ Q_1^\bullet]^{GL(\alpha_1)}[x_1, \ldots, x_k] \simeq \mathbb{C}[\texttt{rep}_{\alpha_2}\ Q_2^\bullet]^{GL(\alpha_2)}[y_1, \ldots, y_l]$$

In the case of quivers, a full classification of all the quiver settings (Q, α) such that the ring of invariants is a polynomial ring was given in [1]. The proof relies on a number of reduction steps which modify the ring of invariants only up to polynomial extensions. We will recall these reduction steps as well as their obvious extensions to marked quivers. In the quiver diagrams below, the vertex-dimension component is depicted in the vertex and the number of multiple arrows between two vertices is given by a superscript, unless this number is ≤ 3 in which case the number of arrows is drawn. In the diagrams below we only depict the quiver-neighborhood of the vertex where a change is made, the remaining part of the quiver setting is left unchanged.

Recall that the *Euler form* χ_Q of a quiver Q is the bilinear form on $\mathbb{Z}^k$ (if Q has k vertices) determined by the integral $k \times k$ matrix having as its (i,j)-entry

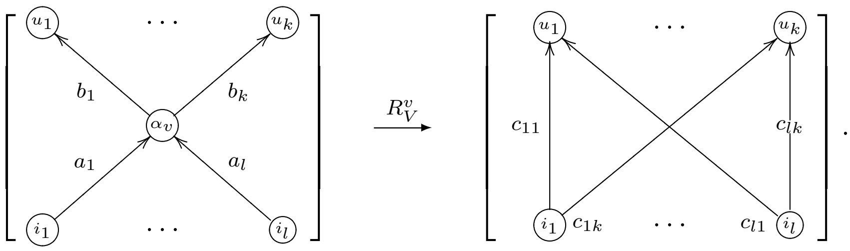
$$\chi_{Q,ij} = \delta_{ij} - \#\{\text{arrows from } v_i \text{ to } v_j\}$$

With ϵ_v we denote the basevector concentrated in vertex v and α_v will denote the vertex dimension component of α in vertex v. There are three types of reduction moves, each with their own condition and effect on the ring of invariants.

Vertex removal : Let $(Q^\bullet, \alpha)$ be a marked quiver setting and v a vertex satisfying the condition C_V^v, that is, v is without (marked) loops and satisfies

$$\chi_Q(\alpha, \epsilon_v) \geq 0 \quad \text{or} \quad \chi_Q(\epsilon_v, \alpha) \geq 0$$

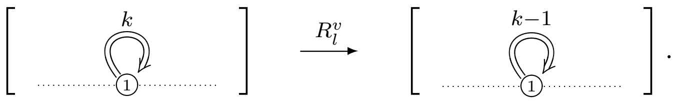
Define the new quiver setting $(Q^{\bullet'}, \alpha')$ obtained by the operation R_V^v which removes the vertex v and composes all arrows through v, the dimensions of the other vertices are unchanged :



where $c_{ij} = a_i b_j$ (observe that some of the incoming and outgoing vertices may be the same so that one obtains loops in the corresponding vertex). In this case we have

$$\mathbb{C}[\mathtt{rep}_\alpha\, Q^\bullet]^{GL(\alpha)} \simeq \mathbb{C}[\mathtt{rep}_{\alpha'}\, Q^{\bullet'}]^{GL(\alpha')}$$

loop removal : Let $(Q^\bullet, \alpha)$ be a marked quiver setting and v a vertex satisfying the condition C_l^v that the vertex-dimension $\alpha_v = 1$ and there are $k \geq 1$ loops in v. Let $(Q^{\bullet'}, \alpha)$ be the quiver setting obtained by the loop removal operation R_l^v



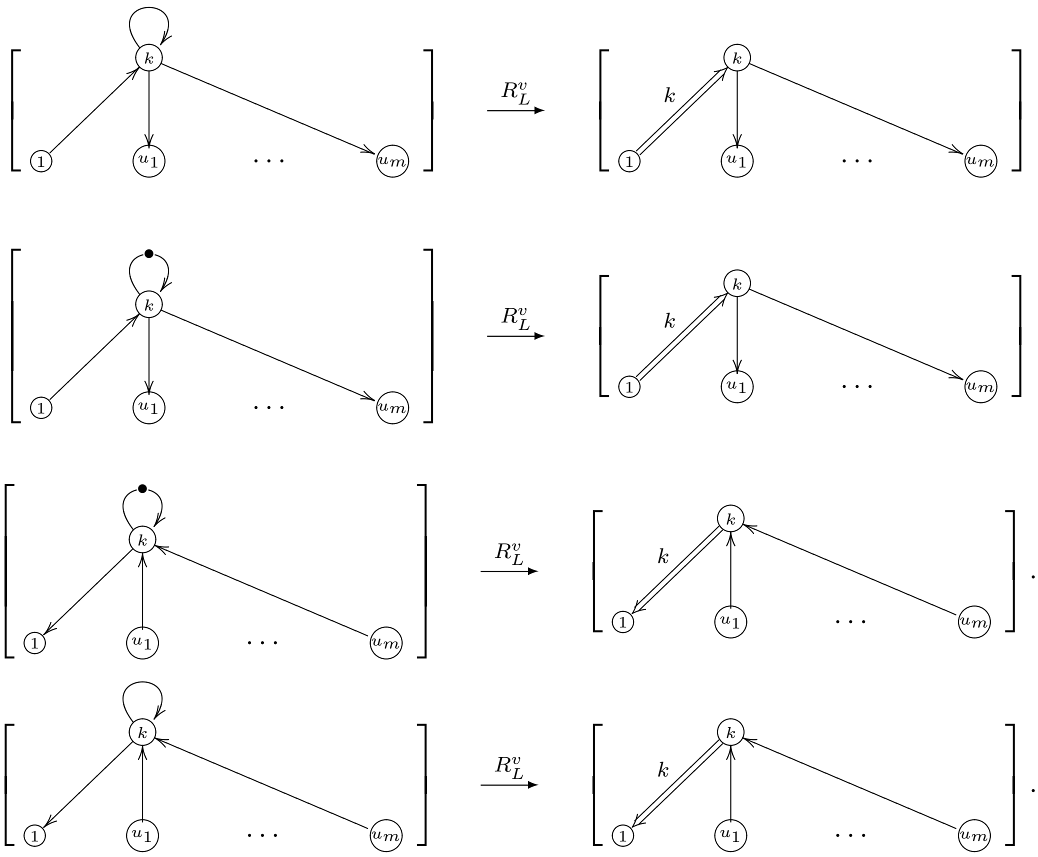
removing one loop in v and keeping the dimension vector the same, then

$$\mathbb{C}[\mathtt{rep}_\alpha\, Q^\bullet]^{GL(\alpha)} \simeq \mathbb{C}[\mathtt{rep}_\alpha\, Q^{\bullet'}]^{GL(\alpha)}[x]$$

Loop removal : Let $(Q^\bullet, \alpha)$ be a marked quiver setting and v a vertex satisfying condition C_L^v, that is, the vertex dimension $\alpha_v \geq 2$, v has precisely one (marked) loop in v and

$$\chi_Q(\epsilon_v, \alpha) = -1 \quad \text{or} \quad \chi_Q(\alpha, \epsilon_v) = -1$$

(that is, there is exactly one other incoming or outgoing arrow from/to a vertex with dimension 1). Let $(Q^{\bullet\prime}, \alpha)$ be the marked quiver setting obtained by changing the quiver as indicated below (depending on whether the incoming or outgoing condition is satisfied and whether there is a loop or a marked loop in v)



and the dimension vector is left unchanged, then we have

$$\mathbb{C}[\mathtt{rep}_\alpha\ Q^\bullet]^{GL(\alpha)} = \begin{cases} \mathbb{C}[\mathtt{rep}_\alpha\ Q^{\bullet\prime}]^{GL(\alpha)}[x_1, \ldots, x_k] & \text{(loop)} \\ \mathbb{C}[\mathtt{rep}_\alpha\ Q^{\bullet\prime}]^{GL(\alpha)}[x_1, \ldots, x_{k-1}] & \text{(marked loop)} \end{cases}$$

Definition 1 *A marked quiver $Q^\bullet$ is said to be strongly connected if for every pair of vertices $\{v, w\}$ there is an oriented path from v to w and an oriented path from w to v.*

A marked quiver setting $(Q^\bullet, \alpha)$ is said to be reduced *if and only if there is* no *vertex v such that one of the conditions C_V^v, C_l^v or C_L^v is satisfied.*

Lemma 1 *Every marked quiver setting $(Q_1^\bullet, \alpha_1)$ can be reduced by a sequence of operations R_V^v, R_l^v and R_L^v to a* reduced *quiver setting $(Q_2^\bullet, \alpha_2)$ such that*

$$\mathbb{C}[\mathtt{rep}_{\alpha_1}\ Q_1^\bullet]^{GL(\alpha_1)} \simeq \mathbb{C}[\mathtt{rep}_{\alpha_2}\ Q_2^\bullet]^{GL(\alpha_2)}[x_1, \ldots, x_z]$$

Moreover, the number z of extra indeterminates is determined by the reduction sequence

$$(Q_2^\bullet, \alpha_2) = R_{X_u}^{v_{i_u}} \circ \ldots \circ R_{X_1}^{v_{i_1}}(Q_1^\bullet, \alpha_1)$$

where for every $1 \leq j \leq u$, $X_j \in \{V, l, L\}$. *More precisely,*

$$z = \sum_{X_j = l} 1 + \sum_{X_j = L}^{(unmarked)} \alpha_{v_{i_j}} + \sum_{X_j = L}^{(marked)} (\alpha_{v_{i_j}} - 1)$$

Proof. As any reduction step removes a (marked) loop or a vertex, any sequence of reduction steps starting with $(Q_1^\bullet, \alpha_1)$ must eventually end in a reduced marked quiver setting. The statement then follows from the discussion above.

As the reduction steps have no uniquely determined inverse, there is no a priori reason why the reduced quiver setting of the previous lemma should be unique. Nevertheless this is true as we will prove in section 4 :

Theorem 2 *Every marked quiver setting* $(Q_1^\bullet, \alpha_1)$ *can be transformed by a sequence of reduction steps* R_V^v, R_l^v *or* R_L^v *to a* uniquely determined *reduced marked quiver setting* $(Q_2^\bullet, \alpha_2)$.

This result shows that it is enough to classify *reduced* marked quiver settings up to smooth equivalence. We can always assume that the quiver Q is strongly connected (if not, the ring of invariants is the tensor product of the rings of invariants of the maximal strongly connected sub-quivers). Our aim is to classify the reduced quiver singularities up to equivalence, so we need to determine the Krull dimension of the rings of invariants.

Lemma 2 *Let* $(Q^\bullet, \alpha)$ *be a reduced marked quiver setting and* Q *strongly connected. Then,*

$$dim \ \mathtt{iss}_\alpha \ Q^\bullet = 1 - \chi_Q(\alpha, \alpha) - m$$

where m *is the total number of marked loops in* $Q^\bullet$.

Proof. Because $(Q^\bullet, \alpha)$ is reduced, none of the vertices satisfies condition C_V^v, whence

$$\chi_Q(\epsilon_v, \alpha) \leq -1 \quad \text{and} \quad \chi_Q(\alpha, \epsilon_v) \leq -1$$

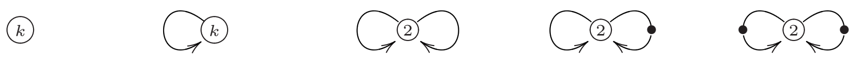
for all vertices v. In particular it follows (because Q is strongly connected) from [5] that α is the dimension vector of a simple representation of Q and that the dimension of the quotient variety

$$dim \ \mathtt{iss}_\alpha \ Q = 1 - \chi_Q(\alpha, \alpha)$$

Finally, separating traces of the loops to be marked gives the required formula.

Applying the main result of [1] we have all marked quiver settings having a regular ring of invariants. This result also describes the smooth locus of the central variety of a Cayley-Hamilton smooth order using the étale local description of section 1.

Theorem 3 *Let* $(Q^\bullet, \alpha)$ *be a marked quiver setting such that* Q *is strongly connected. Then* $\mathtt{iss}_\alpha \ Q^\bullet$ *is smooth if and only if the unique reduced marked quiver setting to which* $(Q^\bullet, \alpha)$ *can be reduced is one of the following five types*



Proof. Because the ring of invariants is graded it suffices to prove smoothness in the origin. Consider the underlying quiver Q, apply the main result of [1] and separate traces of the marked loops.

The next step is to classify for a given dimension d all reduced marked quiver settings $(Q^\bullet, \alpha)$ such that $dim\ \mathtt{iss}_\alpha\ Q^\bullet = d$. The following result limits the possible cases drastically in low dimensions.

Lemma 3 *Let $(Q^\bullet, \alpha)$ be a reduced marked quiver setting on $k \geq 2$ vertices. Then,*

$$dim\ \mathtt{iss}_\alpha\ Q^\bullet \geq 1 + \sum^{a\geq 1} a + \sum^{a>1} (2a-1) + \sum^{a>1} (2a) + \sum^{a>1} (a^2+a-2) +$$

$$\sum^{a>1} (a^2+a-1) + \sum^{a>1} (a^2+a) + \ldots + \sum^{a>1} ((k+l-1)a^2+a-k) + \ldots$$

In this sum the contribution of a vertex v with $\alpha_v = a$ is determined by the number of (marked) loops in v. By the reduction steps (marked) loops only occur at vertices where $\alpha_v > 1$.

Proof. We know that the dimension of $\mathtt{iss}_\alpha\ Q^\bullet$ is equal to

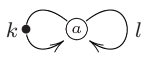
$$1 - \chi_Q(\alpha,\alpha) - m = 1 - \sum_v \chi_Q(\epsilon_v,\alpha)\alpha_v - m$$

If there are no (marked) loops at v, then $\chi_Q(\epsilon_v,\alpha) \leq -1$ (if not we would reduce further) which explains the first sum. If there is exactly one (marked) loop at v then $\chi_Q(\epsilon_v,\alpha) \leq -2$ for if $\chi_Q(\epsilon_v,\alpha) = -1$ then there is just one outgoing arrow to a vertex w with $\alpha_w = 1$ but then we can reduce the quiver setting further. This explains the second and third sums. If there are k marked loops and l ordinary loops in v (and Q has at least two vertices) , then

$$-\chi_Q(\epsilon_v,\alpha)\alpha_v - k \geq ((k+l)\alpha_v - \alpha_v + 1)\alpha_v - k$$

which explains all other sums.

Observe that the dimension of the quotient variety of the one vertex marked quivers



is equal to $(k+l-1)a^2 + 1 - k$ and is singular (for $a \geq 2$) unless $k + l = 2$. We will now classify the reduced singular settings when there are at least two vertices in low dimensions. By the previous lemma it follows immediately that

1. the maximal number of vertices in a reduced marked quiver setting $(Q^\bullet, \alpha)$ of dimension d is $d-1$ (in which case all vertex dimensions must be equal to one)

2. if a vertex dimension in a reduced marked quiver setting is $a \geq 2$, then the dimension $d \geq 2a$.

Lemma 4 *Let $(Q^\bullet, \alpha)$ be a reduced marked quiver setting such that $\mathtt{iss}_\alpha\ Q^\bullet$ is singular of dimension $d \leq 5$, then $\alpha = (1,\ldots,1)$. Moreover, each vertex must have at least two incoming and two outgoing arrows and no loops.*

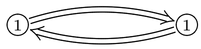
Proof. From the lower bound of the sum formula it follows that if some $\alpha_v > 1$ it must be equal to 2 and must have a unique marked loop and there can only be one other vertex w with $\alpha_w = 1$. If there are x arrows from w to v and y arrows from v to w, then

$$dim\ \mathtt{iss}_\alpha\, Q^\bullet = 2(x+y) - 1$$

whence x or y must be equal to 1 contradicting reducedness. The second statement follows as otherwise we could perform extra reductions.

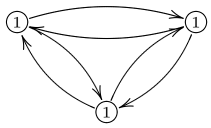
Proposition 1 *The only reduced marked quiver singularity in dimension 3 is*

3_{con} :

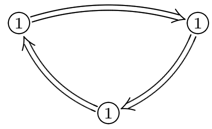


The reduced marked quiver singularities in dimension 4 are

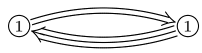
4_{3a} :



4_{3b} :



4_2 :



Proof. All one vertex marked quiver settings with quotient dimension ≤ 5 are smooth, so we are in the situation of lemma 4. If the dimension is 3 there must be two vertices each having exactly two incoming and two outgoing arrows, whence the indicated type is the only one. The resulting singularity is the *conifold singularity*

$$\frac{\mathbb{C}[[x,y,u,v]]}{(xy-uv)}$$

In dimension 4 we can have three or two vertices. In the first case, each vertex must have exactly two incoming and two outgoing arrows whence the first two cases. If there are two vertices, then just one of them has three incoming arrows and one has three outgoing arrows.

In dimensions 5 and 6 one can give a classification of all reduced singularities by hand, see sections 5 and 6. This concludes the first step in our strategy, the next will be to distinguish reduced singularities of the same dimension up to (étale) isomorphism.

3. Fingerprinting singularities

In this section we will outline methods to distinguish two reduced marked quiver settings $(Q_1^\bullet, \alpha_1)$ and $(Q_2^\bullet, \alpha_2)$ having the same quotient dimension d. Recall from [5] that the rings of quiver invariants are generated by taking traces along oriented cycles in the quiver (again separating traces gives the same result for marked quivers). Assume that all vertex dimensions are equal to one, then one can write any (trace of an) oriented cycle as a product of (traces of) *primitive* oriented cycles (that is, those that cannot be decomposed further). From this one deduces immediately :

Lemma 5 *Let* $(Q^\bullet, \alpha)$ *be a reduced marked quiver setting such that all* $\alpha_v = 1$. *Let* m *be the maximal graded ideal of* $\mathbb{C}[\mathtt{rep}_\alpha\, Q^\bullet]^{GL(\alpha)}$, *then a vectorspace basis of*

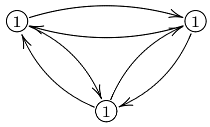
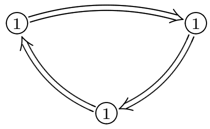
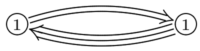
$$\frac{m^i}{m^{i+1}}$$

is given by the oriented cycles in Q *which can be written as a product of* i *primitive cycles but not as a product of* $i+1$ *such cycles.*

Clearly, the dimensions of the quotients m^i/m^{i+1} are (étale) isomorphism invariants. Hence, for $d \leq 5$ this simple minded counting method can be used to separate quiver singularities.

Theorem 4 *There are precisely three reduced quiver singularities in dimension* $d = 4$.

Proof. The number of primitive oriented cycles of the three types of reduced marked quiver settings in dimension four

4_{3a} :  4_{3b} :  4_2 : 

is 5, respectively 8 and 6. Hence, they give nonisomorphic rings of invariants.

In section 5 we will classify the reduced quiver singularities for $d = 5$. If some of the vertex dimensions are ≥ 2 we have no easy description of the vectorspaces m^i/m^{i+1} and we need a more refined argument. The idea is to answer the question "what other singularities can the reduced singularity see ?" by the theory of local quivers of [5].

Let Q be a quiver (we will indicate the necessary changes to be made for marked quivers below) and α a dimension vector. An *α-representation type* is a datum

$$\tau = (e_1, \beta_1; \ldots; e_l, \beta_l)$$

where the e_i are natural numbers ≥ 1, the β_i are dimension vectors of simple representations of Q (for which we have a precise description by [5]) such that $\alpha = \sum_i e_i \beta_i$. Any neighborhood of the trivial representation contains semi-simple representations of Q of type τ for any α-representation type.

To determine the dimension of the corresponding strata and the nature of their singularities we construct a new quiver Q_τ, the *local quiver*, on l vertices (the number of distinct simple components) say $\{w_1, \ldots, w_l\}$ such that the number of oriented arrows (or loops) from w_i to w_j is given by the number

$$\delta_{ij} - \chi_Q(\beta_i, \beta_j)$$

There is an étale local isomorphism between a neighborhood of a semi-simple α-dimensional representation of type τ and a neighborhood of the trivial representation of $\mathtt{iss}_{\alpha_\tau}\ Q_\tau$ where $\alpha_\tau = (e_1, \ldots, e_l)$ is the dimension vector determined by the multiplicities.

As a consequence we see that the dimension of the corresponding strata is equal to the number of loops in Q_τ. Now, assume that $\mathtt{iss}_{\alpha_\tau}\ Q_\tau$ has a singularity, then the couple

$$(\text{dimension of strata, type of singularity})$$

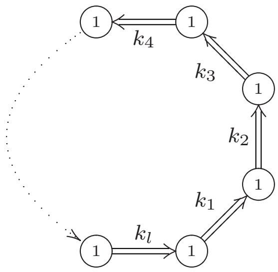
is a characteristic feature of the singularity of $\mathtt{iss}_\alpha\ Q$ and one can often distinguish types by these couples. In the case of a marked quiver one proceeds as before for the underlying quiver and in the final result compensates for the markings (that is, one marks as many loops in the local quiver in the vertices giving a non-zero contribution to the original marked vertex).

Recall from [5] that there is a partial ordering $\tau < \tau'$ on the α-representation types induced by degeneration of representations. The *fingerprint* of a reduced quiver singularity will be the

Hasse diagram of those α-representation types τ such that the local marked quiver setting $(Q^\bullet_\tau, \alpha_\tau)$ can be reduced to a reduced quiver singularity (necessarily occurring in lower dimension and the difference between the two dimensions gives the dimension of the stratum).

Clearly, this method fails in case the marked quiver singularity is an *isolated singularity*. Fortunately, we have a complete classification of such singularities by the work of [2].

Theorem 5 *[2] The only reduced marked quiver settings $(Q^\bullet, \alpha)$ such that the quotient variety is an isolated singularity are of the form*



where Q has l vertices and all $k_i \geq 2$. The dimension of the corresponding quotient is

$$d = \sum_i k_i + l - 1$$

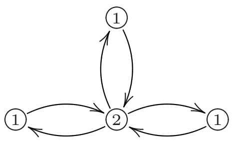
and the unordered l-tuple $\{k_1, \ldots, k_l\}$ *is an (étale) isomorphism invariant of the ring of invariants.*

Not only does this result distinguish among isolated reduced quiver singularities, but it also shows that in all other marked quiver settings we will have additional families of singularities. We will illustrate the method in some detail to separate the reduced marked quiver settings in dimension 6 having one vertex of dimension two.

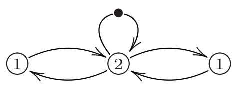
Proposition 2 *The reduced singularities of dimension 6 such that α contains a component equal to 2 are pairwise non-equivalent.*

Proof. In section 6 we will show that the relevant reduced marked quiver setting are the following

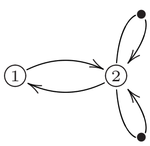
type A



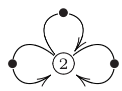
type B



type C

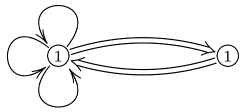


type D



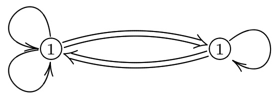
We will order the vertices such that $\alpha_1 = 2$.

type A : There are three different representation types $\tau_1 = (1, (2; 1, 1, 0); 1, (0; 0, 0, 1))$ (and permutations of the 1-vertices). The local quiver setting has the form



because for $\beta_1 = (2;1,1,0)$ and $\beta_2 = (0;0,0,1)$ we have that $\chi_Q(\beta_1,\beta_1) = -2$, $\chi_Q(\beta_1,\beta_2) = -2$, $\chi_Q(\beta_2,\beta_1) = -2$ and $\chi(\beta_2,\beta_2) = 1$. These three representation types each give a three dimensional family of conifold (type 3_{con}) singularities.

Further, there are three different representation types $\tau_2 = (1,(1;1,1,0);1,(1;0,0,1))$ (and permutations) of which the local quiver setting is of the form

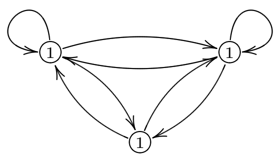


as with $\beta_1 = (1;1,1,0)$ and $\beta_2 = (1;0,0,1)$ we have $\chi_Q(\beta_1,\beta_1) = -1$, $\chi_Q(\beta_1,\beta_2) = -2$, $\chi_Q(\beta_2,\beta_1) = -2$ and $\chi_Q(\beta_2,\beta_2) = 0$. These three representation types each give a three dimensional family of conifold singularities.

Finally, there are the three representation types

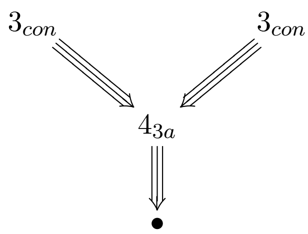
$$\tau_3 = (1,(1;1,0,0);1,(1;0,1,0);1,(0;0,0,1))$$

(and permutations) with local quiver setting



These three types each give a two dimensional family of reduced singularities of type 4_{3a}.

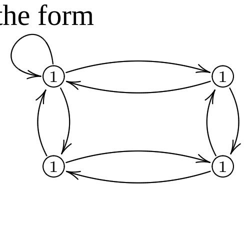
The degeneration order on representation types gives $\tau_1 < \tau_3$ and $\tau_2 < \tau_3$ (but for different permutations) and the *fingerprint* of this reduced singularity can be depicted as



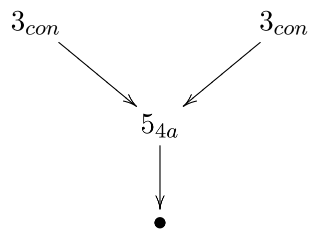
type B : There is one representation type $\tau_1 = (1,(1;1,0);1,(1;0,1))$ giving as above a three dimensional family of conifold singularities, one representation type $\tau_2 = (1,(1;1,1);1,(1;0,0))$ giving a three dimensional family of conifolds and finally one representation type

$$\tau_3 = (1,(1;0,0);1,(1;0,0);1,(0;1,1);1,(0;0,1))$$

of which the local quiver setting has the form



(the loop in the downright corner is removed to compensate for the marking) giving rise to a one-dimensional family of five-dimensional singularities of type 5_{4a}. This gives the fingerprint



type C : We have a three dimensional family of conifold singularities coming from the representation type $(1,(1;1);1,(1;0))$ and a two-dimensional family of type 4_{3a} singularities corresponding to the representation type $(1,(1;0);1,(1,0);1,(0;1))$. Therefore, the fingerprint is depicted as

$$3_{con} \longrightarrow 4_{3a} \longrightarrow \bullet$$

type D : We have just one three-dimensional family of conifold singularities determined by the representation type $(1,(1);1,(1))$ so the fingerprint is $3_{con} \longrightarrow \bullet$. As fingerprints are isomorphism invariants of the singularity, this finishes the proof.

We claim that the minimal number of generators for these invariant rings is 7. The structure of the invariant ring of three 2×2 matrices upto simultaneous conjugation was determined by Ed Formanek [3] who showed that it is generated by 10 elements

$$\{tr(X_1), tr(X_2), tr(X_3), det(X_1), det(X_2), det(X_3), tr(X_1X_2), tr(X_1X_3), tr(X_2X_3), tr(X_1X_2X_3)\}$$

and even gave the explicit quadratic polynomial satisfied by $tr(X_1X_2X_3)$ with coefficients in the remaining generators. The rings of invariants of the four cases of interest to us are quotients of this algebra by the ideal generated by three of its generators : for type A it is $(det(X_1), det(X_2), det(X_3))$, for type B : $(det(X_1), tr(X_2), det(X_3))$, for type C : $(det(X_1), tr(X_2), tr(X_3))$ and for type D : $(tr(X_1), tr(X_2), tr(X_3))$.

4. Uniqueness of reduced setting

In this section we will prove theorem 2. We will say that a vertex v is *reducible* if one of the conditions C_V^v (vertex removal), C_l^v (loop removal in vertex dimension one) or C_L^v (one (marked) loop removal) is satisfied. If we let the specific condition unspecified we will say that v satisfies C_X^v and denote R_X^v for the corresponding marked quiver setting reduction. The resulting marked quiver setting will be denoted by

$$R_X^v(Q^\bullet, \alpha)$$

If $w \neq v$ is another vertex in $Q^\bullet$ we will denote the corresponding vertex in $R_X^v(Q^\bullet)$ also with w. The proof of the uniqueness result relies on three claims :

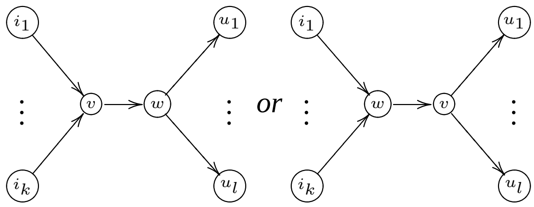
1. If $w \neq v$ satisfies R_Y^w in $(Q^\bullet, \alpha)$, then w virtually always satisfies R_Y^w in $R_X^v(Q^\bullet, \alpha)$.
2. If v satisfies R_X^v and w satisfies R_Y^w, then $R_X^v(R_Y^w(Q^\bullet, \alpha)) = R_Y^w(R_X^v(Q^\bullet, \alpha))$.

3. The previous two facts can be used to prove the result by induction on the minimal length of the reduction chain.

By the *neighborhood* of a vertex v in $Q^\bullet$ we mean the (marked) subquiver on the vertices connected to v. A neighborhood of a set of vertices is the union of the vertex-neighborhoods. *Incoming* resp. *outgoing* neighborhoods are defined in the natural manner.

Lemma 6 *Let $v \neq w$ be vertices in $(Q^\bullet, \alpha)$.*

1. *If v satisfies C_V^v in $(Q^\bullet, \alpha)$ and w satisfies C_X^w, then v satisfies C_V^w in $R_X^w(Q^\bullet, \alpha)$ unless the neighborhood of $\{v, w\}$ looks like*



and $\alpha_v = \alpha_w$. Observe that in this case $R_V^v(Q^\bullet, \alpha) = R_V^w(Q^\bullet, \alpha)$.

2. *If v satisfies C_l^v and w satisfies C_X^w then then v satisfies C_l^v in $R_X^w(Q^\bullet, \alpha)$.*

3. *If v satisfies C_V^v and w satisfies C_X^w then then v satisfies C_V^v in $R_X^w(Q^\bullet, \alpha)$.*

Proof. (1) : If $X = l$ then R_X^w does not change the neighborhood of v so C_V^v holds in $R_l^w(Q^\bullet, \alpha)$. If $X = L$ then R_X^w does not change the neighborhood of v unless $\alpha_v = 1$ and $\chi_Q(\epsilon_w, \epsilon_v) = -1$ (resp. $\chi_Q(\epsilon_v, \epsilon_w) = -1$) depending on whether w satisfies the in- or outgoing condition C_L^w. We only consider the first case, the latter is similar. Then v cannot satisfy the outgoing form of C_V^v in $(Q^\bullet, \alpha)$ so the incoming condition is satisfied. Because the R_L^w-move does not change the incoming neighborhood of v, C_V^v still holds for v in $R_L^w(Q^\bullet, \alpha)$.

If $X = V$ and v and w have disjoint neighborhoods then C_V^v trivially remains true in $R_V^w(Q^\bullet, \alpha)$. Hence assume that there is at least one arrow from v to w (the case where there are only arrows from w to v is similar). If $\alpha_v < \alpha_w$ then the incoming condition C_V^v must hold (outgoing is impossible) and hence w does not appear in the incoming neighborhood of v. But then R_V^w preserves the incoming neighborhood of v and C_V^v remains true in the reduction. If $\alpha_v > \alpha_w$ then the outgoing condition C_V^w must hold and hence w does not appear in the incoming neighborhood of v. So if the incoming condition C_V^v holds in $(Q^\bullet, \alpha)$ it will still hold after the application of R_V^w. If the outgoing condition C_V^v holds, the neighborhoods of v and w in $(Q^\bullet, \alpha)$ and v in $R_V^w(Q^\bullet, \alpha)$ are depicted in figure 1 Let A be the set of arrows in $Q^\bullet$ and A' the set of arrows in the reduction, then because $\sum_{a \in A, s(a)=w} \alpha_{t(a)} \leq \alpha_w$ (the incoming condition for w) we have

$$\begin{aligned}
\sum_{a \in A', s(a)=v} \alpha'_{t(a)} &= \sum_{\substack{a \in A, \\ s(a)=v, t(a) \neq w}} \alpha_{t(a)} + \sum_{\substack{a \in A \\ t(a)=w, s(a)=v}} \sum_{a \in A, s(a)=w} \alpha_{t(a)} \\
&\leq \sum_{\substack{a \in A, \\ s(a)=v, t(a) \neq w}} \alpha_{t(a)} + \sum_{\substack{a \in A \\ t(a)=w, s(a)=w}} \alpha_w \\
&= \sum_{a \in A, s(a)=v} \alpha_{t(a)} \leq \alpha_v
\end{aligned}$$

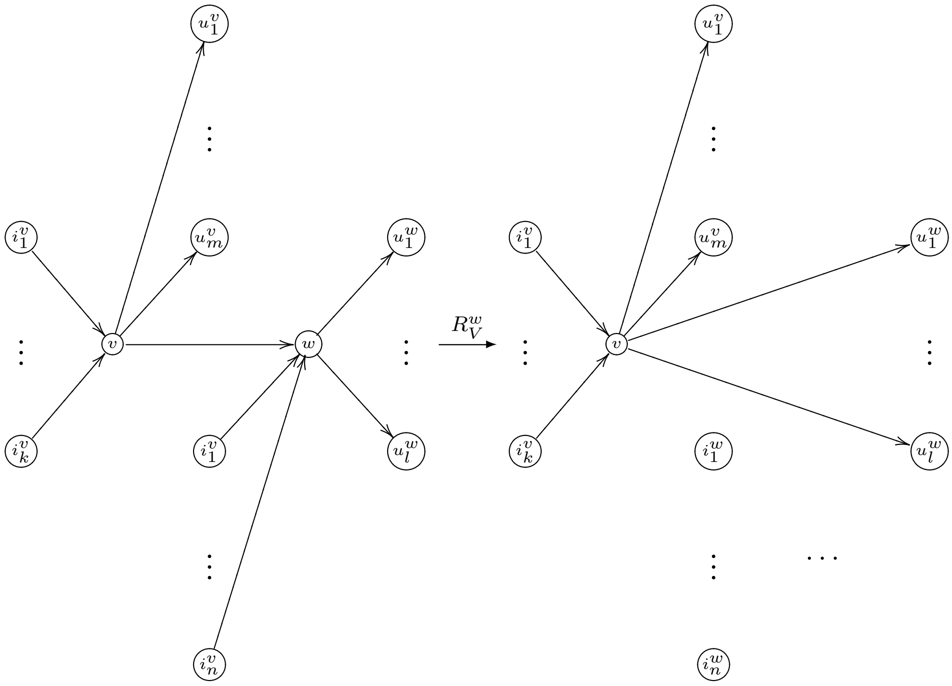


Figure 1: Neighborhoods of v and w

and therefore the outgoing condition C_V^v also holds in $R_V^w(Q^\bullet, \alpha)$. Finally if $\alpha_v = \alpha_w$, it may be that C_V^v does not hold in $R_V^w(Q^\bullet, \alpha)$. In this case $\chi(\epsilon_v, \alpha) < 0$ and $\chi(\alpha, \epsilon_w) < 0$ (C_V^v is false in $R_V^w(Q^\bullet, \alpha)$). Also $\chi(\alpha, \epsilon_v) \geq 0$ and $\chi(\epsilon_w, \alpha) \geq 0$ (otherwise C_V does not hold for v or w in $(Q^\bullet, \alpha)$). This implies that we are in the situation described in the lemma and the conclusion follows.

(2) : None of the R_X^w-moves removes a loop in v nor changes $\alpha_v = 1$.

(3) : Assume that the incoming condition C_L^v holds in $(Q^\bullet, \alpha)$ but not in $R_X^w(Q^\bullet, \alpha)$, then w must be the unique vertex which has an arrow to v and $X = V$. Because $\alpha_w = 1 < \alpha_v$, the incoming condition C_V^w holds. This means that there is also only one arrow arriving in w and this arrow is coming from a vertex with dimension 1. Therefore after applying R_V^w, v will still have only one incoming arrow starting in a vertex with dimension 1. A similar argument holds for the outgoing condition C_L^v.

Lemma 7 *Suppose that $v \neq w$ are vertices in $(Q^\bullet, \alpha)$ and that C_X^v and C_Y^w are satisfied. If C_X^v holds in $R_Y^w(Q^\bullet, \alpha)$ and C_Y^w holds in $R_X^v(Q^\bullet, \alpha)$ then*

$$R_X^v R_Y^w(Q^\bullet, \alpha) = R_Y^w R_X^v(Q^\bullet, \alpha)$$

Proof. If $X, Y \in \{l, L\}$ this is obvious, so let us assume that $X = V$. If $Y = V$ as well, we can calculate the Euler form $\chi_{R_V^w R_V^v Q}(\epsilon_x, \epsilon_y)$. Because

$$\chi_{R_V^v Q}(\epsilon_x, \epsilon_y) = \chi_Q(\epsilon_x, \epsilon_y) - \chi_Q(\epsilon_x, \epsilon_v)\chi_Q(\epsilon_v, \epsilon_y)$$

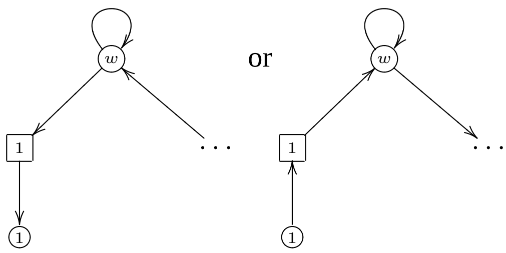
it follows that

$$\begin{aligned}
\chi_{R_V^w R_V^v Q}(\epsilon_x,\epsilon_y) &= \chi_{R_V^v Q}(\epsilon_x,\epsilon_y) - \chi_{R_V^v Q}(\epsilon_x,\epsilon_w)\chi_{R_V^v Q}(\epsilon_v,\epsilon_y) \\
&= \chi_Q(\epsilon_x,\epsilon_y) - \chi_Q(\epsilon_x,\epsilon_v)\chi_Q(\epsilon_v,\epsilon_y) \\
&- \left(\chi_Q(\epsilon_x,\epsilon_w) - \chi_Q(\epsilon_x,\epsilon_v)\chi_Q(\epsilon_v,\epsilon_w)\right)\left(\chi_Q(\epsilon_w,\epsilon_y) - \chi_Q(\epsilon_w,\epsilon_v)\chi_Q(\epsilon_v,\epsilon_y)\right) \\
&= \chi_Q(\epsilon_x,\epsilon_y) - \chi_Q(\epsilon_x,\epsilon_v)\chi_Q(\epsilon_v,\epsilon_y) - \chi_Q(\epsilon_x,\epsilon_w)\chi_Q(\epsilon_w,\epsilon_y) \\
&- \chi_Q(\epsilon_x,\epsilon_v)\chi_Q(\epsilon_v,\epsilon_w)\chi_Q(\epsilon_w,\epsilon_v)\chi_Q(\epsilon_v,\epsilon_y) \\
&+ \chi_Q(\epsilon_x,\epsilon_w)\chi_Q(\epsilon_w,\epsilon_v)\chi_Q(\epsilon_v,\epsilon_y) + \chi_Q(\epsilon_x,\epsilon_v)\chi_Q(\epsilon_v,\epsilon_w)\chi_Q(\epsilon_w,\epsilon_y)
\end{aligned}$$

This is symmetric in v and w and therefore the ordering of R_V^v and R_V^w is irrelevant.
If $Y = l$ we have the following equalities

$$\begin{aligned}
\chi_{R_l^w R_V^v Q}(\epsilon_x,\epsilon_y) &= \chi_{R_V^v Q}(\epsilon_x,\epsilon_y) - \delta_{wx}\delta_{wy} \\
&= \chi_Q(\epsilon_x,\epsilon_y) - \chi_Q(\epsilon_x,\epsilon_v)\chi_Q(\epsilon_v,\epsilon_y) - \delta_{wx}\delta_{wy} \\
&= \chi_Q(\epsilon_x,\epsilon_y) - \delta_{wx}\delta_{wy} - (\chi_Q(\epsilon_x,\epsilon_v) - \delta_{wx}\delta_{wv})(\chi_Q(\epsilon_v,\epsilon_y) - \delta_{wv}\delta_{wy}) \\
&= \chi_{R_l^w Q}(\epsilon_x,\epsilon_y) - \chi_{R_l^w Q}(\epsilon_x,\epsilon_v)\chi_{R_l^w Q}(\epsilon_v,\epsilon_y) \\
&= \chi_{R_V^v R_l^w Q}.
\end{aligned}$$

If $Y = L$, an R_L^w-move commutes with the R_V^v move because it does not change the neighborhood of v except when v is the unique vertex of dimension 1 connected to w. In this case the neighborhood of v looks like



In this case the reduction at v is equivalent to a reduction at v' (i.e. the lower vertex) which certainly commutes with R_L^w.

We are now in a position to prove theorem 2.

Theorem 6 *If $(Q^\bullet,\alpha)$ is a strongly connected marked quiver setting and $(Q_1^\bullet,\alpha_1)$ and $(Q_2^\bullet,\alpha_2)$ are two reduced marked quiver setting obtained by applying reduction moves to $(Q^\bullet,\alpha)$ then*

$$(Q_1^\bullet,\alpha_1) = (Q_2^\bullet,\alpha_2)$$

Proof. We do induction on the length l_1 of the reduction chain R_1 reducing $(Q^\bullet,\alpha)$ to $(Q_1^\bullet,\alpha_1)$. If $l_1 = 0$, then $(Q^\bullet,\alpha)$ has no reducible vertices so the result holds trivially. Assume the result holds for all lengths $< l_1$. There are two cases to consider.

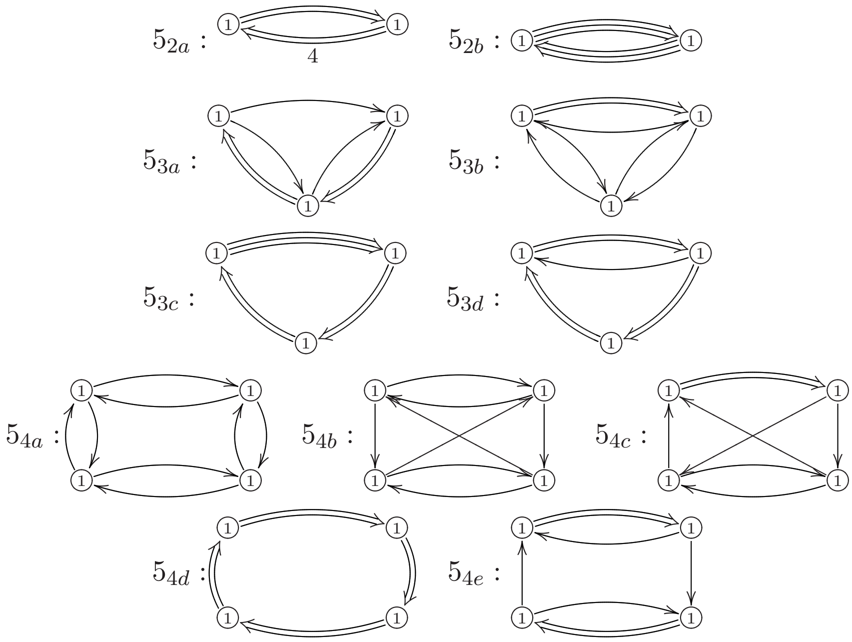
There exists a vertex v satisfying a loop removal condition $C_X^v, X = l$ or L. Then, there is a R_X^v-move in both reduction chains R_1 and R_2. This follows from lemma 6 and the fact that none of the vertices in $(Q_1^\bullet, \alpha_1)$ and $(Q_2^\bullet, \alpha_2)$ are reducible. By the commutation relations from lemma 7, we can bring this reduction to the first position in both chains and use induction.

If there is a vertex v satisfying condition C_V^v, either both chains will contain an R_V^v-move or the neighborhood of v looks like the figure in lemma 6 (1). Then, R_1 can contain an R_V^v-move and R_2 an R_V^w-move. But then we change the R_V^w move into a R_V^v move, because they have the same effect. The concluding argument is similar to that above.

5. Dimension 5 singularities

In this section we classify the reduced marked quiver singularities in dimension $d = 5$ up to isomorphism. First, we classify all reduced marked quiver settings.

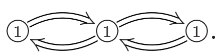
Proposition 3 *The reduced marked quiver settings for $d = 5$ are*



Proof. We are in the situation of lemma 4 and hence know that all vertex-dimensions are equal to one, every vertex has at least two incoming and two outgoing arrows and the total number of arrows is equal to $5 - 1 + k$ where k is the number of arrows which can be at most 4.

$k = 2$ **:** There are 6 arrows and as there must be at least two incoming arrows in each vertex, the only possibilities are types 5_{2a} and 5_{2b}.

$k = 3$ **:** There are seven arrows. Hence every two vertices are connected, otherwise one needs at least 8 arrows:



There is one vertex with 3 incoming arrows and one vertex with 3 outgoing arrows. If these vertices are equal ($= v$), there are no triple arrows. Call x the vertex with 2 arrows coming from v and y the other one. Because there are already two incoming arrows in x, $\chi_Q(\epsilon_y, \epsilon_x) = 0$. This also implies that $\chi_Q(\epsilon_y, \epsilon_v) = -2$ and $\chi_Q(\epsilon_x, \epsilon_v) = \chi_Q(\epsilon_x, \epsilon_y) = -1$. This gives us setting 5_{3a}. If the two vertices are different, we can delete one arrow between them, which leaves us with a singularity of dimension $d = 4$ (because now all vertices have 2 incoming and 2 outgoing vertices). So starting from the types 4_{3a-b} and adding one extra arrow we obtain three new types 5_{3b-d}.

$k = 4$ **:** There are 8 arrows so each vertex must have exactly two incoming and two outgoing arrows. First consider the cases having no double arrows. Fix a vertex v, there is at least one vertex connected to v in both directions. This is because there are 3 remaining vertices and four arrows connected to v (two incoming and two outgoing). If there are two such vertices, w_1 and w_2, the remaining vertex w_3 is not connected to v. Because there are no double arrows we must be in case 5_{4a}. If there is only one such vertex, the quiver contains two disjoint cycles of length 2. This leads to type 5_{4b}.

If there is precisely one double arrow (from v to w), the two remaining vertices must be contained in a cycle of length 2 (if not, there would be 3 arrows leaving v). This leads to type 5_{4c}.

If there are two double arrows, they can be consecutive or disjoint. In the first case, all arrows must be double (if not, there are three arrows leaving one vertex), so this is type 5_{4d}. In the latter case, let v_1 and v_2 be the starting vertices of the double arrows and w_1 and w_2 the end points. As there are no consecutive double arrows, the two arrows leaving w_1 must go to different vertices not equal to w_2. An analogous condition holds for the arrows leaving w_2 and therefore we are in type 5_{4e}.

Next, we have to separate the corresponding rings of invariants up to isomorphism. This is done with the methods of section 3. The proofs of the claims are left to the reader but are similar to the proof of proposition 2.

Theorem 7 *There are exactly ten reduced marked quiver singularities in dimension $d = 5$. Only the types 5_{3a} and 5_{4e} have an isomorphic ring of invariants.*

Proof. Recall that the dimension of m/m^2 is given by the number of primitive cycles in Q. These numbers are

type	$dim\ m/m^2$	type	$dim\ m/m^2$
5_{2a}	8	5_{4a}	6
5_{2b}	9	5_{4b}	6
5_{3a}	8	5_{4c}	9
5_{3b}	7	5_{4d}	16
5_{3c}	12	5_{4e}	8
5_{3d}	10		

Type 5_{4a} can be separated from type 5_{4b} because 5_{4a} contains $2 + 4$ twodimensional families of conifold singularities corresponding to representation types of the form

$$\begin{cases} \begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix} \oplus \begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix} \\ \begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix} \oplus \begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix} \end{cases} \quad \text{and } 4 \times \begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix} \oplus \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}.$$

whereas type 5_{4b} has only $1+4$ such families as the decomposition

$$\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix} \oplus \begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}$$

is not a valid representation type.

Type 5_{2a} and 5_{2b} are both isolated singularities because we have no non- trivial representation types, whereas types 5_{4c}, and 5_{4e} are not as they have representation types of the form

$$\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \oplus \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \oplus \begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}$$

giving local quivers smooth equivalent to type 4_{3b} (in the case of type 5_{4c}) and to type 3_a (in the case of 5_{3e}).

Finally, as we know the algebra generators of the rings of invariants (the primitive cycles) it is not difficult to compute these rings explicitly. Type 5_{3a} and type 5_{4e} have a ring of invariants isomorphic to

$$\frac{\mathbb{C}[X_i,Y_i,Z_{ij}:1\leq i,j\leq 2]}{(Z_{11}Z_{22}=Z_{12}Z_{21},X_1Y_1Z_{22}=X_1Y_2Z_{21}=X_2Y_1Z_{12}=X_2Y_2Z_{11})}$$

6. Dimension 6 singularities

In this section we will classify all reduced quiver singularities in dimension $d=6$. First, we need some information on the reduced marked quiver settings.

Lemma 8 *Let $(Q^\bullet,\alpha)$ be a reduced marked quiver setting on at least two vertices such that the dimension of the quotient variety $\mathtt{iss}_\alpha\ Q^\bullet$ is 6. Then, the maximal vertex dimension is 2 and the only settings having such a vertex dimension are the quivers 6_A, 6_B, 6_C or 6_D of section 3.*

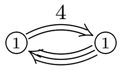
Proof. From the formula of lemma 3 follows that the maximal vertex dimension is 4 and for $\alpha_v \geq 3$, there cannot be a (marked) loop in v. But then, there can be just one other vertex with $\alpha_w = 1$. Reducedness then forces the dimension of the quotient variety to be larger than 6. If there are two vertices with $\alpha_v = \alpha_w = 2$, then at most one of them can have a marked loop (in which case there are no other vertices and reducedness implies again that the dimension $d > 6$), if neither has a marked loop there can be just one more vertex u with $\alpha_u = 1$ and again we obtain $d > 6$ if we impose reducedness. So, there is at most one vertex v with $\alpha_v = 2$ and we can have at most three remaining vertices all of vertex dimension one.

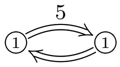
four vertices : There can be no (marked) loop in v and we need that $\chi_Q(\epsilon_v,\alpha) = \chi_Q(\alpha,\epsilon_w)$ for all vertices w giving type 6_A.

three vertices : There can be at most one marked loop in v in which case we must be in type 6_B. If there is no marked loop in v, there must be at least three incoming and three outgoing arrows from v giving a lower bound of seven for the quotient variety.

two vertices : There are at most two marked loops in v in which case we must be in type 6_C. If there is one (marked) loop in v, there must be at least two incoming and two outgoing arrows from/to w (if not we have C_V^v) giving a lower bound of seven for the quotient variety.

Next, we have to classify all reduced quiver settings such that all vertex dimensions are equal to one. In this case, each vertex must have at least two incoming and two outgoing arrows, the maximal number of vertices is bounded by 5 and the total number of arrows is equal to $5+k$ where k is the number of vertices. The case $k=2$ is easy.

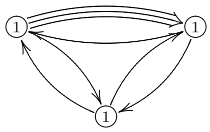
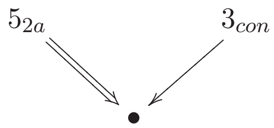
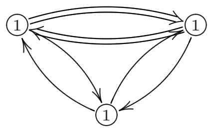
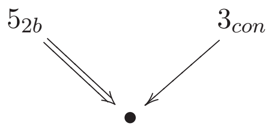
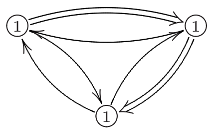
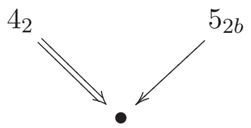
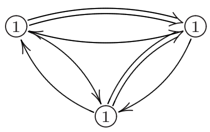
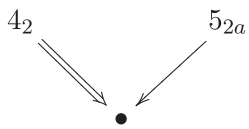
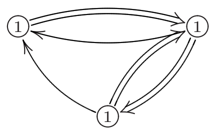
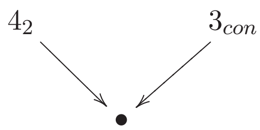
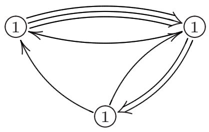
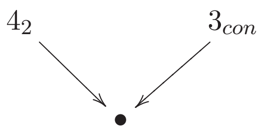
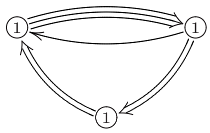
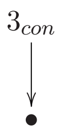
6_{2a} (12) : 

6_{2b} (10) : 

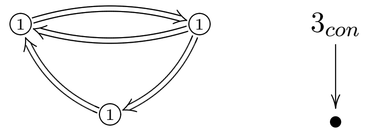
where the number between brackets gives the number of primitive cycles. The cases $2 < k \leq 5$ can be classified either by ad-hoc methods as in the previous section or by using the `vint` procedure of `PORTA` [6] which is an efficient method to find all integral points satisfying a set of (in)equalities. Here, the inequalities are given by the conditions that the number of incoming (outgoing) arrows is at least two and the equality states that the total number of arrows is $5+k$. Taking quiver-isomorphism classes of the obtained list of integral solutions then gives the lists below.

In these lists we indicate the type of singularity, the number of primitive cycles and the fingerprint. Some of these quiver settings give a non-isomorphic quiver setting when we reverse *all* arrows. As this operation has no effect on the ring of invariants we did not list the reversed cases.

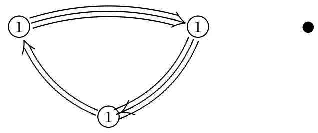
The reduced quiver settings for $d=6$ on three vertices.

6_{3a} (9)		
6_{3b} (10)		
6_{3c} (10)		
6_{3d} (9)		
6_{3e} (10)		
6_{3f} (11)		
6_{3g} (15)		

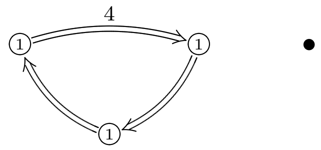
6_{3h} (12)



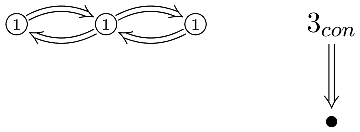
6_{3i} (18)



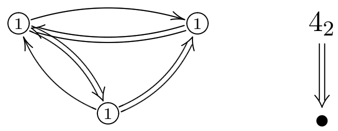
6_{3j} (16)



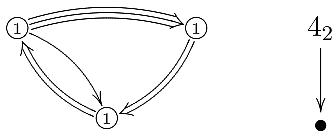
6_{3k} (8)



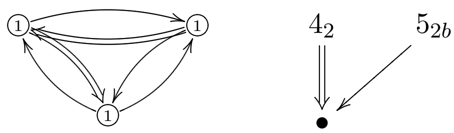
6_{3l} (12)



6_{3m} (14)

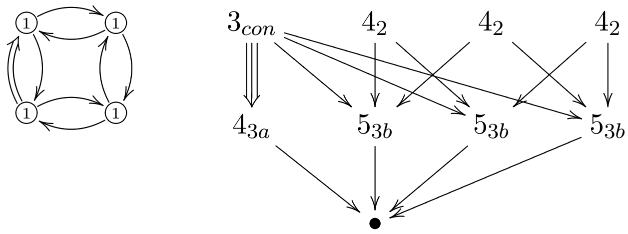


6_{3n} (10)

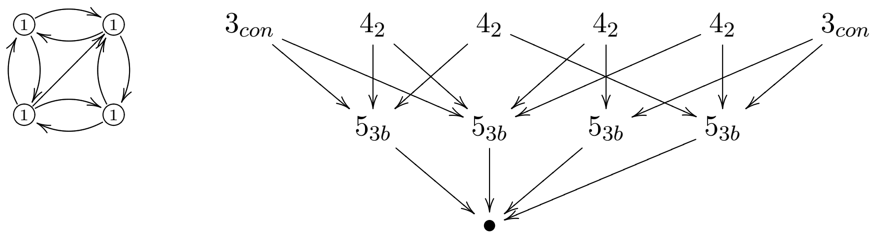


The reduced quiver settings for $d = 6$ on four vertices.

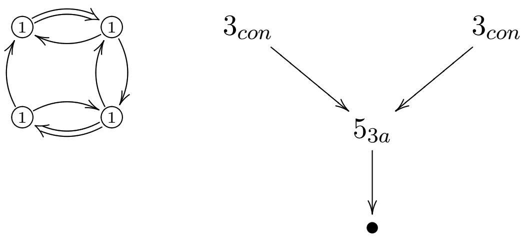
6_{4a} (8)



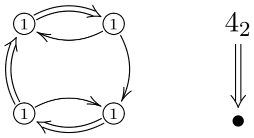
6_{4b} (8)



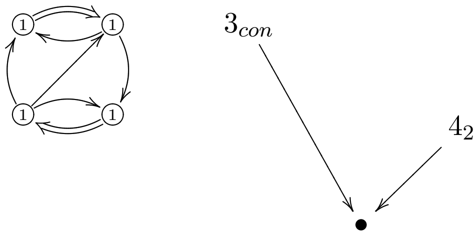
6_{4c} (9)



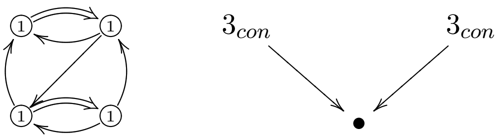
6_{4d} (12)



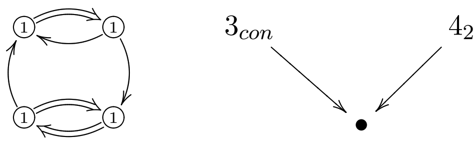
6_{4e} (10)



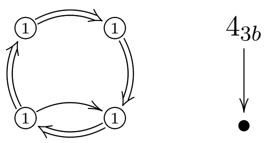
6_{4f} (8)



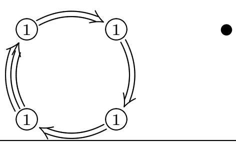
6_{4g} (10)



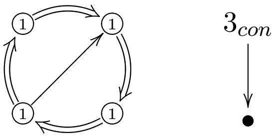
6_{4h} (18)



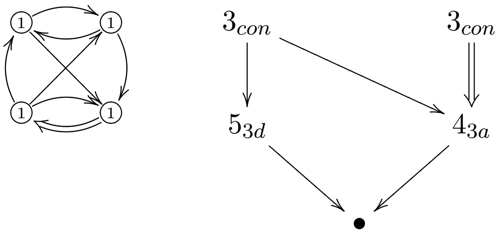
6_{4i} (24)



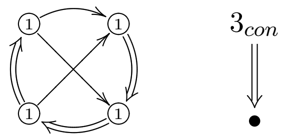
6_{4j} (20)



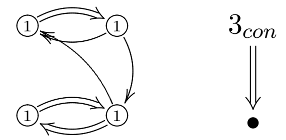
6_{4k} (9)



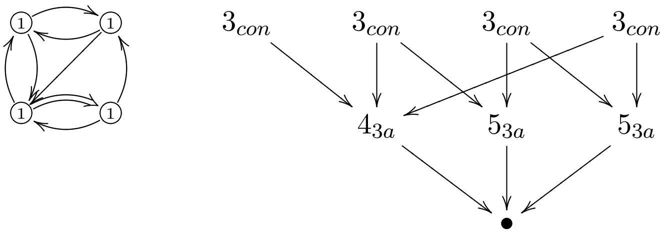
6_{4l} (16)



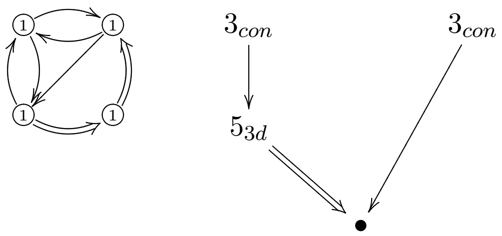
6_{4m} (8)



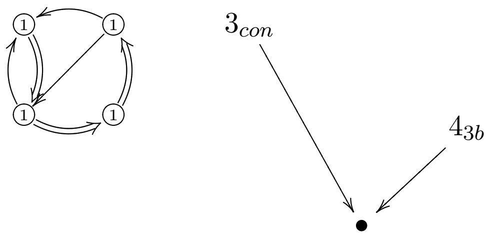
6_{4n} (9)



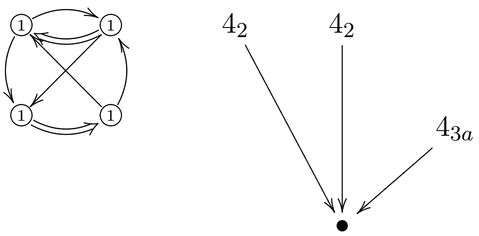
6_{4o} (11)



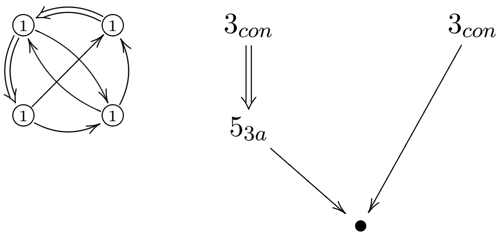
6_{4p} (14)



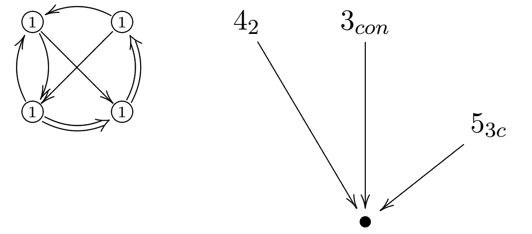
6_{4q} (8)



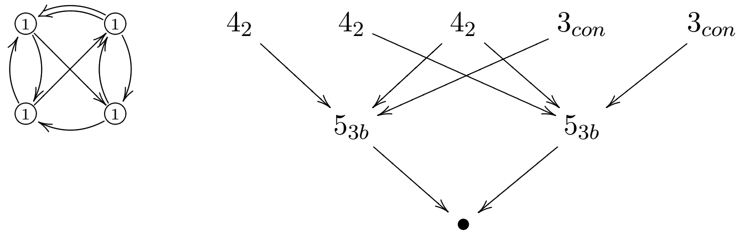
6_{4r} (13)



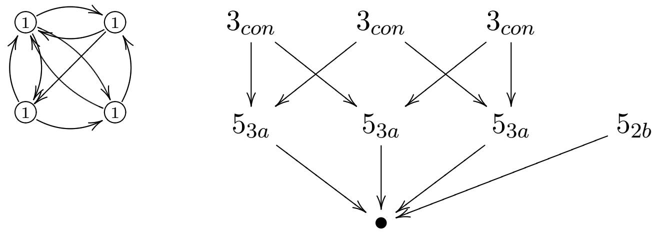
6_{4s} (11)



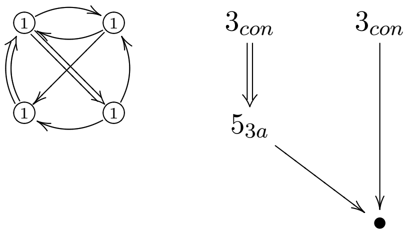
6_{4t} (8)



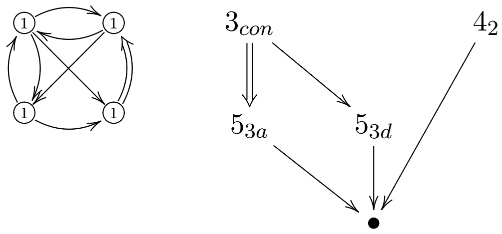
6_{4u} (8)



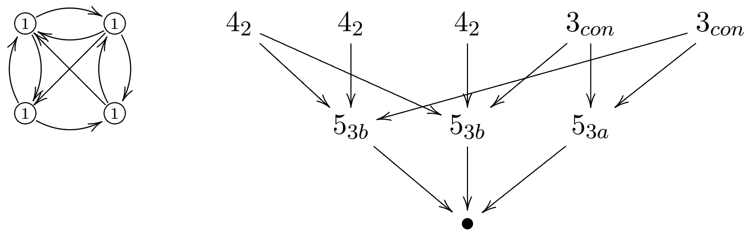
6_{4v} (9)



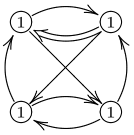
6_{4w} (9)

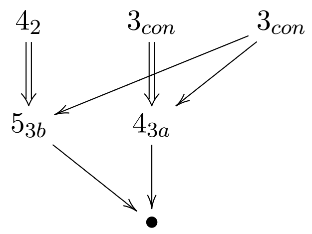


6_{4x} (8)

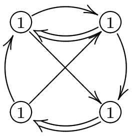


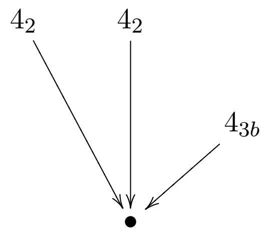
6_{4y} (8)





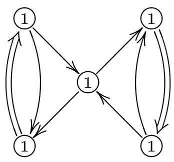
6_{4z} (8)

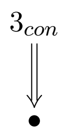




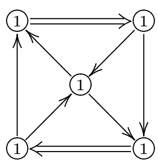
The reduced quiver settings for $d = 6$ on five vertices.

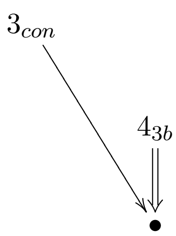
6_{5a} (8)



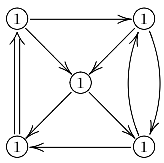


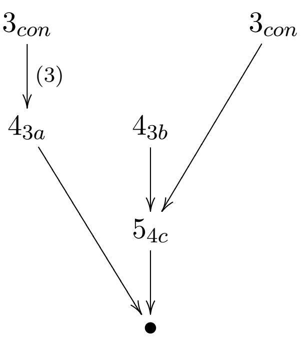
6_{5b} (16)



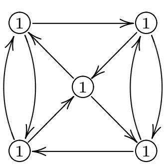


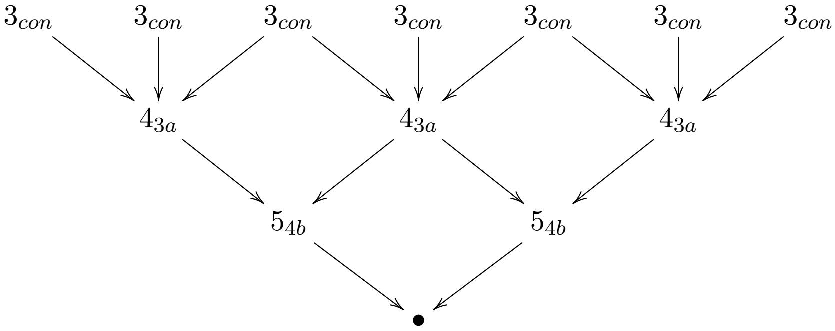
6_{5c} (12)



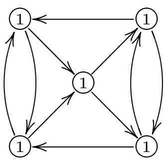


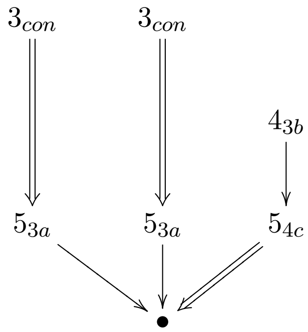
6_{5d} (9)



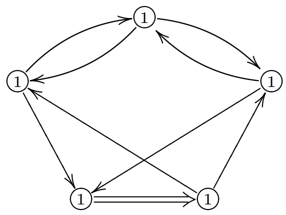


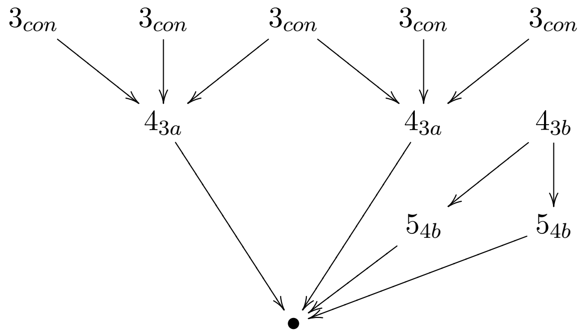
6_{5e} (10)



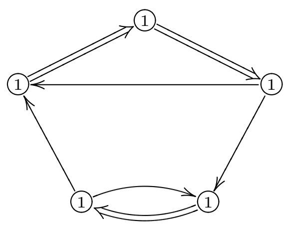


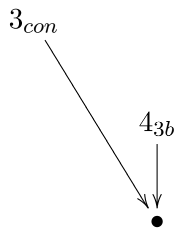
6_{5f} (10)



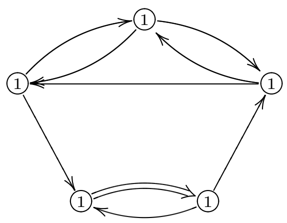


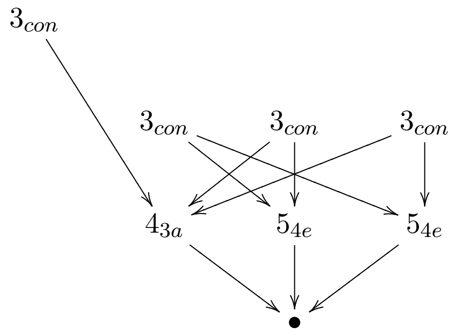
6_{5g} (14)



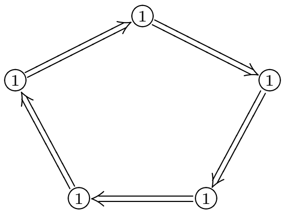


6_{5h} (9)

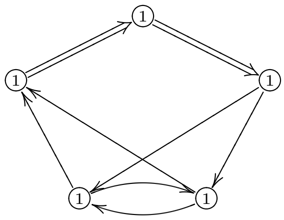


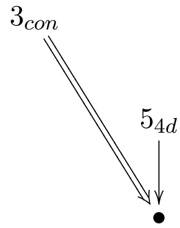


6_{5i} (32)

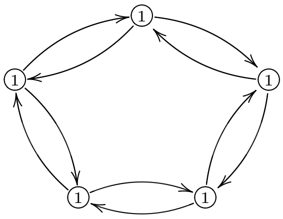


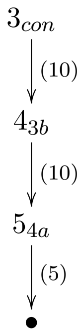
6_{5j} (13)



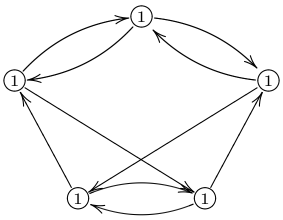


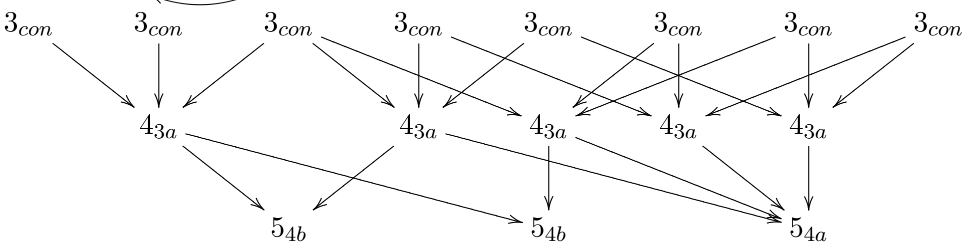
6_{5k} (7)



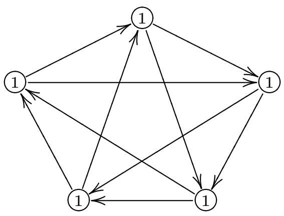


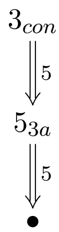
6_{5l} (7)





6_{5m} (7)





Theorem 8 *There are exactly* 53 *nonisomorphic reduced marked quiver singularities in dimension* $d = 6$.

Proof. Using the above lists, combined with the fingerprints of section 3 (and teh fact that these algebras have seven generators) we fail to separate the following sets of marked quiver settings by their number of primitive cycles (the minimal number of generators) and their fingerprints

$$\{6_{3k}, 6_{4f}, 6_{4m}\} \quad \{6_{3e}, 6_{4e}, 6_{4g}\} \quad \{6_{3l}, 6_{4d} \quad \{6_{4q}, 6_{4z}\}\}$$

The first couple is easily seen to be isomorphic comparing cycles, the second and third pair are isomorphic because they are extensions of the isomorphism in dimension 5 and the fourth pair is isomorphic because the settings are obtained from interchanging two vertices. Counting the remaining cases yields the result.

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