THE LOCAL STRUCTURE OF GRADED REPRESENTATIONS

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ABSTRACT. In this article we show that the local structure of the projective representation space of a graded algebra can locally be described by quivers with an automorphism of their path algebra, a twisted weight. We describe the quotient spaces of these twisted weighted quiver settings and determine which of them are smooth.

1. INTRODUCTION

Given a finitely generated algebra A over $\mathbb{C}$ one can look at the set of n-dimensional representations

$$\mathsf{rep}_n A := \{\rho : A \to \mathtt{Mat}_{n\times n}(\mathbb{C}) | \rho \text{ is an algebra morphism}\}.$$

This has the structure of an algebraic variety and it has an additional GL_n-action on it by conjugation. One can take the algebraic quotient of this action to obtain a new variety $\mathsf{iss}_n A$ that classifies all closed GL_n-orbits or equivalently all isomorphism classes of semisimple representations.

If A is a formally smooth algebra, i.e. the kernel of the multiplication map $A \otimes A \to A$ is a projective bimodule, then $\mathsf{rep}_n A$ is a smooth variety and one can describe the étale local structure of the quotient map by means of quivers [6], [5].

A quiver Q is a directed graph consisting of a set of vertices Q_0 and a set of arrows Q_1. The maps $h, t : Q_1 \to Q_0$ will denote the heads and the tails of the arrows. The path algebra of a quiver is the vector space spanned by all paths in Q (including the vertices considered as paths of length 0) equipped with concatenation as multiplication.

A dimension vector α is a map from Q_0 to $\mathbb{N}$ and a couple (Q, α) of a quiver and a dimension vector is called a quiver setting. The space of all α-dimensional representations is denoted by $\mathsf{rep}_\alpha Q$.

$$\mathsf{rep}_\alpha Q := \bigoplus_{a\in A} \mathtt{Mat}_{\alpha_{h(a)}\times\alpha_{t(a)}}(\mathbb{C})$$

To the dimension vector α we can also assign a reductive group

$$\mathsf{GL}_\alpha := \prod_{v\in Q_0} \mathsf{GL}_{\alpha_v}(\mathbb{C}).$$

An element of this group, $g = (g_v)_{v\in Q_0}$, has a natural action on $\mathsf{rep}_\alpha Q$:

$$W := (W_a)_{a\in A},\ W^g := (g_{t(a)} W_a g^{-1}_{s(a)})_{a\in A},$$

and the quotient of this action is denoted by $\mathsf{iss}_\alpha Q$.

Theorem 1 (Le Bruyn). *If $\mathsf{rep}_n A$ is smooth in p, where p corresponds to a semisimple representation $S_1^{\oplus e_1} \oplus \cdots \oplus S_k^{\oplus e_k}$ then there is a quiver setting (Q_p, α_p) such that there are étale neighborhoods making the following diagram commutative.*

$$\begin{array}{ccc} \mathsf{GL}_n \times_{\mathsf{GL}_\alpha} \mathsf{rep}_\alpha Q & \longrightarrow & \mathsf{rep}_n A \\ \downarrow & & \downarrow \\ \mathsf{iss}_\alpha Q & \longrightarrow & \mathsf{iss}_n A \end{array}$$

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This quiver setting is called the local quiver setting. Its vertices $s_1, \dots, s_k$ correspond to the simples $S_1, \dots, S_k$ and the number of arrows between s_i and s_j is

$$\dim \mathsf{Ext}^1(S_i, S_j).$$

Quiver settings provide a powerful combinatorial tool to study the quotient map locally. One can find the smooth points of the quotient [1] and the types of singularities that can occur in $\mathsf{iss}_\alpha Q$ [2].

If p is not a smooth point of the variety $\mathsf{rep}_n A$ the formalism of local quiver settings breaks down. In this paper we explore the possibility to save some of the power of this approach using twisted weighted quiver settings. The ideas presented here are an extension of [4].

Instead of looking at $\mathsf{rep}_n A$ which contains singularities, one can try to find a resolution of $\mathsf{rep}_n A$: a surjective GL_n-equivariant map $V \to \mathsf{rep}_n A$ such that V is a smooth variety and the generic fiber consists of one point. In general this resolution is not an affine variety so it cannot be seen as a representation space of an algebra, but in some cases this resulotion can be obtained by a non-commutative blow-up.

If A has an ideal $\mathfrak{i}$ one can construct the graded algebra

$$\tilde{A} := A \oplus \mathfrak{i} \oplus \mathfrak{i}^2 \oplus \dots.$$

The ring of polynomial functions over the representation space $\mathsf{rep}_n \tilde{A}$ has also a positive grading and one can look at the projective space it describes $\mathsf{grep}_n \tilde{A}$. If this space is smooth one has found a resolution of $\mathsf{rep}_n A$.

The local structure of such a resolution does not always reduce to ordinary quivers, but one needs some extra structure: an automorphism on the path algebra of the quiver. A quiver setting together with such an automorphism is called a twisted weighted quiver setting.

The main part of this paper will investigate how one can reduce the problem sketched above to these twisted weighted quiver settings and study the finite group action, induced by the automorphism on the quotient space $\mathsf{iss}_\alpha Q$.

The paper is structured as follows. In section 2 we introduce the graded representation space $\mathsf{grep}_n A$ of a graded algebra A. In section 3 we calculate the stabilizer of a semisimple representation, given its decomposition (theorem 2). In section 4 we define the notion of a twisted weighted quiver setting and the corresponding finite group action on its representation space. This allows us to formulate the étale local structure in terms of representation spaces of twisted weighted quiver settings.

The next three sections focus on the geometry and combinatorics of twisted weighted quiver settings alone. First we show that the local structure of twisted weighted quiver settings can again be seen as twisted weighted quiver settings. These new local twisted quiver settings can be calculated using adapted versions of the dimension vectors and the Euler form. Secondly we modify the reduction steps defined in [1] to the twisted weighted case. These reduction steps enable us to simplify the settings without changing the quotient. Finally we give a classification of all twisted weighted quiver settings whose quotient space is smooth.

In the last two sections we show that the theory developed above can be easily transferred to the theory of Cayley-Hamilton [10] algebras. We illustrate this with an example: the quantum plane at $q = -1$.

2. GRADED ALGEBRAS AND REPRESENTATION SPACES

Consider A a positively graded algebra

$$A = \mathbb{C}\langle x_1, \dots, x_m \rangle / (I) = A_0 \oplus A_1 \oplus A_2 \oplus \dots,$$

where I is an ideal of homogeneous polynomials in the variables x_i, with the degree of x_i being equal to d_i.

The coordinate ring of the representation space, $\mathbb{C}[\mathsf{rep}_n\ A]$, looks like

$$\mathbb{C}[\mathsf{rep}_n\ A] = \mathbb{C}[X_i^{kl}; i = 1, \ldots, m; k, l = 1, \ldots, n]/(I')$$

which can also be seen as a graded algebra. Indeed, each X_i^{kl} will have degree d_i and the ideal I' is spanned by homogeneous polynomials which originate from the homogeneous polynomials in the ideal I of A. Therefore, $\mathbb{C}[\mathsf{rep}_n\ A]$ has a $\mathbb{C}^*$-action working on it defined by

$$t \cdot f = t^{d_f} f,$$

where $t \in \mathbb{C}^*$ and $f \in \mathbb{C}[\mathsf{rep}_n\ A]$ has degree d_f. This action induces a $\mathbb{C}^*$-action on the maximal ideals and thus on $\mathsf{rep}_n\ A$.

As $\mathbb{C}[\mathsf{rep}_n A]$ is graded, we can define $\mathtt{proj}\ \mathbb{C}[\mathsf{rep}_n\ A]$ and will denote this by $\mathsf{grep}_n A$.

Definition 1. *A representation $\rho : A \to M_n(\mathbb{C})$ is called* semistable *if there exists an $f \in \mathbb{C}[\mathsf{rep}_n\ A]$ such that*

(1) *f is homogeneous and of degree > 0 (i.e., a semi-invariant for the $\mathbb{C}^*$-action).*
(2) *f is GL_n-invariant.*
(3) *f is non-vanishing on ρ.*

We write $\mathsf{rep}_n^{ss}\ A$ for the set of all semistable representations of A.

Remark 1. *If $\rho : A \to M_n(\mathbb{C})$ sends all generators to nilpotent matrices, then ρ is not a semistable representation. The converse is not true in general, but does hold if all generators have degree bigger than zero.*

With the definition of $\mathsf{rep}_n^{ss}\ A$, we can write $\mathsf{grep}_n\ A$ as an algebraic quotient:

$$\mathsf{grep}_n\ A = \mathtt{proj}\ \mathbb{C}[\mathsf{rep}_n\ A] = \mathsf{rep}_n^{ss}\ A /\!\!/ \mathbb{C}^*.$$

The action of GL_n on $\mathsf{rep}_n\ A$ induces an action of GL_n on $\mathsf{grep}_n\ A$ and we denote the algebraic quotient $\mathsf{giss}_n\ A = \mathsf{grep}_n\ A /\!\!/ \mathsf{GL}_n$. This summarizes in the diagram

$$\begin{array}{ccc} \mathsf{rep}_n^{ss}\ A & \xrightarrow{/\!\!/\mathbb{C}^*} & \mathsf{grep}_n\ A \\ \downarrow{\scriptstyle /\!\!/\mathsf{GL}_n} & & \downarrow{\scriptstyle /\!\!/\mathsf{GL}_n} \\ \mathsf{iss}_n^{ss}\ A & \xrightarrow{/\!\!/\mathbb{C}^*} & \mathsf{giss}_n\ A \end{array}$$

which is commutative since the action of $\mathbb{C}^*$ commutes with the action of GL_n.

On the other hand, we can look at the diagram

$$\begin{array}{ccc} \mathsf{rep}_n^{ss}\ A & \hookrightarrow & \mathsf{rep}_n\ A \\ \downarrow{\scriptstyle /\!\!/\mathsf{GL}_n} & & \downarrow{\scriptstyle /\!\!/\mathsf{GL}_n} \\ \mathsf{iss}_n^{ss}\ A & \hookrightarrow & \mathsf{iss}_n\ A \end{array}$$

and check whether the inclusion of $\mathsf{iss}_n^{ss}\ A$ in $\mathsf{iss}_n\ A$ is well defined. We find:

Lemma 1. *A GL_n-orbit in $\mathsf{rep}_n^{ss}\ A$ is closed if and only if the corresponding GL_n-orbit in $\mathsf{rep}_n A$ is closed.*

Proof. This follows directly from the fact that semistability condition is GL_n-invariant. □

Lemma 2. *With a $\xi \in \mathsf{giss}_n\ A$ corresponds a semisimple $M_\xi \in \mathsf{rep}_n^{ss}\ A$, unique up to the action of $\mathsf{GL}_n \times \mathbb{C}^*$*

Proof. If the orbit M^{GL_n} is closed, then also $(t\cdot M)^{\mathsf{GL}_n}$ is a closed orbit. The only thing to verify is that for every 1 parameter subgroup $(g(t))_{t\in\mathbb{C}^*}$,

$$\rho := \lim_{t\to 0}\,(g(t),t^n)\cdot M$$

is not a semistable representation. Suppose ρ is semistable, then there exists an $f\in\mathbb{C}[\mathsf{rep}_n A]$ homogeneous of nonzero degree with $f(\rho(x_1),\dots,\rho(x_m))\neq 0$. This means that among the x_i, at least one has $\rho(x_i)\neq 0$ and $d_i>0$, but this contradicts the definition of ρ. $\square$

Definition 2. *A semistable representation M_1 is called* graded isomorphic *to a representation M_2 if there exists a ζ such that*

$$M_2\cong\zeta\cdot M_1$$

This definition and lemma 2 gives:

Proposition 1. $\mathsf{giss}_n A$ *classifies the (semistable) semisimple n-dimensional representations up to graded isomorphism.*

3. STABILIZERS OF SEMISIMPLE POINTS

We have a nice characterisation for $M\in\mathsf{rep}_n\,A$ to be semistable.

Notation 1. *Suppose M is a representation of A. We write*

$$\mathsf{Aut}_{\mathbb{C}^*}(M)=\{\zeta\in\mathbb{C}^*\mid\zeta\cdot M\cong M\},$$

where the action of ζ on M is given by $\rho_{\zeta M}(x_i)=\zeta^{d_i}\rho_M(x_i)$.

Lemma 3. *Let M be a semisimple representation of A. Then M is semistable if and only if $\mathsf{Aut}_{\mathbb{C}^*}(M)$ is finite.*

Proof. $\boxed{\Rightarrow}$ Let M be a semistable representation of A. This means there exists an f homogeneous and GL_n-invariant, such that $f\neq 0$ on M. Suppose $\zeta\in\mathsf{Aut}_{\mathbb{C}^*}(M)$, then $\zeta\cdot M\cong M$ and as f is GL_n-invariant, we have $\zeta^{d_f}f=f$. Since f is nonzero on M, this yields ζ to be a d_f-th root of unity. Therefore $\mathsf{Aut}_{\mathbb{C}^*}(M)$ is finite.

$\boxed{\Leftarrow}$ Let M be a semisimple representation that is not semistable. $\mathbb{C}[\mathsf{iss}_n A]$ is a finitely generated algebra and we denote $(f_i)_{i=1}^k$ the (homogeneous) generators (with d_i the degree of f_i). Then, $\zeta\cdot M\cong M$ if and only if $f_i(M)=f_i(\zeta\cdot M)$ for all $i\in\{1,\dots,k\}$. $f_i(\zeta\cdot M)$ can be written as $\zeta^{d_i}f_i(M)$ and as M is not semistable, for both the cases $d_k>0$ and $d_k=0$, we have no restriction on the ζ. $\square$

As we are interested in the étale local structure of $\mathsf{giss}_n\,A$, we want to apply the Luna slice theorem [9] to the quotient map

$$\mathsf{rep}_n^{ss}\,A \twoheadrightarrow \mathsf{giss}_n\,A\ ,$$

and therefore we will suppose $\mathsf{rep}_n^{ss}\,A$ to be a smooth variety.

A first problem to encounter is the structure of the stabilizer under the action of $\mathsf{GL}_n\times\mathbb{C}^*$ of such a semisimple representation. First, we look at the simple situation:

Lemma 4. *Suppose $S\in\mathsf{rep}_n^{ss}A$ is a simple representation of A. The $\mathsf{GL}_n\times\mathbb{C}^*$-stabilizer of S is isomorphic to the group $\mathbb{C}^*\times\mu_e$ where the cyclic group μ_e has generator*

$$(g_\zeta,\zeta)\in\mathsf{GL}_n\times\mathbb{C}^*$$

where ζ is a primitive root of unity and

$$g_\zeta=\mathtt{diag}(\underbrace{1,\dots,1}_{m_0},\underbrace{\zeta,\dots,\zeta}_{m_1},\dots,\underbrace{\zeta^{e-1},\dots,\zeta^{e-1}}_{m_{e-1}}). \tag{1}$$

Proof. Write $\mathsf{Aut}_{\mathbb{C}^*}(S)$ as $<\zeta>$ with $\zeta = e^{\frac{2\pi i}{e}}$ for a certain e (with e a divisor of $n.gcd\,(\deg(x_i))$). As $\zeta \cdot S \cong S$, there exists a $g \in \mathsf{GL}_n$, unique up to $\mathbb{C}^*$, such that $S = \zeta \cdot gSg^{-1}$. Since $S = g^e S g^{-e}$ and S is simple, we know by Schur's lemma that g^e is a scalar and we can choose $g^e = \mathbb{1}_n$. But then, g can be seen as a group representation of $\mathbb{Z}_e$, where all simples are 1-dimensional and therefore g is isomorphic to a diagonal matrix g_ζ. As $(g_\zeta)^e = \mathbb{1}_n$, we have that all elements on the diagonal are e-th roots of unity, which can be permuted in the order we want. □

The proof indicates that the $(m_0, \dots, m_{e-1})$ are determined up to cyclic permutation.

Definition 3. *Suppose S has a stabilizer as in lemma 4, we say that S has* period e *and* matrix type $(m_0, \dots, m_{e-1})$.

Let S be a semisimple (semistable) representation with $\mathsf{Aut}_{\mathbb{C}^*}(S) = \mu_e =<\zeta>$, then for each simple component S_i, one can compute

$$k_i = \min\left\{k \in \mathbb{Z}^+ \setminus \{0\} \mid \zeta^k \cdot S_i \cong S_i\right\}.$$

In order to have $S \cong \zeta S$, the components $S_i, \zeta S_i, \zeta^2 S_i, \dots, \zeta^{k_i-1} S_i$ have to occur an equal number of times in the semisimple decomposition of S. Hence, S can be written as

$$S = \bigoplus_{i=1}^{p} \left(S_i^{\oplus\beta_i} \oplus (\zeta S_i)^{\oplus\beta_i} \oplus \dots \oplus (\zeta^{k_i-1} S_i)^{\oplus\beta_i}\right)$$

The isomorphism $\zeta^{k_i} \cdot S_i \cong S_i$ can be expressed with the terminology of definition 3. Suppose S_i has period e_i and matrix type $(m_0^i, \dots, m_{e_i-1}^i)$, with the generator of μ_{e_i} denoted by ζ_i. Because $\zeta^{k_i} = \zeta_i^{l_i}$ with $l_i = e_i k_i / e$, this yields

$$S_i \cong \zeta^{k_i} g_{S_i}^{l_i} S_i (g_{S_i}^{l_i})^{-1}$$

with $g_{S_i} = g_{\zeta_i}$ the diagonal matrix (1) of lemma 4. We will use the notation

$$g_i := \mathtt{diag}(\underbrace{1, \dots, 1}_{m_0^i}, \underbrace{\zeta, \dots, \zeta}_{m_1^i}, \dots, \underbrace{\zeta^{e_i-1}, \dots, \zeta^{e_i-1}}_{m_{e_i-1}^i})$$

For the matrix g_i then holds that $g_i^{k_i} = g_{S_i}^{l_i}$.

We are now ready to determine the stabilizer of S.

Theorem 2. *Let S be an n-dimensional semisimple representation with* $\mathsf{Aut}_{\mathbb{C}^*}(S) =<\zeta>$ *and decomposition*

$$S = \bigoplus_{i=1}^{p} S_i^{\oplus\beta_i} \oplus (\zeta S_i)^{\oplus\beta_i} \oplus \dots \oplus (\zeta^{k_i-1} S_i)^{\oplus\beta_i}$$

with k_i, g_i and γ_i as defined before, then the ζ-component of the stabilizer is isomorphic to

$$\left(\mathsf{GL}_\alpha\cdot\begin{bmatrix}\ddots&&\\&k_i\gamma_i\beta_i\left\{\overbrace{\begin{bmatrix}0&g_i\otimes\mathbb{1}_{\beta_i}&0&\cdots&0\\0&0&g_i\otimes\mathbb{1}_{\beta_i}&\cdots&0\\\vdots&&&\ddots&\vdots\\0&0&\cdots&0&g_i\otimes\mathbb{1}_{\beta_i}\\g_i\otimes\mathbb{1}_{\beta_i}&0&\cdots&0&0\end{bmatrix}}^{k_i\gamma_i\beta_i}\right.&\\&&\ddots\end{bmatrix},\quad\zeta\right)$$

with $\alpha=(\overbrace{\beta_1,\dots,\beta_1}^{k_1},\beta_2,\dots,\beta_{p-1},\overbrace{\beta_p,\dots,\beta_p}^{k_p})$ and the embedding of GL_α in GL_n is given by the block diagonal matrix

$$\begin{bmatrix}\mathsf{GL}_{\beta_1}(\mathbb{C}\otimes\mathbb{1}_{\gamma_1})&&&&&&\\&\ddots&&&&&\\&&\mathsf{GL}_{\beta_1}(\mathbb{C}\otimes\mathbb{1}_{\gamma_1})&&&&\\&&&\ddots&&&\\&&&&\mathsf{GL}_{\beta_p}(\mathbb{C}\otimes\mathbb{1}_{\gamma_p})&&\\&&&&&\ddots&\\&&&&&&\mathsf{GL}_{\beta_p}(\mathbb{C}\otimes\mathbb{1}_{\gamma_p})\end{bmatrix}$$

Proof. The theorem follows directly from the right choice of basis for S_i and ζS_i. Choose the base change between S_i and ζS_i be exactly determined by g_i. □

We are now one step away from the étale local description of S: the introduction of *twisted weighed quiver settings*.

4. Twisted Weighted Quivers and the étale local structure of $\mathsf{giss}_n A$

In the introduction we sketched how one can use representations of quivers to describe the local structure of $\mathsf{iss}_n A$ (theorem 1). In this section we will prove an analogous result for the description of the local structure of $\mathsf{giss}_n A$, where A is a graded algebra. Before we can do this we introduce the useful concept of *twists* and *weights*.

Consider the path algebra $\mathbb{C}Q$ of a quiver Q. We now provide every path in $\mathbb{C}Q$ with degree the length of the path. This makes $\mathbb{C}Q$ in a graded algebra. We distinguish 2 important types of graded automorphisms (i.e., algebra morphisms that send homogeneous elements to homogeneous elements of the same degree) of $\mathbb{C}Q$:

(1) A *twist* ϕ is a couple of invertible maps $\phi_0 : Q_0 \to Q_0$ and $\phi_1 : Q_1 \to Q_1$ such that
- $\forall a \in Q_1 : \quad \phi_0(h(a)) = h(\phi_1(a))$ and $\phi_0(t(a)) = t(\phi_1(a))$
- $\forall k \in \mathbb{N} : \quad \phi_1^k(a) = a \quad \Leftrightarrow \quad h(\phi_1^k(a)) = h(a)$ and $t(\phi_1^k(a)) = t(a)$.

Every twist induces an automorphism of $\mathbb{C}Q$, which we also call a *twist*.

(2) A *weight* is a map $Q_1 \to \mathbb{C}^*$ and this gives an automorphism of $\mathbb{C}Q$ by mapping every vertex on itself and every arrow a on $w(a)a$. This automorphism will also be called a weight.

We can see twists and weights as basic building blocks of graded automorphisms of $\mathbb{C}Q$ of finite order:

Theorem 3. *Every graded automorphism of $\mathbb{C}Q$ of finite order can be conjugated to the product of a weight and a twist. Moreover, this weight and twist commute.*

Proof. Suppose Q has p vertices and suppose ψ a graded automorphism of $\mathbb{C}Q$ of finite order. Denote by V_{ij} the vector space spanned by all paths from i to j, that is,

$$V_{ij} = (e_i \mathbb{C}Q e_j)_1.$$

The action of ψ then sends a V_{ij} to a V_{kl} and we can find a partition of $S := \{V_{ij} \,|\, i, j \in \{1, \dots p\}\}$ corresponding to the orbits of this action. Choose a set of representatives for the orbits of this action. For each of these representatives $(e_i \mathbb{C}Q e_j)_1$, we can find the smallest k such that $\psi^k(e_i \mathbb{C}Q e_j)_1 = (e_i \mathbb{C}Q e_j)_1$. Choose now a basis of V_{ij} that diagonalizes this action of ψ^k on V_{ij} to a matrix A_{ij} and let B_{ij} be a diagonal matrix such that $B_{ij}^k = A_{ij}$ (remark that B_{ij} has roots of unity on its diagonal). Every other space $V_{kl} \in S$ can be seen as $\psi^m V_{ij}$ for some representative. Choose a basis in V_{kl} such that the map ψ^l on V_{ij} is given by the matrix B_{ij}^l. But then we have chosen a basis for $(\mathbb{C}Q)_1$ for which ψ maps every basis element a to a multiple w_a of another basis element $\phi(a)$ (with w_a one of the elements on the diagonal of the corresponding B_{ij}):

$$\psi(a) = w_a \cdot \phi(a).$$

So up to conjugation, ψ can be seen as the product of the weight $w: \; a \mapsto w_a a$ and the twist ϕ (ϕ is a twist since the A_{ij} are diagonalized). This construction shows that $w(a)$ and $w(\phi(a))$ coincide, and therefore ϕ and w commute. □

Remark 2. *From the construction of the proof we can conclude that $w(a)$ is an e-th root of unity if the order of the automorphism is given by e.*

Remark 3. *Another consequence of the construction of the proof is that all arrows lying in the same orbit under ϕ have the same weight. However these weights are not uniquely determined, but depend on the choice of B_{ij}.*

From now on we suppose that the automorphism of $\mathbb{C}Q$ is in a twisted weighted form $\psi = w\phi$ and let e be the order of $w\phi$.

Definition 4. *With the previous notation, we call $\psi = w\phi$ a* twisted weight.

w and ϕ define an action of μ_e on $\mathbb{C}Q$ and we want to look at this action as an action on $\mathsf{rep}_\alpha Q$ for a quiver setting (Q, α). (Q, α) is said to be *compatible with this action* if $\alpha_{\phi_0(v)} = \alpha_v$ for all vertices v. This implies that $\mathsf{rep}_\alpha Q$ also has a corresponding action of μ_e on it. The action of μ_e does not commute with GL_α as ϕ can interchange vertices. We do have that $\mathsf{GL}_\alpha^\psi = \mathsf{GL}_\alpha$, so to describe the total action we must take the semidirect product $\mathsf{GL}_\alpha \rtimes_\psi \mu_e$, where the multiplication is defined by

$$(g_1, m_1).(g_2, m_2) = (g_1 w^{m_2} \phi^{m_2}(g_2), m_1 m_2)$$

with m_1 and m_2 integer representatives of μ_e.

We will call the linear space $\mathsf{rep}_\alpha Q$ with this $\mathsf{GL}_\alpha \rtimes_\phi \mu_e$-action a *twisted weighted quiver representation space*. Because $\mathsf{GL}_\alpha \rtimes \mu_e$ is a semidirect product of two reductive groups we can take the quotient by first taking the quotient of the GL_α-action and then the quotient by the group μ_e. We denote this quotient by $\mathsf{giss}_\alpha^\psi Q$:

$$\mathbb{C}[\mathsf{giss}_\alpha^\psi\, Q] = \mathbb{C}[\mathsf{iss}_\alpha\, Q]^{\mu_e}.$$

The precise action of $(g,i) \in \mathsf{GL}_\alpha \rtimes_\psi \mu_e$ on the representation $(\rho_a)_{a\in Q_1} \in \mathsf{rep}_\alpha\, Q$ is given by

$$(g,i)\cdot(\rho_a)_{a\in Q_1} = \left((\zeta^{w_{\phi^i(a)}})^i\, g_{\phi^i(t(a))}\rho_{\phi^i(a)}g^{-1}_{\phi^i(h(a))}\right)_{a\in Q_1}.$$

and by the previous we can write

$$\mathsf{giss}^{w\phi}_\alpha Q = \mathsf{rep}_\alpha Q /\!\!/ (\mathsf{GL}_\alpha \rtimes_{w\phi} \mu_k).$$

We can use these twisted weighted quiver settings to describe the local structure of $\mathsf{giss}_n\, A$:

Theorem 4. *For a smooth semisimple point $M \in \mathsf{rep}^{ss}_n\, A$, we denote the normal space $T_M\mathsf{rep}_n A/T_M((\mathsf{GL}_n \times \mathbb{C}^*)\cdot M)$ by N_M. There exists a twisted weighted quiver setting (Q,α,ψ) such that*

$$\mathsf{Stab}_M \cong \mathsf{GL}_\alpha \rtimes_\psi \mu_e \text{ and } N_M \oplus \mathbb{C} \cong \mathsf{rep}_\alpha\, Q.$$

Proof. From section 3 we know that a semisimple representation always can be written as

$$M = \bigoplus_{i=1}^{p} S_i^{\oplus\beta_i} \oplus (\zeta S_i)^{\oplus\beta_i} \oplus \ldots \oplus (\zeta^{k_i-1}S_i)^{\oplus\beta_i}$$

and theorem 2 gives us the exact behaviour of the stabilizer, which is isomorphic to $\mathsf{GL}_\alpha \rtimes_\psi \mu_e$. The normal space N with respect to the GL_n-orbit of M is given by $\mathsf{Ext}(M,M)$ and can be identified by $\mathsf{rep}_\alpha\, Q$ for a certain quiver Q and dimension vector α (see [5]). As M is semistable, we know that the extra action of $\mathbb{C}^*$ is never trivial and that

$$T_M(\mathsf{GL}_n\cdot M)\oplus\mathbb{C} \cong T_M((\mathsf{GL}_n \rtimes \mu_e)\cdot M)$$

and this yields $N \cong \mathbb{C}\oplus N_M$.

Q is constructed in such a way that $\mathbb{C}Q_1 = \mathsf{Ext}(\bigoplus_j S_j, \bigoplus_j S_j)$ (where the direct sum varies over *all* simple components) and we get that the path algebra $\mathbb{C}Q$ is given by the tensor algebra

$$\mathbb{C}Q = \mathsf{T}_{\mathbb{C}^k}\left(\bigoplus_{i,j}\mathsf{Ext}(S_i,S_j)\right)$$

with $k = \sum_{i=1}^p k_i$ the number of vertices of Q. The action of the element

$$\left(\bigoplus_{i=1}^{p}\begin{bmatrix} 0 & g_i\otimes \mathbb{1}_{\beta_i} & \cdots & 0\\ 0 & 0 & \ddots & \vdots\\ \vdots & & \ddots & g_i\otimes\mathbb{1}_{\beta_i}\\ g_i\otimes\mathbb{1}_{\beta_i} & 0 & \cdots & 0\end{bmatrix},\zeta\right)$$

then determines a quiver automorphism ψ, as $\psi(\mathsf{Ext}(S_i,S_j)) = \mathsf{Ext}(\psi S_i, \psi S_j)$. □

Definition 5. *We call (Q,α,ψ) from the previous theorem a* twisted weighted local quiver setting *of ξ and will denote it by $(Q_\xi,\alpha_\xi,\psi_\xi)$.*

If we now apply the Luna slice theorem [9], we arrive at

Theorem 5. *Let ξ be an element of $\mathsf{giss}_n\, A$, such that M_ξ is smooth in $\mathsf{rep}^{ss}_n\, A$. Then, for all $t\in\mathbb{C}$, there is a neighborhood of $(\xi,t)\in\mathsf{giss}_n\, A\times\mathbb{C}$, which is étale isomorphic to a neighborhood of $0\in\mathsf{giss}^{\psi_\xi}_{\alpha_\xi}\, Q_\xi$.*

To draw twisted weighted quiver settings we will use the following conventions.

(1) The dimension vector will be written inside the vertices.
(2) The *weight* w associates $w_a\in\mu_e$ to each arrow a and weights will be denoted by their power of ζ (i.e., integers from 0 to $e-1$) and will be depicted in a small square on the arrow they correspond with. Weight 0 will not be depicted.

(3) The action of the *twist* ϕ permutes the vertices and the permutation will be indicated by dotted arrows. The way ϕ permutes the arrows is fully determined by the permutation of the vertices, as arrows in the same ϕ-orbit always have the same weight.

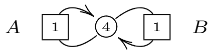
5. LOCAL SETTINGS FOR TWISTED WEIGHTED QUIVER SETTING

Like in the case of quivers, twisted weighted quivers are closed under étale slicing. This means that if we have a twisted weighted setting and we look at the étale local structure of $\mathsf{giss}_\alpha^\psi\, Q$ around a point ξ, then this can described by a neighborhood of the $0 \in \mathsf{giss}_{\alpha_\xi}^{\psi_\xi}\, Q_\xi$ of a new weighted twisted quiver setting.

We illustrate this with an example:

Example

Let us look at the weighted quiver setting



with the action of μ_4

(A and B are 4×4-matrices) and we want to look in a neighborhood of a semisimple representation ρ, with two components $\rho_1 \oplus \zeta\rho_1$

The stabilizer then can be given by the generator

$$\left(\begin{bmatrix} 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & i\lambda \\ \mu & 0 & 0 & 0 \\ 0 & i\mu & 0 & 0 \end{bmatrix}, \zeta\right) \tag{2}$$

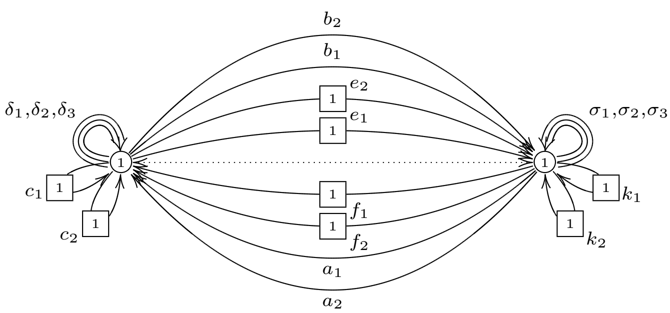
and in that case ρ can be written in the following basis:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

If we calculate the normal space N (based on the orthogonal complement of the tangent space to the orbit in the total tangent space), we can write N in 18 variables in the following way:

$$\begin{bmatrix} c_1 & \delta_1 & f_1 & a_1 \\ \delta_1 - \delta_2 + \delta_3 & c_1 & a_2 & f_2 \\ e_1 & -b_1 & k_1 & -\sigma_1 \\ b_2 & e_2 & \sigma_1 - \sigma_2 + \sigma_3 & k_1 \end{bmatrix}, \begin{bmatrix} c_2 & -\delta_2 & -f_2 & -a_1 \\ \delta_3 & c_2 & a_2 & -f_1 \\ -e_2 & b_1 & k_2 & \sigma_2 \\ b_2 & -e_1 & \sigma_3 & k_2 \end{bmatrix}$$

The action of the stabilizer as in (2) then result in the following twisted weighted local quiver:



with weights in μ_4.

The local twisted weighted quiver setting that occur in a given twisted weighted quiver setting can be calculated using the theory of *isomorphism invariant representations*.

Following [3], we define an isomorphism invariant representation (ii-representation) of a twisted weighted quiver setting as a representation S such that $S^{w\phi} \cong S$. Let σ be an automorphism of S corresponding to that isomorphism:

$$\rho_S(a)\sigma = \sigma\rho_S(w\phi(a)).$$

Note that σ is not uniquely defined but for two choices σ_1, σ_2 we have that $\sigma_1\sigma_2^{-1} \in \mathsf{Aut}_{\mathbb{C}Q}S$. This enables us to choose σ such that it has the same order as $w\phi$.

If k is the order of $w\phi$ then we can use σ to define a $\mathbb{Z}_k$-action on S. This σ makes S into a representation of the smash product

$$\mathbb{C}Q\#\mathbb{Z}_k,$$

and the category of $\mathbb{C}Q\#\mathbb{Z}_k$-representations is equivalent to the category of couples of an ii-representation of Q and a σ.

We will also need a straightforward adaptation of a result from [3] and [11]:

Theorem 6. *The algebra $\mathbb{C}Q\#\mathbb{Z}_k$ is Morita equivalent to the path algebra of a new quiver $\tilde{Q}$. This new quiver is constructed in the following way:*

- *For every ϕ-orbit of vertices $v^\phi \subset Q_0$ there are $k/|v^\phi|$ vertices in $\tilde{Q}_0$ parametrized by the characters of* $\mathsf{Stab}\, v$. *Denote these vertices by $\tilde{v}_\chi,\ \chi \in \mathsf{Stab}\, v^\vee$.*
- *For every ϕ-orbit of arrows $a^\phi \subset Q_1$ there are $|\mathsf{Stab}\, s(a)||\mathsf{Stab}\, t(a)|/|\mathsf{Stab}\, a|$ arrows in $\tilde{Q}_1$ parametrized by couples $(\chi_1,\chi_2) \in \mathsf{Stab}\, h(a)^\vee \times \mathsf{Stab}\, t(a)^\vee$ such that $\chi_1|_{\mathsf{Stab}\, a} = w(a)|_{\mathsf{Stab}\, a}\chi_2|_{\mathsf{Stab}\, a}$.*

If $\tilde{\alpha}$ is the dimension vector of a $\mathbb{C}\tilde{Q}$-representation then the corresponding $\mathbb{C}Q\#\mathbb{Z}_k$ representation is

$$\alpha_v = \sum_{\chi\in\mathsf{Stab}\, v^\vee} \tilde{\alpha}_{v_\chi}$$

Moreover as a $\mathbb{Z}_k$-representation its character is determined by

$$X(\tilde{\alpha}) := \sum_{\tilde{v}_\chi\in\tilde{Q}} \tilde{\alpha}(\tilde{v}_\chi) \sum_{\varphi\in\mathbb{Z}_k{}^\vee,\varphi|_{\mathsf{Stab}\, v}=\chi} \chi.$$

Remark 4. *This new quiver is identical to the one quiver defined by L. Le Bruyn in* [7].

Now suppose that S and T are two simple ii-representations of $(Q, w\phi)$ and let σ and τ be two automorphisms that make $\bar{S} = (S,\sigma)$ and $\bar{T} = (T,\tau)$ into $\mathbb{C}Q\#\mathbb{Z}_k$ representations. We denote the dimension vectors of S and T by α and β and the corresponding $\tilde{Q}$-dimension vectors by $\tilde{\alpha}$ and $\tilde{\beta}$.

We can now define three operations on these dimension vectors. To decrease the notation load we will denote the semisimple subalgebra of $\mathbb{C}Q$ generated by the vertices by $\mathbb{C}I$ instead of $(\mathbb{C}Q)_0$.

1. The *dual dimension vector* $\tilde{\alpha}^\vee$ is the dimension vector of the dual representation $\bar{S}^\vee = \mathrm{Hom}_{\mathbb{C}I}(\bar{S},\mathbb{C}I)$:
$$\tilde{\alpha}^\vee(v_\chi) := \tilde{\alpha}(v_{\chi^{-1}}).$$
2. The *flow dimension vector* $|\tilde{\alpha}\rangle$ shifts all dimensions along the arrows:
$$|\tilde{\alpha}\rangle(v_\chi) = \sum_{a\in\tilde{Q}_1,h(a)=v_\chi} \tilde{\alpha}(s(a)).$$
It is easy to check that this is also the dimension vector of the $\mathbb{C}I\#\mathbb{Z}_k$-representation $\bar{S}\otimes_{\mathbb{C}I}(\mathbb{C}Q)_1$ (where $(\mathbb{C}Q)_1$ is the $\mathbb{C}I\#\mathbb{Z}_k$-bimodule spanned by the arrows in Q.
3. The *convolution product* $\tilde{\alpha}\star\tilde{\beta}$ is the dimension of the tensor product $\bar{S}\otimes_{\mathbb{C}I}\bar{T}$:
$$\tilde{\alpha}\star\tilde{\beta}(v_\chi) := \sum_{\varphi\in\mathsf{Stab}\, v^\vee} \tilde{\alpha}(v_\varphi)\tilde{\beta}(v_{\varphi^{-1}\chi}).$$

Lemma 5. *The character of* $\mathsf{Ext}_{\mathbb{C}Q}(S,T)$ *as a* $\mathbb{Z}_k$*-representation is*

$$1\delta_{ST} - X(\tilde{\alpha}^\vee \star \tilde{\beta} - |\tilde{\alpha}^\vee\rangle \star \beta),$$

where 1 *is the trivial character.*

Proof. We have the following exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{C}Q}(S,T) \to S^\vee \otimes_{\mathbb{C}I} T \to S^\vee \otimes_{\mathbb{C}I} (\mathbb{C}Q)_1 \otimes_{\mathbb{C}I} T \to \mathsf{Ext}_{\mathbb{C}Q}(S,T) \to 0,$$

which also an exact sequence of $\mathbb{Z}_k$-representations. Using the definitions of the operations on the dimension vectors and the fact that $\mathbb{Z}_k$-representations are semisimple one obtains the formula above. □

Note that for $k=1$ one obtains the usual formula involving the Euler form because then X reduces then to taking the length $|\alpha| = \sum_v \alpha_v$ of the vector and $\star$ reduces to the product in $\mathbb{C}I$.

Lemma 6. *If* $\bar{S} = (S,\sigma)$ *is a* $\mathbb{C}Q\#\mathbb{Z}_k$*-representation with setting* $(\tilde{Q},\tilde{\alpha})$ *and* $q \in \mathbb{N}$ *then* (S,σ^q) *is a* $\mathbb{C}Q\#\mathbb{Z}_{\frac{k}{\gcd(k,q)}}$*-representation. The corresponding new setting will be denoted by* $(\tilde{Q}^q,\tilde{\alpha}^q)$*. This new setting can be derived from* $(\tilde{Q},\tilde{\alpha})$ *as follows: Split every vertex* $\tilde{v}_\chi \in \tilde{Q}$ *in* $\ell = \gcd(|v^\phi|,q)$ *vertices* $\tilde{v}_{\chi^q,0},\dots,\tilde{v}_{\chi^q,\ell}$*, then identify vertices labeled with the same character:* $\tilde{v}_{\chi_1^q,i} = \tilde{v}_{\chi_2^q,i}$ *iff* $\chi_1^q = \chi_2^q$*. For the dimension vector* $\tilde{\alpha}^q$ *we obtain the formula*

$$\tilde{\alpha}^q(v_{\chi^q,i}) = \sum_{\varphi \in \mathsf{Stab}\ v, \varphi^q = \chi^q} \tilde{\alpha}(v_\varphi).$$

Proof. Straightforward. □

Now if S is a semisimple ii-representation of $(Q,w\phi)$ we can decompose S as follows

$$(S_1 \oplus w\phi S_1 \oplus \dots \oplus (w\phi)^{k_1-1}S_1)^{\oplus e_1} \oplus \dots \oplus (S_p \oplus w\phi S_p \oplus \dots \oplus (w\phi)^{k_p-1}S_p)^{\oplus e_p}$$

where S_i is a simple ii-representation of the twisted weighted quiver $(Q, w\phi^{k_i})$. Choose corresponding σ_i to make the S_i into $\mathbb{C}Q\#k_i\mathbb{Z}_k$-representations and let $\tilde{\alpha}_i$ be the corresponding dimension vector for the quiver $\tilde{Q}^{k_i}$. Note that $w\phi S_i$ has dimension vector $\phi\tilde{\alpha}_i$, where ϕ acts on the vertices of $\tilde{Q}^{k_i}$ by $v_\chi \mapsto (\phi v)_\chi$.

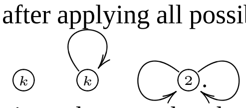
Theorem 7. *The local twisted quiver setting of* S *is a quiver* Q_L *with one vertex* s_i^μ *for each simple* $(w\phi)^\mu S_i$ *in the decomposition of* S*. The local twist twists the vertices according to* $w\phi$*. The number of arrows between the vertices* s_i^μ *and* s_j^ν *correspond to the dimension of* $\mathsf{Ext}_{\mathbb{C}Q}(S_i^\mu, S_j\nu)$*. The* $\mathbb{Z}_k$*-weights of the arrows correspond to lifts of* $\mathrm{lcm}(k_i,k_j)\mathbb{Z}_k$*-weights of the character of* $\mathsf{Ext}_{\mathbb{C}Q}(S_i^\mu, S_j^\nu)$ *viewed as a* $\mathrm{lcm}(k_i,k_j)\mathbb{Z}_k = \mathbb{Z}_{k/\mathrm{lcm}(k_i,k_j)}$*-representation:*

$$\delta_{ij}\delta_{\mu\nu}1 - X(\phi^\mu\tilde{\alpha}_i{}^\vee \star \phi^\nu\tilde{\alpha}_j - |\phi^\mu\tilde{\alpha}_i{}^\vee\rangle \star \phi^\nu\alpha_j)$$

Proof. This is a direct consequence of the previous lemmas. □

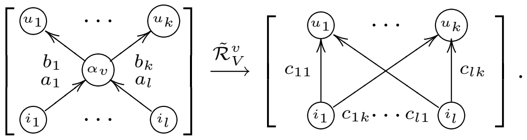
6. Adjusted reduction steps

The first thing we can ask about the quotient space $\mathsf{giss}_\alpha^\psi\, Q$ is for which twisted weighted settings this quotient space is smooth. In the case of ordinary quivers this has been done in [1]. The first author introduces there 3 reduction steps, that reduce the complexity of the quiver while keeping the quotient space invariant. The quotient space of a strongly connected quiver setting (this means that between every two vertices there is a path in both directions) is smooth if and only if after applying all possible reductions, the quiver setting is one of the three below:

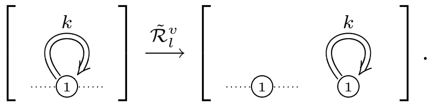


We will adjust these reduction steps in such a way that the quotient space is invariant under the $\mathsf{GL}_\alpha \rtimes \mu_e$-action. As in [1], we will call a quiver that can not be simplified using such moves, is called *reduced*. We will first state the reduction steps and after that we indicate how they alter the weights.

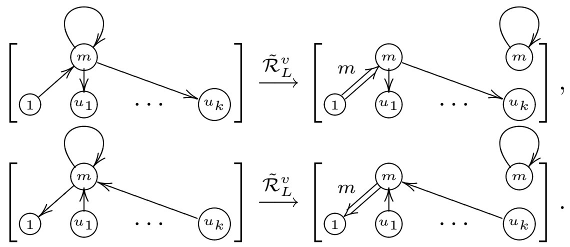
(1) **Vertex removal.** If $\sum_{j=1}^{k} i_j \leq \alpha_v$ or $\sum_{j=1}^{l} u_j \leq \alpha_v$ we delete the vertex v.



(2) **loop removal.** Uncouple the loops on a vertex with dimension 1.



(3) **Loop removal.** Uncouple the only loop or marked loops on a vertex with dimension $m \geq 2$ which has a neighborhood as shown in one of the pictures below.



Because the twist is an automorphism of the quiver setting, the reducibility conditions commute with the twist: if a vertex of a loop is reducible for a quiver setting, then its image under the twist is also reducible.

For the reduction steps $\tilde{\mathcal{R}}_l^v$ and $\tilde{\mathcal{R}}_L^v$ this means that we can perform the reductions of all these loops simultaneously and obtain a new quiver setting with the same twisted weight extended in the obvious way to the new vertices and arrows. For $\tilde{\mathcal{R}}_L^v$ the $m-1$ new arrows have weights $w_a + w_\ell, \dots w_a + (m-1)w_\ell$, where ℓ is the loop and a the unique arrow leaving or arriving.

For reduction step $\tilde{\mathcal{R}}_V^v$ we must take care. If there are no arrows between vertices in the ϕ-orbit of the reducible vertex v, then we can again reduce all the vertices simultaneously. The new weights are the sums of the weights of couples of the original arrows.

If there are arrows between two of the vertices in the same ϕ-orbit, then the reducibility condition on the vertices implies that these arrows are the only arrows that arrive in or leave these vertices. So Q is extended Dynkin of type $\tilde{A}_n$ and we can reduce all vertices except one, to obtain a one vertex quiver with as weight a multiple of the original weight.

Theorem 8. *If (Q, α) is a quiver setting with twisted weight ψ and (Q_R, α_R) is a reduced setting obtained from (Q, α), then there is a reduced twisted weight ψ_R such that*

$$\mathsf{giss}_\alpha^\psi Q = \mathsf{giss}_{\alpha_R}^{\psi_R} Q_R$$

From now on, we only consider reduced settings.

7. Reduced settings for which $\mathsf{giss}_\alpha Q$ is smooth

Because $\mathsf{rep}_\alpha Q$ is a linear $\mathsf{GL}_\alpha \rtimes \mu_e$-representation the quotient is a cone, so its worst singularity is situated in the origin. This has the following easy to check consequences:

- $\mathsf{giss}_\alpha^\psi Q$ is smooth iff the minimal number of graded generators of $\mathbb{C}[\mathsf{giss}_\alpha^\psi Q]$ equals the dimension of $\mathsf{giss}_\alpha^\psi Q$.
- If $\mathsf{giss}_\alpha^\psi Q$ is smooth then $\mathsf{giss}_\alpha^\psi Q'$ is smooth for all subquivers Q' that are closed under the twist, and their reductions.
- If $\mathsf{giss}_\alpha^\psi Q$ is smooth then $\mathsf{giss}_{\alpha_L}^{\psi_L} Q_L$ is smooth for all local quiver settings (Q_L, α_L) and their reductions.

These consequences can be used to give a complete classification of all twisted weighted quiver settings with a smooth quotient space. The trick is to rule out all other settings by looking at local quivers or subquivers and their reductions which are already known to contain singularities. In order to do that we first need to investigate some simple cases.

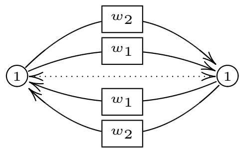
Lemma 7. *Suppose that $\mathbb{C}[X_1, \dots, X_k]$ has a faithful action of μ_e on it and the weight of X_i is w_i. Then $\mathbb{C}[X_1, \dots, X_k]^{\mu_e}$ is a polynomial ring if and only if for every prime power $p^m|e$ at most one of the weights is not trivial modulo p^k.*

Proof. As the ring of invariants has the same dimension, it is smooth if and only if it is generated by the functions $X_i^{e/\gcd(e,w_i)}$.

First let $e = p^m$ for some prime and suppose that w_1 and w_2 are non-trivial. Up to an isomorphism of μ_e we can assume that $w_1 = 1$. Therefore $X_1^{p^m - w_2} X_2$ must be a generator and the ring of invariants cannot be regular. The reverse implication is trivial.

The case $e = p_1^{n_1} \dots p_t^{n_t}$ follows from the Chinese remainder theorem. □

Lemma 8. *For*

$Q :=$ 

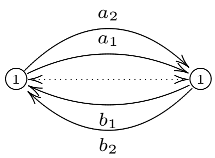
the only weights (w_1, w_2) that make $\mathsf{giss}^\psi_\alpha Q$ smooth, are the trivial weights (i.e. $2w_1, 2w_2 = 0 \mod e$).

Proof. The ring of polynomial functions $\mathbb{C}[\mathsf{iss}_\alpha Q]$ is a free module of rank 2 over the polynomial ring

$$R = \mathbb{C}[X_{11}, X_{22}, X_{12} + X_{21}]$$

The twisted weight gives an action of μ_e on this ring with weights $2w_1, 2w_2, w_1 + w_2$. If one of the weights w_1, w_2 is non-trivial then the ring R^{μ_e} is not polynomial because of lemma 7. As the generators of R^{μ_e} form a subset of those of $\mathbb{C}[\mathsf{giss}^\psi_\alpha Q]$ the latter cannot be smooth either.

The twisted quiver setting



with the action of $\mu_2 =< \zeta >$,

where ζ switches the 2 vertices and maps a_i to b_i is smooth. The twist maps $X_{ij} = a_i b_j$ (i.e., the generators of $\mathbb{C}[\mathsf{iss}_\alpha\ Q]$) to X_{ji} and this means that $\mathbb{C}[\mathsf{giss}^\psi_\alpha\ Q]$ is given by

$$\mathbb{C}[\mathsf{iss}_\alpha\ Q]^{\mu_e} = \mathbb{C}[X_{11}, X_{22}, X_{12} + X_{21}, X_{12}X_{21}]$$

and as $X_{12}X_{21} = X_{11}X_{22}$, we obtain that $\mathsf{giss}^\psi_\alpha\ Q$ is just $\mathbb{A}^3$. □

In order to use the technique of local quivers we also need a lemma concerning the dimension vectors of simples:

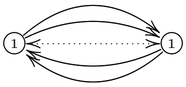
Lemma 9. *If (Q, α) is a strongly connected reduced quiver setting and v is the vertex with highest dimension then both $\mathsf{iss}_\alpha Q$ and $\mathsf{iss}_{\alpha - \epsilon_v} Q$ contain simple representations. (ϵ_v assigns 1 to v and 0 to the other vertices)*

Proof. This is a straightforward combination of the definition of reducedness in [1] and the characterization of dimension vectors of simple quiver representations in [8]. □

Let us now investigate the structure of the quotient space $\mathsf{giss}^{\psi}_{\alpha}Q$. Because this quotient factors through $\mathsf{iss}_{\alpha}Q$, we can delete all arrows that are not contained in any cycle.

First we assume that (Q, α) is strongly connected.

Theorem 9. *If a strongly connected reduced twisted weighted quiver setting is smooth then either the untwisted unweighted setting is smooth or it is equal to*

$\Theta :=$ 

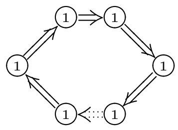
with trivial weights.

Proof. If the setting is only weighted and not twisted, then the setting can only be smooth if the unweighted setting is smooth: if the primitive cycle c is a generator for $\mathbb{C}[\mathsf{iss}_{\alpha}Q]$ and its weight is a k^{th} root of one, then c^k must be a generator for $\mathbb{C}[\mathsf{giss}^{\psi}_{\alpha}Q]$. The number of generators of $\mathbb{C}[\mathsf{giss}^{\psi}_{\alpha}Q]$ is at least as big as that of $\mathbb{C}[\mathsf{iss}_{\alpha}Q]$. If $\mathbb{C}[\mathsf{iss}_{\alpha}Q]$ is not smooth and $\mathbb{C}[\mathsf{giss}^{\psi}_{\alpha}Q]$ has the same dimension, the latter cannot be smooth either.

Now suppose (Q, α, w, ϕ) has a nontrivial twist.

If $\alpha = \mathbf{1}$, we discern two cases

- *All cycles run through all the vertices of Q.* In that case Q looks like

  $\Theta_{p,q} =$ 

  where p is the number of vertices and $q \geq 2$ the number of arrows between 2 consecutive vertices. The order of ϕ must be a divisor of p.

  The number of generators of $\mathbb{C}[\mathsf{iss}_{\mathbf{1}}Q]$ is q^p. The twist permutes these generators and for every orbit we have at least one generator of $\mathbb{C}[\mathsf{giss}^{\psi}_{\mathbf{1}}Q]$: the sum of the elements in the orbit raised to the appropriate power in order to let the weight vanish. Because the dimension of $\mathsf{giss}^{\psi}_{\mathbf{1}}Q$ is $p(q-1)+1$, it can only be smooth if the number of orbits is smaller than $p+1$. However the number of orbits is bigger than $(q^p - q)/p + q$ because an orbit has at most p elements and there are at least q fixed cycles. This condition implies that $p = 2, 3$ and $q = 2$. If $p = 3$ we can find more generators: take an orbit of order 3, sum the three products of 2 factors' and raise it to the appropriate power.

  From lemma 8 we know that the only possibility for $p = 2$ is $\Theta := \Theta_{2,2}$ with trivial weights.
- *There is a cycle that runs not through all the vertices of Q.* This means that there is a strongly connected full subquiver not containing all vertices. Let Q_M be maximal in the subset of all such subquivers. Consider the local quiver Q_L coming from the representation that assigns ones to the arrows in Q_M and zeros to the other arrows. The number of vertices of this quiver is smaller than the original quiver and the local twist fixes the vertex coming from Q_M.

  If this vertex can be reduced then it has exactly one arrow arriving and one arrow leaving. These two arrows must connect the vertex to two different vertices otherwise Q_M was not maximal. This means that the twist fixes these vertices as well and after the reduction no other vertex can be reduced anymore.

  So in all cases the local quiver Q_L reduces to a reduced quiver with less vertices and at least one vertex that is fixed by the local twist. Therefore it cannot be Θ and we can use induction on the vertices.

Finally, suppose that (Q, α) is reduced and $\alpha \neq \mathbf{1}$. Let v be a vertex with the highest dimension and consider the decomposition $(Q, \alpha - \epsilon_v) \oplus (Q, \epsilon_v)$. Lemma 9 implies that there exist indeed simple quivers with dimension vector $\alpha - \epsilon_v$ and that the number of arrows between the two vertices in the local quiver is bigger than one. The local quiver is thus untwisted and reduced (up to loops). □

If (Q, α) is not strongly connected then we can assume that there is at most one component that is smooth because we can change the smooth parts to a quiver with one vertex and a number of loops equal to the dimension of the smooth parts. The action of the automorphism on the smooth part can be diagonalized to give weights on the loops.

If (Q, α) contains two or more components $Q_1 \sqcup \cdots \sqcup Q_s$, then we will prove that $\mathsf{giss}_\alpha^\psi Q$ cannot be smooth, unless the twisted weight acts trivially on all $\mathsf{iss}_\alpha Q_i$ except one. By the previous paragraph, we can take a component Q_i such that one of the other components is not smooth. If the finite action on this component $\mathsf{iss}_\alpha Q_i$ is not trivial, then there exists a representation not fixed by $w\phi$. Because Q_i is reduced, $\mathsf{iss}_\alpha Q_i$ contains simples (lemma 9) and we can choose a simple with trivial stabilizer. Look at the local quiver corresponding to the direct sum of this simple with the trivial simples of the other components. This setting (Q_L, α_L) is untwisted and it contains the same components as the original quiver except Q_i. Because one of the components is not smooth, $\mathsf{giss}_{\alpha_L}^{\psi_L} Q_L = \mathsf{iss}_{\alpha_L} Q_L$ is also not smooth.

8. Graded Cayley-Hamilton algebras

Let us switch to the setting where A is a graded Cayley-Hamilton algebra and where we are interested in the trace preserving properties of this algebras. Recall from [10] that an n^{th} Cayley-Hamilton algebra is a finitely generated algebra A equipped with a trace, that is a $Z(A)$-linear map $\mathsf{tr} : A \to Z(A)$, such that for all a, b in A

- $\mathsf{tr}(ab) = \mathsf{tr}(ba)$,
- $\mathsf{tr}(1) = n$,
- $\chi_{n,a}(a) = 0$,

where $\chi_{n,a}(X)$ is the n^{th} Cayley-Hamilton identity expressed in the traces of powers of a. The matrix algebra $\mathsf{Mat}_{n\times n}(\mathbb{C})$ with the natural trace is the simplest example of such a Cayley-Hamilton algebra.

Analogous to the previous situation we define the representation varieties:

Definition 6. *Let $A \in \mathsf{CH}_n$ be a graded Cayley-Hamilton order. Then we define*

(1) $\mathsf{trep}_n^{ss}\ A$ *for the set of semistable trace preserving representations.*
(2) $\mathsf{gtrep}_n\ A = \mathtt{proj}\ \mathbb{C}[\mathsf{trep}_n\ A] = \mathsf{trep}_n^{ss}\ A /\!\!/ \mathbb{C}^*$.
(3) $\mathsf{gtiss}_n\ A = \mathsf{gtrep}_n\ A /\!\!/ \mathsf{GL}_n$.

Again, $\mathsf{gtiss}_n\ A$ classifies the isoclasses of trace preserving (semistable) semisimple representations up to graded isomorphism.

Similar to the observation that the $\mathsf{tiss}_n\ A$ can be seen as $\mathtt{spec}\ Z(A)$, we have that $\mathsf{gtiss}_n\ A$ exactly coincide with $\mathtt{proj}\ Z(A)$.

The only extra feature we encounter here, is that the local structure of a Cayley-Hamilton algebra is described by marked quivers. These are quivers of which some of the loops are marked. Marked loops are represented by traceless matrices. If we introduce weights and twists this situation may lead to loops $\ell_1, \ldots, \ell_s$ on different vertices in the same orbit of the twist. As these loops originate from a bigger trace 0 representation, this means that the sum of the traces of $\ell_1, \ldots, \ell_s$ is 0. This situation can be undone by adding to the (twisted weighted) local quiver setting an extra vertex v with dimension 1 and $s-1$ loops on v and marking the loops $\ell_1, \ldots, \ell_s$.

9. Non-commutative blow-up of the quantum plane

In the commutative case, a blow-up $\tilde{\mathbb{A}}^2$ of $\mathbb{A}^2$ in the origin can be seen as the variety $\mathbb{V}(xY - yX) \subset \mathbb{A}^2 \times \mathbb{P}^1$, with x and y the affine coordinates and X and Y the projective coordinates. The coordinate ring of $\mathbb{A}^2$ is given by $\mathbb{C}[x, y]$ and we can extend this ring to the graded algebra

$$R = \mathbb{C}[x, y] \oplus (x, y)t \oplus (x, y)^2 t^2 \oplus \ldots \hookrightarrow \mathbb{C}[x, y][t].$$

R is generated by two elements x, y of degree zero and two elements $X = xt, Y = yt$ of degree one, obeying the homogeneous relation $xY - yX$. We see that $\tilde{\mathbb{A}}^2 = \texttt{proj}\ R$ and that the projection morphism $\tilde{\mathbb{A}^2} \twoheadrightarrow \mathbb{A}^2$ is given by the inclusion (in degree zero) $\mathbb{C}[x, y] \hookrightarrow R$.

We will try to make a non-commutative equivalent of this by looking at a similar construction, but now starting with the quantum plane $\mathbb{C}_q[u, v]$ at $q = -1$ instead of the affine plane. With

$$B = \mathbb{C}_q[u, v] = \mathbb{C}\langle u, v\rangle/(uv + vu)$$

and center $Z(B) = \mathbb{C}[u^2, v^2]$, we can associate the graded algebra

$$A = B \oplus (u, v)t \oplus (u, v)^2 t^2 \oplus \dots \hookrightarrow \mathbb{C}_q[u, v][t].$$

This algebra can be seen as the non-commutative blow-up in $(0, 0)$ of the ring B. A is generated by two elements u and v of degree zero and two elements $U = ut$ and $V = vt$ satisfying a number of relations and with u^2, v^2, U^2 and V^2 central homogeneous elements.

Let us now concentrate on the closed orbits over the point $(0, 0)$. For those, either U^2 or V^2 must be invertible and we consider the case where U^2 is invertible.

If we take the graded localization at U^2, we obtain

$$A' = A^g_{U^2} = \mathbb{C}_{-1}[u, w][U, U^{-1}]$$

where $w = VU^{-1}$. This means we want to look at the positively graded ring

$$A' = \frac{\mathbb{C}\langle u, w, U\rangle}{(uw + wu, wU + Uw, uU - Uu)}$$

with the degree of u and w equal to zero and the degree of U equal to 1. We need to check whether $\mathsf{trep}_2\ A'$ is smooth in the closed orbits lying over $(0, 0) \in \mathbb{A}^2$. Representatives of these orbits are of one of the following types:

(1) simple type: ρ_s sending (u, w, U) to

$$\left(\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}a & 0\\ 0 & -a\end{bmatrix}, \begin{bmatrix}0 & 1\\ b & 0\end{bmatrix}\right)$$

(2) semisimple type: ρ_{ss} sending (u, w, U) to

$$\left(\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}a & 0\\ 0 & -a\end{bmatrix}\right)$$

with $a, b \in \mathbb{C}^*$. $\mathsf{trep}_2\ A'$ is of dimension 6, so we need to verify that the tangent space to $\mathsf{trep}_2\ A'$ is of dimension 6 in the indicated points. In order to compute the tangent space, we work in $M_2(\mathbb{C}[\varepsilon])$ and consider

$$u_s = \rho_s(u) + \varepsilon \begin{bmatrix}\alpha^u_s & \beta^u_s\\ \gamma^u_s & -\alpha^u_s\end{bmatrix}, \quad w_s = \rho_s(w) + \varepsilon \begin{bmatrix}\alpha^w_s & \beta^w_s\\ \gamma^w_s & -\alpha^w_s\end{bmatrix},$$
$$U_s = \rho_s(U) + \varepsilon \begin{bmatrix}\alpha^U_s & \beta^U_s\\ \gamma^U_s & -\alpha^U_s\end{bmatrix}.$$

The conditions on (u_s, w_s, U_s) are the defining relations of A':

$$u_s w_s + w_s u_s = w_s U_s + U_s w_s = u_s U_s - U_s u_s = 0$$

and with $\varepsilon^2 = 0$, this leads to

$$\alpha^u_s = 0, \quad \gamma^u_s = b\beta^u_s, \quad \alpha^U_s = \frac{\gamma^w_s + b\beta^w_s}{2a},$$

and with 3 linear relations, this yields a 6-dimensional tangent space. A similar construction (and notation) lead for the semisimple points to

$$\beta^u_{ss} = \gamma^u_{ss} = \alpha^w_{ss} = 0.$$

which again leads to a six dimensional subspace and the smoothness of $\mathsf{trep}_2\ A'$ is verified.

We wonder how the (twisted weighted) local quiver in these orbits look like. The stabilizer subgroups of $\mathsf{GL}_2 \times \mathbb{C}^*$ are the following:

- in simple points x the stabilizer G_x looks like $\mathbb{C}^* \times \mu_2$ where the generator of μ_2 is given by

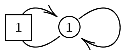
$$\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, -1 \right)$$

- in semisimple points x the stabilizer G_x looks like $(\mathbb{C}^* \times \mathbb{C}^*) \rtimes \mu_2$ where the component of $-1 \in \mu_2$ is embedded by

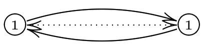
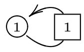
$$\left(\begin{bmatrix} 0 & \delta \\ \eta & 0 \end{bmatrix}, -1 \right).$$

Proposition 2. *The (weighted) local quivers are as follows:*

(1) *In simple points they look like*

 *with action of* μ_2*.*

(2) *In semisimple points they look like*

  *with the action of* μ_2*.*

Proof. In a simple point x, the action of the stabilizer $\mathbb{C}^* \times \mu_2$ on a tangent vector is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \left(\begin{bmatrix} 0 & \beta_s^u \\ b\beta_s^u & 0 \end{bmatrix} \quad \begin{bmatrix} \alpha_s^w & \beta_s^w \\ \gamma_s^w & -\alpha_s^w \end{bmatrix} \quad - \begin{bmatrix} \alpha_s^U & \beta_s^U \\ \gamma_s^U & -\alpha_s^U \end{bmatrix} \right) \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The action on the individual coefficients can therefore be expressed by

$$\beta_s^u \mapsto -\beta_s^u, \quad \alpha_s^w \mapsto \alpha_s^w,$$

$$\beta_s^w \mapsto -\beta_s^w, \quad \gamma_s^w \mapsto -\gamma_s^w, \quad \beta_s^U \mapsto \beta_s^U, \quad \gamma_s^U \mapsto \gamma_s^U. \tag{3}$$

We now need to calculate the submodule corresponding to the tangent space of the orbit of x. This can be done by calculating

$$(1 + \varepsilon t^{\text{deg}}) \left(\mathbb{1}_2 + \varepsilon \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \right) Z \left(\mathbb{1}_2 - \varepsilon \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \right) \tag{4}$$

with $Z = \rho_s(u), \rho_s(w), \rho_s(U)$ and $\varepsilon^2 = 0$. This result in the space

$$\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2ax_2 \\ 2ax_3 & 0 \end{bmatrix}, \begin{bmatrix} bx_2 - x_3 & t + x_1 - x_4 \\ b(t + x_4 - x_1) & x_3 - bx_2 \end{bmatrix} \right),$$

which under the action of G_x has the mapping

$$x_2 \mapsto -x_2, \quad x_3 \mapsto -x_3, \quad t \mapsto t, \quad (x_1 - x_4) \mapsto (x_1 - x_4)$$

and by identifying variables, we can remove line (3), to arrive at the desired twisted weighted local quiver setting.

In semisimple points, the action of $(\mathbb{C}^* \times \mathbb{C}^*) \rtimes \mu_2$ on a tangent vector is given by

$$\begin{bmatrix} 0 & \delta \\ \eta & 0 \end{bmatrix} \cdot \left(\begin{bmatrix} \alpha_{ss}^u & 0 \\ 0 & -\alpha_{ss}^u \end{bmatrix} \quad \begin{bmatrix} 0 & \beta_{ss}^w \\ \gamma_{ss}^w & 0 \end{bmatrix} \quad - \begin{bmatrix} \alpha_{ss}^U & \beta_{ss}^U \\ \gamma_{ss}^U & -\alpha_{ss}^U \end{bmatrix} \right) \cdot \begin{bmatrix} 0 & \delta^{-1} \\ \eta^{-1} & 0 \end{bmatrix}$$

which leads to

$$\alpha_{ss}^u \mapsto -\alpha_{ss}^u, \quad \beta_{ss}^w \mapsto \delta\gamma_{ss}^w\eta^{-1}, \quad \gamma_{ss}^w \mapsto \eta\beta_{ss}^w\delta^{-1},$$

$$\alpha_{ss}^U \mapsto \alpha_{ss}^U, \quad \beta_{ss}^U \mapsto -\delta\gamma_{ss}^U\eta^{-1}, \quad \gamma_{ss}^U \mapsto -\eta\beta_{ss}^U\delta^{-1}. \tag{5}$$

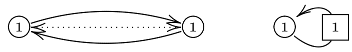
If we do the calculation of (4), we have the space

$$\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} at & -2ax_2 \\ 2ax_3 & -at \end{bmatrix} \right),$$

and the action of G_x yields

$$t \mapsto t, \quad x_2 \mapsto -\delta x_3\eta^{-1}, \quad -\eta x_3\delta^{-1}.$$

This makes us delete the variables of (5) and the remaining variables with the action on them can be encoded in the twisted weighted local quiver setting



with the action of μ_2. $\square$

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