THE LOCAL STRUCTURE OF GRADED REPRESENTATIONS

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ABSTRACT. In this article we show that the local structure of the projective representation space of a graded algebra can locally be described by quivers with an automorphism of their path algebra, a twisted weight. We describe the quotient spaces of these twisted weighted quiver settings and determine which of them are smooth.

1. INTRODUCTION

Given a finitely generated algebra $A$ over $\mathbb{C}$ one can look at the set of $n$-dimensional representations

$$\text{rep}_n A := \{ \rho : A \to \text{Mat}_{n \times n}(\mathbb{C}) \mid \rho \text{ is an algebra morphism} \}.$$ 

This has the structure of an algebraic variety and it has an additional $\text{GL}_n$-action on it by conjugation. One can take the algebraic quotient of this action to obtain a new variety $\text{iss}_n A$ that classifies all closed $\text{GL}_n$-orbits or equivalently all isomorphism classes of semisimple representations.

If $A$ is a formally smooth algebra, i.e. the kernel of the multiplication map $A \otimes A \to A$ is a projective bimodule, then $\text{rep}_n A$ is a smooth variety and one can describe the étale local structure of the quotient map by means of quivers [6], [5].

A quiver $Q$ is a directed graph consisting of a set of vertices $Q_0$ and a set of arrows $Q_1$. The maps $h, t : Q_1 \to Q_0$ will denote the heads and the tails of the arrows. The path algebra of a quiver is the vector space spanned by all paths in $Q$ (including the vertices considered as paths of length $0$) equipped with concatenation as multiplication.

A dimension vector $\alpha$ is a map from $Q_0$ to $\mathbb{N}$ and a couple $(Q, \alpha)$ of a quiver and a dimension vector is called a quiver setting. The space of all $\alpha$-dimensional representations is denoted by $\text{rep}_\alpha Q$.

$$\text{rep}_\alpha Q := \bigoplus_{a \in A} \text{Mat}_{\alpha_h(a) \times \alpha_t(a)}(\mathbb{C})$$

To the dimension vector $\alpha$ we can also assign a reductive group

$$\text{GL}_\alpha := \prod_{v \in Q_0} \text{GL}_{\alpha_v}(\mathbb{C}).$$

An element of this group, $g = (g_v)_{v \in Q_0}$, has a natural action on $\text{rep}_\alpha Q$:

$$W := (W_a)_{a \in A}, \quad W^g := (g_{t(a)}W_ag_{a(a)})_{a \in A},$$

and the quotient of this action is denoted by $\text{iss}_\alpha Q$.

Theorem 1 (Le Bruyn). If $\text{rep}_p A$ is smooth in $p$, where $p$ corresponds to a semisimple representation $S^1_{\alpha_1} \oplus \cdots \oplus S^k_{\alpha_k}$ then there is a quiver setting $(Q_p, \alpha_p)$ such that there are étale neighborhoods making the following diagram commutative.

$$\begin{array}{ccc}
\text{GL}_n \times_{\text{GL}_\alpha} \text{rep}_\alpha Q & \longrightarrow & \text{rep}_n A \\
\downarrow & & \downarrow \\
\text{iss}_\alpha Q & \longrightarrow & \text{iss}_n A
\end{array}$$

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This quiver setting is called the local quiver setting. Its vertices \( s_1, \ldots, s_k \) correspond to the simples \( S_1, \ldots, S_k \) and the number of arrows between \( s_i \) and \( s_j \) is

\[
\dim \text{Ext}^1(S_i, S_j).
\]

Quiver settings provide a powerful combinatorial tool to study the quotient map locally. One can find the smooth points of the quotient \([1]\) and the types of singularities that can occur in \( \text{iss}_\alpha Q \) \([2]\).

If \( p \) is not a smooth point of the variety \( \text{rep}_n A \) the formalism of local quiver settings breaks down. In this paper we explore the possibility to save some of the power of this approach using twisted weighted quiver settings. The ideas presented here are an extension of \([4]\).

Instead of looking at \( \text{rep}_n A \) which contains singularities, one can try to find a resolution of \( \text{rep}_n A \): a surjective \( \text{GL}_n \)-equivariant map \( V \to \text{rep}_n A \) such that \( V \) is a smooth variety and the generic fiber consists of one point. In general this resolution is not an affine variety so it cannot be seen as a representation space of an algebra, but in some cases this resolution can be obtained by a non-commutative blow-up.

If \( A \) has an ideal \( i \) one can construct the graded algebra

\[
\tilde{A} := A \oplus i \oplus i^2 \oplus \ldots.
\]

The ring of polynomial functions over the representation space \( \text{grep}_n \tilde{A} \) has also a positive grading and one can look at the projective space it describes \( \text{grep}_n A \). If this space is smooth one has found a resolution of \( \text{rep}_n A \).

The local structure of such a resolution does not always reduce to ordinary quivers, but one needs some extra structure: an automorphism on the path algebra of the quiver. A quiver setting together with such an automorphism is called a twisted weighted quiver setting.

The main part of this paper will investigate how one can reduce the problem sketched above to these twisted weighted quiver settings and study the finite group action, induced by the automorphism on the quotient space \( \text{iss}_\alpha Q \).

The paper is structured as follows. In section 2 we introduce the graded representation space \( \text{grep}_n A \) of a graded algebra \( A \). In section 3 we calculate the stabilizer of a semisimple representation, given its decomposition (theorem 2). In section 4 we define the notion of a twisted weighted quiver setting and the corresponding finite group action on its representation space. This allows us to formulate the étale local structure in terms of representation spaces of twisted weighted quiver settings.

The next three sections focus on the geometry and combinatorics of twisted weighted quiver settings alone. First we show that the local structure of twisted weighted quiver settings can again be seen as twisted weighted quiver settings. These new local twisted quiver settings can be calculated using adapted versions of the dimension vectors and the Euler form. Secondly we modify the reduction steps defined in \([1]\) to the twisted weighted case. These reduction steps enable us to simplify the settings without changing the quotient. Finally we give a classification of all twisted weighted quiver settings whose quotient space is smooth.

In the last two sections we show that the theory developed above can be easily transferred to the theory of Cayley-Hamilton \([10]\) algebras. We illustrate this with an example: the quantum plane at \( q = -1 \).

2. GRADED ALGEBRAS AND REPRESENTATION SPACES

Consider \( A \) a positively graded algebra

\[
A = \mathbb{C}(x_1, \ldots, x_m)/(I) = A_0 \oplus A_1 \oplus A_2 \oplus \ldots,
\]

where \( I \) is an ideal of homogeneous polynomials in the variables \( x_i \), with the degree of \( x_i \) being equal to \( d_i \).
The coordinate ring of the representation space, \( \mathbb{C}[\text{rep}_n A] \), looks like
\[
\mathbb{C}[\text{rep}_n A] = \mathbb{C}[X_{kl}^i; i = 1, \ldots, m; k, l = 1, \ldots, n]/(I')
\]
which can also be seen as a graded algebra. Indeed, each \( X_{kl}^i \) will have degree \( d_i \) and the ideal \( I' \) is spanned by homogeneous polynomials which originate from the homogeneous polynomials in the ideal \( I \) of \( A \). Therefore, \( \mathbb{C}[\text{rep}_n A] \) has a \( \mathbb{C}^* \)-action working on it defined by
\[
t \cdot f = t^{d_i} f,
\]
where \( t \in \mathbb{C}^* \) and \( f \in \mathbb{C}[\text{rep}_n A] \) has degree \( d_f \). This action induces a \( \mathbb{C}^* \)-action on the maximal ideals and thus on \( \text{rep}_n A \).

As \( \mathbb{C}[\text{rep}_n A] \) is graded, we can define \( \text{proj} \mathbb{C}[\text{rep}_n A] \) and will denote this by \( \text{grep}_n A \).

**Definition 1.** A representation \( \rho : A \to M_n(\mathbb{C}) \) is called semistable if there exists an \( f \in \mathbb{C}[\text{rep}_n A] \) such that
1. \( f \) is homogeneous and of degree \( > 0 \) (i.e., a semi-invariant for the \( \mathbb{C}^* \)-action).
2. \( f \) is \( \text{GL}_n \)-invariant.
3. \( f \) is non-vanishing on \( \rho \).

We write \( \text{rep}^{ss}_n A \) for the set of all semistable representations of \( A \).

**Remark 1.** If \( \rho : A \to M_n(\mathbb{C}) \) sends all generators to nilpotent matrices, then \( \rho \) is not a semistable representation. The converse is not true in general, but does hold if all generators have degree bigger than zero.

With the definition of \( \text{rep}^{ss}_n A \), we can write \( \text{grep}_n A \) as an algebraic quotient:
\[
\text{grep}_n A = \text{proj} \mathbb{C}[\text{rep}_n A] = \text{rep}^{ss}_n A / \mathbb{C}^*.
\]
The action of \( \text{GL}_n \) on \( \text{rep}_n A \) induces an action of \( \text{GL}_n \) on \( \text{grep}_n A \) and we denote the algebraic quotient \( \text{iss}^{ss}_n A = \text{grep}_n A / \text{GL}_n \). This summarizes in the diagram
\[
\begin{array}{ccc}
\text{rep}^{ss}_n A & \xrightarrow{\mathbb{C}^*} & \text{grep}_n A \\
\downarrow{\mathbb{C}^*} & & \downarrow{\mathbb{C}^*} \\
\text{iss}^{ss}_n A & \xrightarrow{\mathbb{C}^*} & \text{giss}_n A \\
\end{array}
\]
which is commutative since the action of \( \mathbb{C}^* \) commutes with the action of \( \text{GL}_n \).

On the other hand, we can look at the diagram
\[
\begin{array}{ccc}
\text{rep}^{ss}_n A & \xleftarrow{\text{GL}_n} & \text{rep}_n A \\
\downarrow{\text{GL}_n} & & \downarrow{\text{GL}_n} \\
\text{iss}^{ss}_n A & \xleftarrow{\text{GL}_n} & \text{iss}_n A \\
\end{array}
\]
and check whether the inclusion of \( \text{iss}^{ss}_n A \) in \( \text{iss}_n A \) is well defined. We find:

**Lemma 1.** A \( \text{GL}_n \)-orbit in \( \text{rep}^{ss}_n A \) is closed if and only if the corresponding \( \text{GL}_n \)-orbit in \( \text{rep}_n A \) is closed.

**Proof.** This follows directly from the fact that semistability condition is \( \text{GL}_n \)-invariant. □

**Lemma 2.** With a \( \xi \in \text{giss}_n A \) corresponds a semisimple \( M_\xi \in \text{rep}^{ss}_n A \), unique up to the action of \( \text{GL}_n \times \mathbb{C}^* \).
Proof. If the orbit $M^{GL_n}$ is closed, then also $(t \cdot M)^{GL_n}$ is a closed orbit. The only thing to verify is that for every 1 parameter subgroup $(g(t))_{t \in C^*}$,
\[ \rho := \lim_{t \to 0} (g(t), t^n) \cdot M \]
is not a semistable representation. Suppose $\rho$ is semistable, then there exists an $f \in \mathbb{C}[\text{rep}_n A]$ homogeneous of nonzero degree with $f(\rho(x_1), \ldots, \rho(x_m)) \neq 0$. This means that among the $x_i$, at least one has $\rho(x_i) \neq 0$ and $d_i > 0$, but this contradicts the definition of $\rho$. \hfill \Box

Definition 2. A semistable representation $M_1$ is called graded isomorphic to a representation $M_2$ if there exists a $\zeta$ such that
\[ M_2 \cong \zeta \cdot M_1 \]
This definition and lemma 2 gives:

Proposition 1. $\text{giss}_n A$ classifies the (semistable) semisimple $n$-dimensional representations up to graded isomorphism.

3. STABILIZERS OF SEMISIMPLE POINTS

We have a nice characterisation for $M \in \text{rep}_n A$ to be semistable.

Notation 1. Suppose $M$ is a representation of $A$. We write
\[ \text{Aut}_{C^*}(M) = \{ \zeta \in C^* \mid \zeta \cdot M \cong M \}, \]
where the action of $\zeta$ on $M$ is given by $\rho_{gM}(x_i) = \zeta^d_i \rho_M(x_i)$.

Lemma 3. Let $M$ be a semisimple representation of $A$. Then $M$ is semistable if and only if $\text{Aut}_{C^*}(M)$ is finite.

Proof. \([\rightarrow]\) Let $M$ be a semistable representation of $A$. This means there exists an $f$ homogeneous and $GL_n$-invariant, such that $f \neq 0$ on $M$. Suppose $\zeta \in \text{Aut}_{C^*}(M)$, then $\zeta \cdot M \cong M$ and as $f$ is $GL_n$-invariant, we have $\zeta^d_i f = f$. Since $f$ is nonzero on $M$, this yields $\zeta$ to be a $d_i$-th root of unity. Therefore $\text{Aut}_{C^*}(M)$ is finite.

\([\leftarrow]\) Let $M$ be a semisimple representation that is not semistable. $\mathbb{C}[[\text{iss}_n A]$ is a finitely generated algebra and we denote $(f_i)_{i=1}^k$ the (homogeneous) generators (with $d_i$ the degree of $f_i$). Then, $\zeta \cdot M \cong M$ if and only if $f_i(\zeta \cdot M) = f_i(M)$ for all $i \in \{1, \ldots, k\}$. $f_i(\zeta \cdot M)$ can be written as $\zeta^{d_i} f_i(M)$ and as $M$ is not semistable, for both the cases $d_k > 0$ and $d_k = 0$, we have no restriction on the $\zeta$. \hfill \Box

As we are interested in the étale local structure of $\text{giss}_n A$, we want to apply the Luna slice theorem [9] to the quotient map
\[ \text{rep}^{ss}_n A \longrightarrow \text{giss}_n A, \]
and therefore we will suppose $\text{rep}^{ss}_n A$ to be a smooth variety.

A first problem to encounter is the structure of the stabilizer under the action of $GL_n \times C^*$ of such a semisimple representation. First, we look at the simple situation:

Lemma 4. Suppose $S \in \text{rep}^{ss}_n A$ is a simple representation of $A$. The $GL_n \times C^*$-stabilizer of $S$ is isomorphic to the group $C^* \times \mu_c$ where the cyclic group $\mu_c$ has generator
\[ (g_\zeta, \zeta) \in GL_n \times C^* \]
where $\zeta$ is a primitive root of unity and
\[ g_\zeta = \text{diag}(1, \ldots, 1, \zeta, \ldots, \zeta, \ldots, \zeta^{-1}, \ldots, \zeta^{-1}). \]
Proof. Write $\text{Aut}_{\mathbb{C}}(S)$ as $< \zeta >$ with $\zeta = e^{2\pi i / n}$ for a certain $e$ (with $e$ a divisor of $n, \gcd(\deg(x))$). As $\zeta \cdot S \cong S$, there exists a $g \in \text{GL}_n$, unique up to $\mathbb{C}^*$, such that $S = \zeta \cdot gSg^{-1}$. Since $S = g^eSg^{-e}$ and $S$ is simple, we know by Schur's lemma that $g^e$ is a scalar and we can choose $g^e = 1$. But then, $g$ can be seen as a group representation of $\mathbb{Z}_e$, where all simples are 1-dimensional and therefore $g$ is isomorphic to a diagonal matrix $g_\zeta$. As $(g_\zeta)^e = 1$, we have that all elements on the diagonal are $e$-th roots of unity, which can be permuted in the order we want. $\square$

The proof indicates that the $(m_0, \ldots, m_{e-1})$ are determined up to cyclic permutation.

Definition 3. Suppose $S$ has a stabilizer as in lemma 4, we say that $S$ has period $e$ and matrix type $(m_0, \ldots, m_{e-1})$.

Let $S$ be a semisimple (semistable) representation with $\text{Aut}_{\mathbb{C}}(S) = \mu_e = < \zeta >$, then for each simple component $S_i$, one can compute

$$k_i = \min \{ k \in \mathbb{Z}^+ \setminus \{0\} \mid \zeta^k \cdot S_i \cong S_i \} .$$

In order to have $S \cong \zeta S$, the components $S_i, \zeta S_i, \zeta^2 S_i, \ldots, \zeta^{k_i-1} S_i$ have to occur an equal number of times in the semisimple decomposition of $S$. Hence, $S$ can be written as

$$S = \bigoplus_{i=1}^p \left( S_i^\oplus \beta_i \oplus (\zeta S_i)^\oplus \beta_i \oplus \ldots \oplus (\zeta^{k_i-1} S_i)^\oplus \beta_i \right)$$

The isomorphism $\zeta^{k_i} \cdot S_i \cong S_i$ can be expressed with the terminology of definition 3. Suppose $S_i$ has period $e_i$ and matrix type $(m_0^i, \ldots, m_{e_i-1}^i)$, with the generator of $\mu_{e_i}$ denoted by $\zeta_i$. Because $\zeta_i^{k_i} = \zeta_i^{l_i}$ with $l_i = e_i k_i / e$, this yields

$$S_i \cong \zeta_i^{k_i} g_{S_i}^{l_i} S_i (g_{S_i}^{l_i})^{-1}$$

with $g_{S_i} = g_{\zeta_i}$ the diagonal matrix (1) of lemma 4. We will use the notation

$$g_i := \text{diag}(\underbrace{1, \ldots, 1}_{m_0^i}, \underbrace{\zeta, \ldots, \zeta}_{m_1^i}, \ldots, \underbrace{\zeta^{e_i-1}, \ldots, \zeta^{e_i-1}}_{m_{e_i-1}^i})$$

For the matrix $g_i$, then holds that $g_i^{k_i} = g_{S_i}^{l_i}$.

We are now ready to determine the stabilizer of $S$.

Theorem 2. Let $S$ be an $n$-dimensional semisimple representation with $\text{Aut}_{\mathbb{C}}(S) = < \zeta >$ and decomposition

$$S = \bigoplus_{i=1}^p S_i^\oplus \beta_i \oplus (\zeta S_i)^\oplus \beta_i \oplus \ldots \oplus (\zeta^{k_i-1} S_i)^\oplus \beta_i$$
with \( k_i, g_i \) and \( \gamma_i \) as defined before, then the \( \zeta \)-component of the stabilizer is isomorphic to

\[
\begin{pmatrix}
\cdot & \cdot & \cdot & \cdots & \cdot & k_i \gamma_i \beta_i \\
\cdot & \cdot & \cdot & \cdots & \cdot & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdot & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdot & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdot & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdot & 0 \\
\end{pmatrix}
\]

with \( \alpha = (\beta_1, \ldots, \beta_1, \beta_2, \ldots, \beta_{p-1}, \beta_p, \ldots, \beta_p) \) and the embedding of \( GL_\alpha \) in \( GL_n \) is given by the block diagonal matrix

\[
\begin{pmatrix}
GL_{\beta_1} (C \otimes \mathbb{1}_{k_1}) \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
GL_{\beta_p} (C \otimes \mathbb{1}_{k_p})
\end{pmatrix}
\]

**Proof.** The theorem follows directly from the right choice of basis for \( S_i \) and \( \zeta S_i \). Choose the base change between \( S_i \) and \( \zeta S_i \) be exactly determined by \( g_i \). \( \square \)

We are now one step away from the étale local description of \( S \): the introduction of *twisted weighted quiver settings*.

### 4. Twisted Weighted Quivers and the Étale Local Structure of \( giss_n A \)

In the introduction we sketched how one can use representations of quivers to describe the local structure of \( iss_n A \) (theorem 1). In this section we will prove an analogous result for the description of the local structure of \( giss_n A \), where \( A \) is a graded algebra. Before we can do this we introduce the useful concept of *twists* and *weights*.

Consider the path algebra \( \mathbb{C}Q \) of a quiver \( Q \). We now provide every path in \( \mathbb{C}Q \) with degree the length of the path. This makes \( \mathbb{C}Q \) a graded algebra. We distinguish 2 important types of graded automorphisms (i.e., algebra morphisms that send homogeneous elements to homogeneous elements of the same degree) of \( \mathbb{C}Q \):

1. A *twist* \( \phi \) is a couple of invertible maps \( \phi_0 : Q_0 \to Q_0 \) and \( \phi_1 : Q_1 \to Q_1 \) such that
   - \( \forall a \in Q_1 : \phi_0(h(a)) = h(\phi_1(a)) \) and \( \phi_0(t(a)) = t(\phi_1(a)) \)
   - \( \forall k \in \mathbb{N} : \phi_k^h(a) = a \iff h(\phi^t_k(a)) = h(a) \) and \( t(\phi^t_k(a)) = t(a) \).
   
   Every twist induces an automorphism of \( \mathbb{C}Q \), which we also call a *twist*. 
Every graded automorphism of \( CQ \) of finite order can be conjugated to the product of a weight and a twist. Moreover, this weight and twist commute.

**Theorem 3.**

**Proof.** Suppose \( Q \) has \( p \) vertices and suppose \( \psi \) a graded automorphism of \( CQ \) of finite order. Denote by \( V_{ij} \) the vector space spanned by all paths from \( i \) to \( j \), that is,

\[
V_{ij} = (e_i C Q e_j)_1.
\]

The action of \( \psi \) then sends a \( V_{ij} \) to a \( V_{kl} \) and we can find a partition of \( S := \{ V_{ij} \mid i, j \in \{1, \ldots, p\} \} \) corresponding to the orbits of this action. Choose a set of representatives for the orbits of this action. For each of these representatives \( (e_i C Q e_j)_1 \), we can find the smallest \( k \) such that \( \psi^k (e_i C Q e_j)_1 = (e_i C Q e_j)_1 \). Choose now a basis of \( V_{ij} \) that diagonalizes this action of \( \psi \) on \( V_{ij} \) to a matrix \( A_{ij} \) and let \( B_{ij} \) be a diagonal matrix such that \( B_{ij}^k = A_{ij} \) (remark that \( B_{ij} \) has roots of unity on its diagonal). Every other space \( V_{kl} \in S \) can be seen as \( \psi^m V_{ij} \) for some representative. Choose a basis in \( V_{kl} \) such that the map \( \psi \) on \( V_{ij} \) is given by the matrix \( B_{ij}^k \). But then we have chosen a basis for \( (CQ)_1 \) for which \( \psi \) maps every basis element \( a \) to a multiple \( w_a \) of another basis element \( \phi(a) \) (with \( w_a \) one of the elements on the diagonal of the corresponding \( B_{ij} \)):

\[
\psi(a) = w_a \cdot \phi(a).
\]

So up to conjugation, \( \psi \) can be seen as the product of the weight \( w : a \mapsto w_a a \) and the twist \( \phi : \phi(a) \) is a twist since the \( A_{ij} \) are diagonalized. This construction shows that \( w(a) \) and \( w(\phi(a)) \) coincide, and therefore \( \phi \) and \( w \) commute. \( \square \)

**Remark 2.** From the construction of the proof we can conclude that \( w(a) \) is an \( \varepsilon \)-th root of unity if the order of the automorphism is given by \( \varepsilon \).

**Remark 3.** Another consequence of the construction of the proof is that all arrows lying in the same orbit under \( \phi \) have the same weight. However these weights are not uniquely determined, but depend on the choice of \( B_{ij} \).

From now on we suppose that the automorphism of \( CQ \) is in a twisted weighted form \( \psi = w \phi \) and let \( \varepsilon \) be the order of \( w \phi \).

**Definition 4.** With the previous notation, we call \( \psi = w \phi \) a twisted weight.

\( w \) and \( \phi \) define an action of \( \mu_\varepsilon \) on \( CQ \) and we want to look at this action as an action on \( \text{rep}_\alpha Q \) for a quiver setting \( (Q, \alpha) \). \( (Q, \alpha) \) is said to be compatible with this action if \( \alpha_{\phi(v)} = \alpha_v \) for all vertices \( v \). This implies that \( \text{rep}_\alpha Q \) also has a corresponding action of \( \mu_\varepsilon \) on it. The action of \( \mu_\varepsilon \) does not commute with \( \text{GL}_\alpha \) as \( \phi \) can interchange vertices. We do have that \( \text{GL}_\alpha^w = \text{GL}_\alpha \), so to describe the total action we must take the semidirect product \( \text{GL}_\alpha \rtimes \psi \mu_\varepsilon \), where the multiplication is defined by

\[
(g_1, m_1) \cdot (g_2, m_2) = (g_1 w^{m_2} \phi^{m_2}(g_2), m_1 m_2)
\]

with \( m_1 \) and \( m_2 \) integer representatives of \( \mu_\varepsilon \).

We will call the linear space \( \text{rep}_\alpha Q \) with this \( \text{GL}_\alpha \rtimes \phi \mu_\varepsilon \)-action a twisted weighted quiver representation space. Because \( \text{GL}_\alpha \rtimes \mu_\varepsilon \) is a semidirect product of two reductive groups we can take the quotient by first taking the quotient of the \( \text{GL}_\alpha \)-action and then the quotient by the group \( \mu_\varepsilon \). We denote this quotient by \( \text{giss}_\alpha^w Q \):

\[
\mathbb{C}[\text{giss}_\alpha^w Q] = \mathbb{C}[\text{iss}_\alpha Q]^{\mu_\varepsilon}.
\]
The precise action of \((g, t) \in \text{GL}_n \rtimes \mu_e\) on the representation \((\rho_a)_{a \in Q_1} \in \text{rep}_Q\) is given by
\[
(g, t) \cdot (\rho_a)_{a \in Q_1} = \left( (\zeta^{w \cdot \omega(a)})^i \rho_{\phi^i(t(a))} \rho_{\phi^i(a)} \rho_{\phi^i(h(a))}^{-1} \right)_{a \in Q_1},
\]
and by the previous we can write
\[
giss_{w \cdot \omega} Q = \text{rep}_Q/(\text{GL}_n \rtimes \mu_k).
\]
We can use these twisted weighted quiver settings to describe the local structure of \(\text{giss}_n\).

**Theorem 4.** For a smooth semisimple point \(M \in \text{rep}_n\), we denote the normal space \(T_M \text{rep}_n/\text{M}((\text{GL}_n \times \mathbb{C}^*) \cdot M)\) by \(N_M\). There exists a twisted weighted quiver setting \((Q, \alpha, \psi)\) such that
\[
\text{Stab}_M \cong \text{GL}_n \rtimes \mu_e \text{ and } N_M \oplus \mathbb{C} \cong \text{rep}_Q.\]

**Proof.** From section 3 we know that a semisimple representation always can be written as
\[
M = \bigoplus_{i=1}^p S_i \oplus (\zeta S_1) \oplus \ldots \oplus (\zeta^{k_i-1} S_1) \oplus S_i,
\]
and theorem 2 gives us the exact behaviour of the stabilizer, which is isomorphic to \(\text{GL}_n \rtimes \mu_e\). The normal space \(N\) with respect to the \(\text{GL}_n\)-orbit of \(M\) is given by \(\text{Ext}(M, M)\) and can be identified by \(\text{rep}_Q\) for a certain quiver \(Q\) and dimension vector \(\alpha\) (see [5]). As \(M\) is semistable, we know that the extra action of \(\mathbb{C}^*\) is never trivial and that
\[
T_M((\text{GL}_n \cdot M) \oplus \mathbb{C}) \cong T_M((\text{GL}_n \rtimes \mu_e) \cdot M)
\]
and this yields \(N \cong \mathbb{C} \oplus N_M\).

\(Q\) is constructed in such a way that \(\mathbb{C}Q_1 = \text{Ext}(\bigoplus S_j, \bigoplus S_j)\) (where the direct sum varies over all simple components) and we get that the path algebra \(\mathbb{C}Q\) is given by the tensor algebra
\[
\mathbb{C}Q = T_{\mathbb{C}^k} \left( \bigoplus_{i,j} \text{Ext}(S_i, S_j) \right)
\]
with \(k = \sum_{i=1}^p k_i\) the number of vertices of \(Q\). The action of the element
\[
\bigoplus_{i=1}^p \left[ \begin{array}{c c c}
0 & g_i \otimes 1_{k_i} & \cdots & 0 \\
0 & 0 & \ddots & \vdots \\
\vdots & \ddots & g_i \otimes 1_{k_i} & 0 \\
g_i \otimes 1_{k_i} & 0 & \cdots & 0
\end{array} \right], \zeta
\]
then determines a quiver automorphism \(\psi\), as \(\psi(\text{Ext}(S_i, S_j)) = \text{Ext}(\psi S_i, \psi S_j)\).

**Definition 5.** We call \((Q, \alpha, \psi)\) from the previous theorem a twisted weighted local quiver setting of \(\xi\) and will denote it by \((Q_\xi, \alpha_\xi, \psi_\xi)\).

If we now apply the Luna slice theorem [9], we arrive at

**Theorem 5.** Let \(\xi\) be an element of \(\text{giss}_n\), such that \(M_\xi\) is smooth in \(\text{rep}_n\). Then, for all \(t \in \mathbb{C}\), there is a neighborhood of \((\xi, t) \in \text{giss}_n \times \mathbb{C}\), which is étale isomorphic to a neighborhood of \(0 \in \text{giss}_n\).

To draw twisted weighted quiver settings we will use the following conventions.

1. The dimension vector will be written inside the vertices.
2. The weight \(w\) associates \(w_a / \mu_e\) to each arrow \(a\) and weights will be denoted by their power of \(\zeta\) (i.e., integers from 0 to \(e - 1\)) and will be depicted in a small square on the arrow they correspond with. Weight 0 will not be depicted.
The action of the twist $\phi$ permutes the vertices and the permutation will be indicated by dotted arrows. The way $\phi$ permutes the arrows is fully determined by the permutation of the vertices, as arrows in the same $\phi$-orbit always have the same weight.

5. LOCAL SETTINGS FOR TWISTED WEIGHTED QUIVER SETTING

Like in the case of quivers, twisted weighted quivers are closed under étale slicing. This means that if we have a twisted weighted setting and we look at the étale local structure of $\text{giss}^\psi Q$ around a point $\xi$, then this can described by a neighborhood of the $0 \in \text{giss}^\psi Q{\xi}$ of a new weighted twisted quiver setting.

We illustrate this with an example:

**Example**

Let us look at the weighted quiver setting

$$
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & \\
\downarrow & \\
\downarrow & \\
\downarrow & \\
\downarrow & \\
\downarrow & \\
\downarrow & \\
\downarrow & \\
\downarrow & \\
\end{array}
$$

$(A$ and $B$ are $4 \times 4$-matrices) and we want to look in a neighborhood of a semisimple representation $\rho$, with two components $\rho_1 \oplus \zeta \rho_1$.

The stabilizer then can be given by the generator

$$
\begin{pmatrix}
0 & 0 & \lambda & 0 \\
0 & 0 & 0 & i\lambda \\
\mu & 0 & 0 & 0 \\
i\mu & 0 & 0 & 0
\end{pmatrix}, \zeta
$$

and in that case $\rho$ can be written in the following basis:

$$
A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0
\end{pmatrix}
$$

If we calculate the normal space $N$ (based on the orthogonal complement of the tangent space to the orbit in the total tangent space), we can write $N$ in 18 variables in the following way:

$$
\begin{pmatrix}
c_1 \\
\delta_1 - \delta_2 + \delta_3 \\
c_1 \\
\delta_1 \\
e_1 \\
b_1 \\
c_2 \\
\delta_2 \\
e_2 \\
b_2 \\
c_1 - \sigma_2 + \sigma_3 \\
f_1 \\
\delta_1 - \delta_2 - f_1 \\
f_1 \\
\delta_2 \\
f_2 \\
\delta_3 \\
\delta_1 - \sigma_2 - \sigma_3 \\
\sigma_1 \\
-\sigma_1 \\
\sigma_2 \\
k_1 \\
k_2 \\
k_1
\end{pmatrix}
$$

The action of the stabilizer as in (2) then result in the following twisted weighted local quiver:

$$
\begin{array}{ccc}
\delta_1, \delta_2, \delta_3 & \rightarrow & \sigma_1, \sigma_2, \sigma_3 \\
\downarrow & \\
\downarrow & \\
\downarrow & \\
\downarrow & \\
\downarrow & \\
\downarrow & \\
\downarrow & \\
\downarrow & \\
\downarrow & \\
\end{array}
$$

with weights in $\mu_4$. 

---

**THE LOCAL STRUCTURE OF GRADED REPRESENTATIONS**

(3) The action of the twist $\phi$ permutes the vertices and the permutation will be indicated by dotted arrows. The way $\phi$ permutes the arrows is fully determined by the permutation of the vertices, as arrows in the same $\phi$-orbit always have the same weight.
The local twisted weighted quiver setting that occur in a given twisted weighted quiver setting can be calculated using the theory of isomorphism invariant representations.

Following [3], we define an isomorphism invariant representation (ii-representation) of a twisted weighted quiver setting as a representation $S$ such that $S^{|φ|} ≅ S$. Let $σ$ be an automorphism of $S$ corresponding to that isomorphism:

$$ρ_S(σ) = σρ_S(wφ(a)).$$

Note that $σ$ is not uniquely defined but for two choices $σ_1, σ_2$ we have that $σ_1σ_2^{−1} ∈ Aut_{CQ}S$. This enables us to choose $σ$ such that it has the same order as $wφ$.

If $k$ is the order of $wφ$ then we can use $σ$ to define a $Z_k$-action on $S$. This $σ$ makes $S$ into a representation of the smash product $CQ#Z_k$.

and the category of $CQ#Z_k$-representations is equivalent to the category of couples of an ii-representation of $Q$ and a $σ$.

We will also need a straightforward adaptation of a result from [3] and [11]:

**Theorem 6.** The algebra $CQ#Z_k$ is Morita equivalent to the path algebra of a new quiver $Q$. This new quiver is constructed in the following way:

- For every $φ$-orbit of vertices $v^φ \subset Q_0$ there are $k/|v^φ|$ vertices in $Q_0$ parametrized by the characters of $Stab v$. Denote these vertices by $v^φ$, $χ ∈ Stab v^φ$.

- For every $φ$-orbit of arrows $a^φ \subset Q_1$ there are $|Stab a^φ|/|Stab a|$ arrows in $Q_1$ parametrized by couples $(χ_1,χ_2) ∈ Stab h(a)^V × Stab t(a)^V$ such that $χ_1|Stab a = w(a)|Stab aχ_2|Stab a$.

If $α$ is the dimension vector of a $CQ$-representation then the corresponding $CQ#Z_k$-representation is

$$α = \sum_{χ \in Stab v^φ} α_v χ.$$

Moreover as a $Z_k$-representation its character is determined by

$$X(α) := \sum_{v^φ ∈ Q} α(v^φ) \sum_{χ \in Stab v^φ} χ.$$

**Remark 4.** This new quiver is identical to the one quiver defined by L. Le Bruyn in [7].

Now suppose that $S$ and $T$ are two simple ii-representations of $(Q, wφ)$ and let $σ$ and $τ$ be two automorphisms that make $S = (S, σ)$ and $T = (T, τ)$ into $CQ#Z_k$ representations. We denote the dimension vectors of $S$ and $T$ by $α$ and $β$ and the corresponding $Q$-dimension vectors by $α$ and $β$.

We can now define three operations on these dimension vectors. To decrease the notation load we will denote the semisimple subalgebra of $CQ$ generated by the vertices by $CI$ instead of $(CQ)_0$.

1. The dual dimension vector $α^V$ is the dimension vector of the dual representation $S^V = Hom_{CI}(S, CI)$:

$$α^V(v_χ) := α(v_{χ^{-1}}).$$

2. The flow dimension vector $|α|$ shifts all dimensions along the arrows:

$$|α|(v_χ) = \sum_{a ∈ Q_1, h(a) = v_χ} α(s(a)).$$

It is easy to check that this is also the dimension vector of the $CI#Z_k$-representation $S ⊗_{CI} (CQ)_1$ (where $(CQ)_1$ is the $CI#Z_k$-bimodule spanned by the arrows in $Q$).

3. The convolution product $α ⋆ β$ is the dimension of the tensor product $S ⊗_{CI} T$:

$$α ⋆ β(v_χ) := \sum_{φ ∈ Stab v^φ} α(φ)β(φ_{χ^{-1}}).$$
Lemma 5. The character of $\text{Ext}_{\mathbb{C}Q}(S, T)$ as a $\mathbb{Z}_k$-representation is

$$1\delta_{ST} - X(\tilde{\alpha}^\vee \ast \tilde{\beta} - |\tilde{\alpha}^\vee| \ast \tilde{\beta}),$$

where 1 is the trivial character.

Proof. We have the following exact sequence

$$0 \rightarrow \text{Hom}_{\mathbb{C}Q}(S, T) \rightarrow S^\vee \otimes_{\mathbb{C}I} T \rightarrow S^\vee \otimes_{\mathbb{C}I} (\mathbb{C}Q)_1 \otimes_{\mathbb{C}I} T \rightarrow \text{Ext}_{\mathbb{C}Q}(S, T) \rightarrow 0,$$

which also an exact sequence of $\mathbb{Z}_k$-representations. Using the definitions of the operations on the dimension vectors and the fact that $\mathbb{Z}_k$-representations are semisimple one obtains the formula above. □

Note that for $k = 1$ one obtains the usual formula involving the Euler form because then $X$ reduces then to taking the length $|\alpha| = \sum\alpha_v$ of the vector and $\ast$ reduces to the product in $\mathbb{C}I$.

Lemma 6. If $\tilde{\delta} = (S, \sigma)$ is a $\mathbb{C}Q \# \mathbb{Z}_k$-representation with setting $Q = (\tilde{Q}, \tilde{\alpha})$ and $q \in \mathbb{N}$ then $(S, \sigma^q)$ is a $\mathbb{C}Q \# \mathbb{Z}_k^{q}$-representation. The corresponding new setting will be denoted by $(\tilde{Q}^q, \tilde{\alpha}^q)$. This new setting can be derived from $(\tilde{Q}, \tilde{\alpha})$ as follows: Split every vertex $\tilde{v}_\chi \in \tilde{Q}$ in $\ell = \gcd(|\tilde{\phi}|, q)$ vertices $\tilde{v}_{\chi,0}, \ldots, \tilde{v}_{\chi,q-1}$, then identify vertices labeled with the same character: $\tilde{v}_{\chi,i}^q = \tilde{v}_{\chi, i \pmod{q}}$. For the dimension vector $\tilde{\alpha}^q$ we obtain the formula

$$\tilde{\alpha}^q(v_{\chi,i}) = \sum_{\varphi \in \text{Stab}_{\tilde{v}} \chi = \chi^q} \tilde{\alpha}(v_{\varphi}).$$

Proof. Straightforward. □

Now if $S$ is a semisimple $i$-representation of $(Q, \tilde{w})$ we can decompose $S$ as follows

$$(S_1 \oplus \tilde{w}S_1 \oplus \cdots \oplus (\tilde{w})^{k_1-1}S_1) \oplus (\cdots \oplus (S_p \oplus \tilde{w}S_p \oplus \cdots \oplus (\tilde{w})^{k_p-1}S_p) \oplus \cdots) \oplus (\tilde{w})^{k_p}S_p),$$

where $S_i$ is a simple $i$-representation of the twisted weighted quiver $(Q, \tilde{w}^{k_i})$. Choose corresponding $\sigma_i$ to make the $S_i$ into $\mathbb{C}Q \# k_i\mathbb{Z}_k$-representations and let $\tilde{\alpha}_i$ be the corresponding dimension vector for the quiver $\tilde{Q}^{k_i}$. Note that $\tilde{w}^{k_i}S_i$ has dimension vector $\phi\tilde{\alpha}_i$, where $\phi$ acts on the vertices of $\tilde{Q}^{k_i}$, by $v_\chi \mapsto (\phi v)_\chi$.

Theorem 7. The local twisted quiver setting of $S$ is a quiver $Q_L$ with one vertex $s_L^i$ for each simple $(\tilde{w})^{k_i}S_i$ in the decomposition of $S$. The local twist twists the vertices according to $\tilde{w}$. The number of arrows between the vertices $s_L^i$ and $s_L^j$ correspond to the dimension of $\text{Ext}_{\mathbb{C}Q}(S_L^i, S_L^j)$. The $\mathbb{Z}_k$-weights of the arrows correspond to lifts of $\text{lcm}(k_i, k_j)\mathbb{Z}_k$-weights of the character of $\text{Ext}_{\mathbb{C}Q}(S_L^i, S_L^j)$ viewed as a $\mathbb{Z}_k/\text{lcm}(k_i, k_j)$-representation.

$$\delta_{ij} \delta_{\mu \nu} 1 - X(\phi^{\mu} \tilde{\alpha}_i^\vee \ast \phi^{\nu} \tilde{\alpha}_j - |\phi^{\mu} \tilde{\alpha}_i^\vee| \ast \phi^{\nu} \tilde{\alpha}_j).$$

Proof. This is a direct consequence of the previous lemmas. □

6. ADJUSTED REDUCTION STEPS

The first thing we can ask about the quotient space $\text{giss}_k^\vee Q$ is for which twisted weighted settings this quotient space is smooth. In the case of ordinary quivers this has been done in [1]. The first author introduces there 3 reduction steps, that reduce the complexity of the quiver while keeping the quotient space invariant. The quotient space of a strongly connected quiver setting (this means that between every two vertices there is a path in both directions) is smooth and only if after applying all possible reductions, the quiver setting is one of the three below:

We will adjust these reduction steps in such a way that the quotient space is invariant under the $\text{GL}_n \rtimes \mu_t$-action. As in [1], we will call a quiver that can not be simplified using such moves, is called reduced. We will first state the reduction steps and after that we indicate how they alter the weights.
(1) **Vertex removal.** If \( \sum_{j=1}^{k} i_j \leq \alpha_v \) or \( \sum_{j=1}^{l} u_j \leq \alpha_v \) we delete the vertex \( v \).

\[
\begin{bmatrix}
\ell_1 & \cdots & \ell_k \\
\mu_1 & \cdots & \mu_l
\end{bmatrix}
\] \( \xrightarrow{\psi} \)

\[
\begin{bmatrix}
\ell_1 & \cdots & \ell_k \\
\mu_1 & \cdots & \mu_l
\end{bmatrix}
\]

(2) **loop removal.** Uncouple the loops on a vertex with dimension 1.

\[
\begin{bmatrix}
\ell_1 & \cdots & \ell_k \\
\mu_1 & \cdots & \mu_l
\end{bmatrix}
\] \( \xrightarrow{\psi} \)

\[
\begin{bmatrix}
\ell_1 & \cdots & \ell_k \\
\mu_1 & \cdots & \mu_l
\end{bmatrix}
\]

(3) **Loop removal.** Uncouple the only loop or marked loops on a vertex with dimension \( m \geq 2 \) which has a neighborhood as shown in one of the pictures below.

\[
\begin{bmatrix}
\ell_1 & \cdots & \ell_k \\
\mu_1 & \cdots & \mu_l
\end{bmatrix}
\] \( \xrightarrow{\psi} \)

\[
\begin{bmatrix}
\ell_1 & \cdots & \ell_k \\
\mu_1 & \cdots & \mu_l
\end{bmatrix}
\]

Because the twist is an automorphism of the quiver setting, the reducibility conditions commute with the twist: if a vertex of a loop is reducible for a quiver setting, then its image under the twist is also reducible.

For the reduction steps \( \tilde{R}_v \) and \( \tilde{R}_v \) this means that we can perform the reductions of all these loops simultaneously and obtain a new quiver setting with the same twisted weight extended in the obvious way to the new vertices and arrows. For \( \tilde{R}_v \) the \( m - 1 \) new arrows have weights \( w_1 + w_\ell, \ldots, w_a + (m - 1)w_\ell \), where \( \ell \) is the loop and \( a \) the unique arrow leaving or arriving.

For reduction step \( \tilde{R}_v \) we must take care. If there are no arrows between vertices in the \( \phi \)-orbit of the reducible vertex \( v \), then we can again reduce all the vertices simultaneously. The new weights are the sums of the weights of couples of the original arrows.

If there are arrows between two of the vertices in the same \( \phi \)-orbit, then the reducibility condition on the vertices implies that these arrows are the only arrows that arrive in or leave these vertices. So \( Q \) is extended Dynkin of type \( A_n \) and we can reduce all vertices except one, to obtain a one vertex quiver with as weight a multiple of the original weight.

**Theorem 8.** If \( (Q, \alpha) \) is a quiver setting with twisted weight \( \psi \) and \( (Q_R, \alpha_R) \) is a reduced setting obtained from \( (Q, \alpha) \), then there is a reduced twisted weight \( \psi_R \) such that

\[
\text{giss}^\psi_{\alpha} Q = \text{giss}^\psi_{\alpha_R} Q_R
\]

From now on, we only consider reduced settings.

# 7. Reduced settings for which giss\(_{\alpha} Q \) is smooth

Because \( \text{rep}_\alpha Q \) is a linear \( GL_n \times \mu_c \)-representation the quotient is a cone, so its worst singularity is situated in the origin. This has the following easy to check consequences:

- \( \text{giss}^\psi_{\alpha} Q \) is smooth iff the minimal number of graded generators of \( \mathbb{C}[\text{giss}^\psi_{\alpha} Q] \) equals the dimension of \( \text{giss}^\psi_{\alpha} Q \).
- If \( \text{giss}^\psi_{\alpha} Q \) is smooth then \( \text{giss}^\psi_{\alpha} Q' \) is smooth for all subquivers \( Q' \) that are closed under the twist, and their reductions.
- If \( \text{giss}^\psi_{\alpha} Q \) is smooth then \( \text{giss}^\psi_{\alpha_L} Q_L \) is smooth for all local quiver settings \( (Q_L, \alpha_L) \) and their reductions.
These consequences can be used to give a complete classification of all twisted weighted quiver settings with a smooth quotient space. The trick is to rule out all other settings by looking at local quivers or subquivers and their reductions which are already known to contain singularities. In order to do that we first need to investigate some simple cases.

Lemma 7. Suppose that $\mathbb{C}[X_1, \ldots, X_k]$ has a faithful action of $\mu_e$ on it and the weight of $X_i$ is $w_i$. Then $\mathbb{C}[X_1, \ldots, X_k]^\mu_e$ is a polynomial ring if and only if for every prime power $p^n | e$ at most one of the weights is not trivial modulo $p^n$.

Proof. As the ring of invariants has the same dimension, it is smooth if and only if it is generated by the functions $X_i$. Suppose that $w_i$ contain singularities. In order to do that we first need to investigate some simple cases. In order to use the technique of local quivers we also need a lemma concerning the dimension vectors of simples.

Lemma 8. For $Q := \begin{array}{c} w_2 \\ w_1 \\ w_1 \\ w_2 \end{array}$ the only weights $(w_1, w_2)$ that make $\text{giss}_\alpha^Q$ smooth, are the trivial weights (i.e. $2w_1, 2w_2 = 0 \mod e$).

Proof. The ring of polynomial functions $\mathbb{C}[\text{iss}_\alpha Q]$ is a free module of rank 2 over the polynomial ring $R = \mathbb{C}[X_{11}, X_{22}, X_{12} + X_{21}]$ the twisted weight gives an action of $\mu_e$ on this ring with weights $2w_1, 2w_2, w_1 + w_2$. If one of the weights $w_1, w_2$ is non-trivial then the ring $R^{\mu_e}$ is not polynomial because of lemma 7. As the generators of $R^{\mu_e}$ form a subset of those of $\mathbb{C}[\text{giss}_\alpha^Q]$ the latter cannot be smooth either.

The twisted quiver setting

$\begin{array}{c} 1 \\ a_2 \\ a_1 \\ b_1 \\ b_2 \end{array}$ with the action of $\mu_2 = < \zeta >$, where $\zeta$ switches the 2 vertices and maps $a_i$ to $b_i$ is smooth. The twist maps $X_{ij} = a_i b_j$ (i.e., the generators of $\mathbb{C}[\text{iss}_\alpha Q]$) to $X_{ji}$ and this means that $\mathbb{C}[\text{giss}_\alpha^Q]$ is given by $\mathbb{C}[\text{iss}_\alpha Q]^{\mu_e} = \mathbb{C}[X_{11}, X_{22}, X_{12} + X_{21}, X_{12}X_{21}]$ and as $X_{12}X_{21} = X_{11}X_{22}$, we obtain that $\text{giss}_\alpha^Q$ is just $\mathbb{A}^3$. □

In order to use the technique of local quivers we also need a lemma concerning the dimension vectors of simples:

Lemma 9. If $(Q, \alpha)$ is a strongly connected reduced quiver setting and $v$ is the vertex with highest dimension then both $\text{iss}_\alpha Q$ and $\text{iss}_{\alpha - e_v} Q$ contain simple representations. ($e_v$ assigns 1 to $v$ and 0 to the other vertices)

Proof. This is a straightforward combination of the definition of reducedness in [1] and the characterization of dimension vectors of simple quiver representations in [8]. □
Let us now investigate the structure of the quotient space $\text{giss}^\psi_\alpha Q$. Because this quotient factors through $\text{iss}_\alpha Q$, we can delete all arrows that are not contained in any cycle.

First we assume that $(Q, \alpha)$ is strongly connected.

**Theorem 9.** If a strongly connected reduced twisted weighted quiver setting is smooth then either the untwisted unweighted setting is smooth or it is equal to

$$\Theta := \begin{tikzpicture}[baseline=(current bounding box.center) , auto,  mark size = 0.5mm]  
  
  \node[fill=white,shape=circle,inner sep=2pt] (1) at (0,0) {};  \node[fill=white,shape=circle,inner sep=2pt] (2) at (0.5,0.5) {1};  \node[fill=white,shape=circle,inner sep=2pt] (3) at (0.5,-0.5) {1};  \node[fill=white,shape=circle,inner sep=2pt] (4) at (2.5,0) {1};  \node[fill=white,shape=circle,inner sep=2pt] (5) at (0.5,1.5) {a};  \node[fill=white,shape=circle,inner sep=2pt] (6) at (2.5,-0.5) {a};  
  
  
  \draw[thick,->] (1) to (5);  \draw[thick,->] (5) to (1);  \draw[thick,->] (1) to (2);  \draw[thick,->] (2) to (3);  \draw[thick,->] (2) to (6);  \draw[thick,->] (6) to (1);  \draw[thick,->] (2) to (4);  \draw[thick,->] (4) to (3);  \draw[thick,->] (4) to (6);  \draw[thick,->] (5) to (4);  \draw[thick,->] (5) to (6);  \draw[thick,->] (6) to (3);  \draw[thick,->] (3) to (2);  
\end{tikzpicture}$$

with trivial weights.

**Proof.** If the setting is only weighted and not twisted, then the setting can only be smooth if the unweighted setting is smooth: if the primitive cycle $c$ is a generator for $\mathbb{C}[\text{iss}_\alpha Q]$ and its weight is a $k^{th}$ root of one, then $c^k$ must be a generator for $\mathbb{C}[\text{giss}^\psi_\alpha Q]$. The number of generators of $\mathbb{C}[\text{giss}^\psi_\alpha Q]$ is at least as big as that of $\mathbb{C}[\text{iss}_\alpha Q]$. If $\mathbb{C}[\text{iss}_\alpha Q]$ is not smooth and $\mathbb{C}[\text{giss}^\psi_\alpha Q]$ has the same dimension, the latter cannot be smooth either.

Now suppose $(Q, \alpha, w, \phi)$ has a nontrivial twist.

If $\alpha = 1$, we discern two cases

- **All cycles run through all the vertices of $Q$**. In that case $Q$ looks like

  $$\Theta_{p,q} = \begin{tikzpicture}[baseline=(current bounding box.center) , auto,  mark size = 0.5mm]  
  
  \node[fill=white,shape=circle,inner sep=2pt] (1) at (0,0) {1};  \node[fill=white,shape=circle,inner sep=2pt] (2) at (1,1) {1};  \node[fill=white,shape=circle,inner sep=2pt] (3) at (1,-1) {1};  \node[fill=white,shape=circle,inner sep=2pt] (4) at (2,0) {1};  
  
  \draw[thick,->] (1) to (2);  \draw[thick,->] (2) to (3);  \draw[thick,->] (3) to (4);  \draw[thick,->] (4) to (1);  
\end{tikzpicture}$$

  where $p$ is the number of vertices and $q \geq 2$ the number of arrows between 2 consecutive vertices. The order of $\phi$ must be a divisor of $p$.

  The number of generators of $\mathbb{C}[\text{iss}_Q]$ is $q^p$. The twist permutes these generators and for every orbit we have at least one generator of $\mathbb{C}[\text{giss}^\psi_\alpha Q]$: the sum of the elements in the orbit raised to the appropriate power in order to let the weight vanish. Because the dimension of $\text{giss}^\psi_\alpha Q$ is $p(q - 1) + 1$, it can only be smooth if the number of orbits is smaller than $p + 1$. However the number of orbits is bigger than $(p^p - q)/p + q$ because an orbit has at most $p$ elements and there are at least $q$ fixed cycles. This condition implies that $p = 2, 3$ and $q = 2$. If $p = 3$ we can find more generators: take an orbit of order 3, sum the three products of 2 factors’ and raise it to the appropriate power.

  From lemma 8 we know that the only possibility for $p = 2$ is $\Theta := \Theta_{2,2}$ with trivial weights.

- **There is a cycle that runs not through all the vertices of $Q$**. This means that there is a strongly connected full subquiver not containing all vertices. Let $Q_M$ be maximal in the subset of all such subquivers. Consider the local quiver $Q_L$ coming from the representation that assigns ones to the arrows in $Q_M$ and zeros to the other arrows.

  The number of vertices of this quiver is smaller than the original quiver and the local twist fixes the vertex coming from $Q_M$.

  If this vertex can be reduced then it has exactly one arrow arriving and one arrow leaving. These two arrows must connect the vertex to two different vertices otherwise $Q_M$ was not maximal. This means that the twist fixes these vertices as well and after the reduction no other vertex can be reduced anymore.

  So in all cases the local quiver $Q_L$ reduces to a reduced quiver with less vertices and at least one vertex that is fixed by the local twist. Therefore it cannot be $\Theta$ and we can use induction on the vertices.

  Finally, suppose that $(Q, \alpha)$ is reduced and $\alpha \neq 1$. Let $v$ be a vertex with the highest dimension and consider the decomposition $(Q, \alpha - \epsilon_v) \oplus (Q, \epsilon_v)$. Lemma 9 implies that there exist indeed simple quivers with dimension vector $\alpha - \epsilon_v$ and that the number of arrows between the two vertices in the local quiver is bigger than one. The local quiver is thus untwisted and reduced (up to loops).
If \((Q, \alpha)\) is not strongly connected then we can assume that there is at most one component that is smooth because we can change the smooth parts to a quiver with one vertex and a number of loops equal to the dimension of the smooth parts. The action of the automorphism on the smooth part can be diagonalized to give weights on the loops.

If \((Q, \alpha)\) contains two or more components \(Q_1 \sqcup \cdots \sqcup Q_s\), then we will prove that \(\text{giss}_s^\alpha Q\) cannot be smooth, unless the twisted weight acts trivially on all \(\text{iss}_s Q_i\) except one. By the previous paragraph, we can take a component \(Q_i\) such that one of the other components is not smooth. If the finite action on this component \(\text{iss}_s Q_i\) is not trivial, then there exists a representation not fixed by \(\psi_0\). Because \(Q_i\) is reduced, \(\text{iss}_s Q_i\) contains simples (lemma 9) and we can choose a simple with trivial stabilizer. Look at the local quiver corresponding to the direct sum of this simple with the trivial simples of the other components. This setting \((Q_L, \alpha_L)\) is untwisted and it contains the same components as the original quiver except \(Q_i\). Because one of the components is not smooth, \(\text{giss}_n^\psi Q_L = \text{iss}_\alpha Q_L\) is also not smooth.

8. GRADED CAYLEY-HAMILTON ALGEBRAS

Let us switch to the setting where \(A\) is a graded Cayley-Hamilton algebra and where we are interested in the trace preserving properties of this algebras. Recall from [10] that an \(n^{\text{th}}\) Cayley-Hamilton algebra is a finitely generated algebra \(A\) equipped with a trace, that is a \(Z(A)\)-linear map \(\text{tr} : A \to Z(A)\), such that for all \(a, b\) in \(A\)

- \(\text{tr}(ab) = \text{tr}(ba)\),
- \(\text{tr}(1) = n\),
- \(\chi_{n,a}(a) = 0\),

where \(\chi_{n,a}(X)\) is the \(n^{\text{th}}\) Cayley-Hamilton identity expressed in the traces of powers of \(a\). The matrix algebra \(\text{Mat}_{n \times n}(\mathbb{C})\) with the natural trace is the simplest example of such a Cayley-Hamilton algebra.

Analogous to the previous situation we define the representation varieties:

**Definition 6.** Let \(A \in \text{CH}_n\) be a graded Cayley-Hamilton order. Then we define

1. \(\text{trep}_n^\alpha A\) for the set of semistable trace preserving representations.
2. \(\text{gtrep}_n A = \text{proj} C[\text{trep}_n A] = \text{trep}_n^\alpha A \sslash \text{GL}_n\).
3. \(\text{gtiss}_n A = \text{gtrep}_n A \sslash \text{GL}_n\).

Again, \(\text{gtiss}_n A\) classifies the isoclasses of trace preserving (semistable) semisimple representations up to graded isomorphism.

Similar to the observation that the \(\text{tiss}_n A\) can be seen as \(\text{spec} Z(A)\), we have that \(\text{gtiss}_n A\) exactly coincide with \(\text{proj} Z(A)\).

The only extra feature we encounter here, is that the local structure of a Cayley-Hamilton algebra is described by marked quivers. These are quivers of which some of the loops are marked. Marked loops are represented by traceless matrices. If we introduce weights and twists this situation may lead to loops \(\ell_1, \ldots, \ell_s\) on different vertices in the same orbit of the twist. As these loops originate from a bigger trace 0 representation, this means that the sum of the traces of \(\ell_1, \ldots, \ell_s\) is 0. This situation can be undone by adding to the (twisted weighted) local quiver setting an extra vertex \(v\) with dimension 1 and \(s - 1\) loops on \(v\) and marking the loops \(\ell_1, \ldots, \ell_s\).

9. NON-COMMUTATIVE BLOW-UP OF THE QUANTUM PLANE

In the commutative case, a blow-up \(\mathbb{A}^2\) of \(\mathbb{A}^2\) in the origin can be seen as the variety \(\mathbb{V}(xY - yX) \subset \mathbb{A}^2 \times \mathbb{P}^1\), with \(x\) and \(y\) the affine coordinates and \(X\) and \(Y\) the projective coordinates. The coordinate ring of \(\mathbb{A}^2\) is given by \(\mathbb{C}[x, y]\) and we can extend this ring to the graded algebra

\[ R = \mathbb{C}[x, y] \oplus (x, y)t \oplus (x, y)^2t^2 \oplus \cdots \subset \mathbb{C}[x, y][t].\]
$R$ is generated by two elements $x, y$ of degree zero and two elements $X = xt, Y = yt$ of degree one, obeying the homogeneous relation $xY = yX$. We see that $\mathbb{k}^2 = \text{proj } R$ and that the projection morphism $\mathbb{k}^2 \rightarrow \mathbb{k}^2$ is given by the inclusion (in degree zero) $\mathbb{C}[x, y] \subseteq \mathbb{R}$.

We will try to make a non-commutative equivalent of this by looking at a similar construction, but now starting with the quantum plane $\mathbb{C}_q[u, v]$ at $q = -1$ instead of the affine plane. With $B = \mathbb{C}_q[u, v] = \mathbb{C}(u, v)/(uv + vu)$ and center $Z(B) = \mathbb{C}[u^2, v^2]$, we can associate the graded algebra

$$A = B \oplus (u, v)t \oplus (u, v)^2t^2 \oplus \ldots \mathbb{C}_q[u, v][t].$$

This algebra can be seen as the non-commutative blow-up in $(0, 0)$ of the ring $B$. $A$ is generated by two elements $u$ and $v$ of degree zero and two elements $U = ut$ and $V = vt$ satisfying a number of relations and with $u^2, v^2, U^2$ and $V^2$ central homogeneous elements.

Let us now concentrate on the closed orbits over the point $(0, 0)$. For those, either $U^2$ or $V^2$ must be invertible and we consider the case where $U^2$ is invertible.

If we take the graded localization at $U^2$, we obtain

$$A' = A_{U^2}^0 = \mathbb{C}_{-1}[u, w][U, U^{-1}]$$

where $w = VU^{-1}$. This means we want to look at the positively graded ring

$$A' = \frac{\mathbb{C}(u, w, U)}{(uw + wu, uU + Uw, uU - Uu)}$$

with the degree of $u$ and $w$ equal to zero and the degree of $U$ equal to 1. We need to check whether $\text{trep}_2 A'$ is smooth in the closed orbits lying over $(0, 0) \in \mathbb{k}^2$. Representatives of these orbits are of one of the following types:

1. simple type: $\rho_s$ sending $(u, w, U)$ to

   $$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ b & 0 \end{bmatrix}$$

2. semisimple type: $\rho_{ss}$ sending $(u, w, U)$ to

   $$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}$$

with $a, b \in \mathbb{C}^*$. $\text{trep}_2 A'$ is of dimension 6, so we need to verify that the tangent space to $\text{trep}_2 A'$ is of dimension 6 in the indicated points. In order to compute the tangent space, we work in $M_2(\mathbb{C}[[\varepsilon]])$ and consider

$$u_s = \rho_s(u) + \varepsilon \begin{bmatrix} \alpha_s^w & \beta_s^w \\ \gamma_s^w & -\alpha_s^w \end{bmatrix}, \quad w_s = \rho_s(w) + \varepsilon \begin{bmatrix} \alpha_s^w & \beta_s^w \\ \gamma_s^w & -\alpha_s^w \end{bmatrix}, \quad U_s = \rho_s(U) + \varepsilon \begin{bmatrix} \alpha_s^U & \beta_s^U \\ \gamma_s^U & -\alpha_s^U \end{bmatrix}.$$

The conditions on $(u_s, w_s, U_s)$ are the defining relations of $A'$:

$$u_s w_s + w_s u_s = w_s U_s + U_s w_s = u_s U_s - U_s u_s = 0$$

and with $\varepsilon^2 = 0$, this leads to

$$\alpha_s^w = 0, \quad \gamma_s^w = b \beta_s^w, \quad \alpha_s^U = \frac{\gamma_s^w + b \beta_s^w}{2a},$$

and with 3 linear relations, this yields a 6-dimensional tangent space. A similar construction (and notation) lead for the semisimple points to

$$\beta_s^w = \gamma_s^w = -\alpha_s^w = 0,$$

which again leads to a six dimensional subspace and the smoothness of $\text{trep}_2 A'$ is verified.

We wonder how the (twisted weighted) local quiver in these orbits look like. The stabilizer subgroups of $\text{GL}_2 \times \mathbb{C}^*$ are the following:
Proposition 2. The (weighted) local quivers are as follows:

1. In simple points $x$ the stabilizer $G_x$ looks like $\mathbb{C}^* \times \mu_2$ where the generator of $\mu_2$ is given by

$$
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}, -1
$$

2. In semisimple points $x$ the stabilizer $G_x$ looks like $(\mathbb{C}^* \times \mathbb{C}^*) \times \mu_2$ where the component of $-1 \in \mu_2$ is embedded by

$$
\begin{bmatrix}
0 & \delta \\
\eta & 0
\end{bmatrix}, -1
$$

Proof. In a simple point $x$, the action of the stabilizer $\mathbb{C}^* \times \mu_2$ on a tangent vector is given by

$$
\left[
\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}
\right]
\left[
\begin{array}{ccc}
0 & \beta^w_s & \alpha^w_s \\
\beta^w_s & 0 & -\alpha^w_s \\
\alpha^U_s & \beta^U_s & -\alpha^U_s
\end{array}
\right]
\left[
\begin{array}{c}
1 \\
0 \\
-1
\end{array}
\right].
$$

The action on the individual coefficients can therefore be expressed by

$$
\beta^w_s \mapsto -\beta^w_s, \quad \alpha^w_s \mapsto -\alpha^w_s,
$$

(3) $\gamma^w_s \mapsto -\gamma^w_s, \quad \beta^U_s \mapsto \beta^U_s, \quad \gamma^U_s \mapsto \gamma^U_s.$

We now need to calculate the submodule corresponding to the tangent space of the orbit of $x$. This can be done by calculating

$$
(1 + \varepsilon^{\deg} \left[ x_1 \ x_2 \\
x_3 \ x_4 \right]) Z \left[ x_1 \ x_2 \\
x_3 \ x_4 \right]
$$

with $Z = \rho_s(u), \rho_s(w), \rho_s(U)$ and $\varepsilon^2 = 0$. This result in the space

$$
\left[
\begin{array}{cccc}
0 & 0 & -2ax_2 & -2ax_3 \\
0 & 0 & b(t+x_4-x_1) & t+x_1-x_4 \\
2ax_3 & 0 & x_3-bx_2 & 0
\end{array}
\right],
$$

which under the action of $G_x$ has the mapping

$$
x_2 \mapsto -x_2, \quad x_3 \mapsto -x_3, \quad t \mapsto t, \quad (x_1-x_4) \mapsto (x_1-x_4)
$$

and by identifying variables, we can remove line (3), to arrive at the desired twisted weighted local quiver setting.

In semisimple points, the action of $(\mathbb{C}^* \times \mathbb{C}^*) \times \mu_2$ on a tangent vector is given by

$$
\left[
\begin{array}{cc}
0 & \delta \\
\eta & 0
\end{array}
\right]
\left[
\begin{array}{ccc}
\alpha^u_{ss} & 0 & \alpha^w_{ss} \\
0 & -\alpha^u_{ss} & \beta^w_{ss} \\
\alpha^U_{ss} & \beta^U_{ss} & -\alpha^U_{ss}
\end{array}
\right]
\left[
\begin{array}{c}
0 \\
\delta^{-1} \\
\eta^{-1}
\end{array}
\right].
$$

which leads to

$$
\alpha^u_{ss} \mapsto -\alpha^u_{ss}, \quad \beta^w_{ss} \mapsto \delta^w_{ss}\eta^{-1}, \quad \gamma^w_{ss} \mapsto \eta\beta^w_{ss}\delta^{-1},
$$

(5) $\alpha^U_{ss} \mapsto \alpha^U_{ss}, \quad \beta^U_{ss} \mapsto -\delta^U_{ss}\eta^{-1}, \quad \gamma^U_{ss} \mapsto -\eta\beta^U_{ss}\delta^{-1}.$

If we do the calculation of (4), we have the space

$$
\left[
\begin{array}{cccc}
0 & 0 & at & -2ax_2 \\
0 & 0 & 2ax_3 & -at
\end{array}
\right],
$$

and the action of $G_x$ yields

$$
t \mapsto t, \quad x_2 \mapsto -\delta x_3\eta^{-1}, \quad -\eta x_3\delta^{-1}.$$

THE LOCAL STRUCTURE OF GRADED REPRESENTATIONS 17
This makes us delete the variables of (5) and the remaining variables with the action on them can be encoded in the twisted weighted local quiver setting

\[
\begin{array}{c}
1 \\
\downarrow \\
2 \\
\downarrow \\
3
\end{array}
\]

with the action of \( \mu_2 \).

REFERENCES