QUIVER QUOTIENT VARIETIES AND COMPLETE INTERSECTIONS

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ABSTRACT. In this paper we classify all the symmetric quivers and corresponding dimension vectors whose quotient space, classifying the semisimple representation classes, is a complete intersection. The result we obtain is that such quivers can be reduced to a few number of basic quivers, using some elementary types of reduction.

1. Motivation

Quiver varieties and their quotients often appear in problems of representation theory and invariant theory. For many interesting classes of algebras, like formally smooth algebras, the variety classifying its isomorphism classes of n-dimensional semisimple representations, can locally be seen as the quotient variety of a quiver.

To be more precise, Let A be a finitely generated algebra and let $\mathsf{Rep}_n A$ denote the variety of its n-dimensional complex representations. On this space we have the standard action of $\mathsf{GL}_n(\mathbb{C})$ by conjugation and if we take the algebraic quotient of this action we obtain a new space classifying all equivalence classes of semisimple representations, $\mathsf{iss}_n A = \mathsf{Rep}_n A/\mathsf{GL}_n(\mathbb{C})$.

Suppose now that A is n-Cayley-Smooth, which is equivalent to the demand that $\mathsf{Rep}_n A$ is a smooth variety, and let $p \in \mathsf{iss}_n A$ correspond to a semisimple representation class. One can construct a quiver setting (Q, α) such that there is an étale neighborhood of p isomorphic to a neighborhood of the zero representation in the quotient variety $\mathsf{iss}_\alpha Q$ (for the definition we refer to paragraph 2). This quiver setting is called the local quiver of p and its structure depends on the decomposition of p in simple representations (see [5] and [11]).

This technique for Cayley-Smooth algebras can be extended to many moduli space problems (see [9]) because using the technique of universal localizations (see [1] and [12]), moduli spaces can locally be seen as quotient spaces of Cayley-Smooth algebras.

Because the local structure of $\mathsf{iss}_n A$ is always given by quiver quotient varieties, it is interesting to look at the local structures of these varieties. In [3] and [4] all the quiver settings are determined for which $\mathsf{iss}_\alpha Q$ is a smooth variety. Quiver settings with this property are called *coregular*. The classification of coregular quivers reduces every coregular quiver to 3 basic coregular types using three reduction moves: $\mathcal{R}_I, \dots, \mathcal{R}_{III}$ (see 3.6).

In this paper we classify the symmetric quiver settings the corresponding $\mathsf{iss}_\alpha Q$ is a complete intersection.

Definition 1.1. A variety V of dimension n is called a complete intersection (C.I.) if

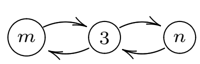
$$\mathbb{C}[V] \cong \mathbb{C}[X_1, \dots, X_k]/(f_1, \dots, f_l)$$

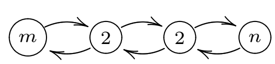
such that $k - l = n$.

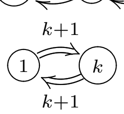
It is impossible to give a list of all symmetric quiver settings whose $\mathsf{iss}_\alpha Q$ is a C.I. Therefore we restrict to strongly connected quiver settings without loops that cannot be reduced using the reduction moves $\mathcal{R}_I, \dots, \mathcal{R}_{III}$ and that are not the connected sum of two smaller quiver settings (see 3.1). Such settings are called prime reduced without loops.

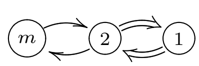
The main result of the paper is the following:

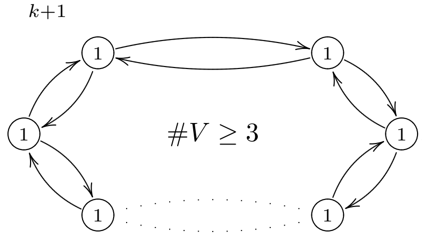
Theorem 1.1. Let (Q, α) be a symmetric prime reduced quiver setting without loops. If $\mathsf{iss}_\alpha Q$ is a complete intersection then (Q, α) is either coregular or one of the following list.

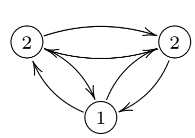
- $I_a =$ 

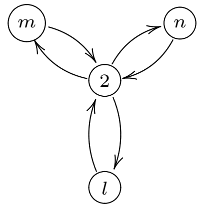
, $I_b =$ 

,
- $II_a =$ 

$II_b =$ 

- $III_a =$ 

, $III_b =$ 

- $IV =$  .

2. DEFINITIONS AND NOTATIONS

A *quiver* $Q = (V, A, s, t)$ consists of a set of vertices V, a set of arrows A between those vertices and maps $s, t : A \to V$ which assign to each arrow its starting and terminating vertex. We also denote this as

$$\textcircled{t(a)} \xleftarrow{a} \textcircled{s(a)} .$$

A *dimension vector* of a quiver is a map $\alpha : V \to \mathbb{N}$, the size of a dimension vector is defined as $|\alpha| := \sum_{v \in V} \alpha_v$. A couple (Q, α) consisting of a quiver and a dimension vector is called a *quiver setting* and for every vertex $v \in V$, α_v is referred to as the dimension of v. The dimension of the vertex is usually written inside the vertex. If no vertex has dimension zero, the quiver setting is called *sincere*.

The space of dimension vectors admits a canonical basis of dimension vectors of the form

$$\epsilon_v : V \to \mathbb{N} : w \mapsto \delta_{vw},$$

where δ is the Kronecker delta. On the space of dimension vectors we can also define a bilinear form $\chi_Q : \mathbb{N}^{\#V} \times \mathbb{N}^{\#V} \to \mathbb{Z}$. This form is called the Euler form and it is determined by the following matrix:

$$m_{vw} = \chi_Q(\epsilon_v, \epsilon_w) := \delta_{vw} - \#\{a | \textcircled{v} \xleftarrow{a} \textcircled{w}\}.$$

A quiver $Q = (V, A, s, t)$ is *symmetric* if and only if its Euler form is symmetric which implies that the number of arrows between two vertices is the same in either direction.

An α*-dimensional complex representation* W of Q assigns to each vertex v a linear space $\mathbb{C}^{\alpha_v}$ and to each arrow a a matrix

$$W_a \in \mathsf{Mat}_{\alpha_{t(a)} \times \alpha_{s(a)}}(\mathbb{C})$$

The space of all α-dimensional representations is denoted by $\mathsf{Rep}_\alpha Q$.

$$\mathsf{Rep}_\alpha Q := \bigoplus_{a \in A} \mathsf{Mat}_{\alpha_{t(a)} \times \alpha_{s(a)}}(\mathbb{C})$$

To the dimension vector α we can also assign a reductive group

$$\mathsf{GL}_\alpha := \bigoplus_{v\in V} \mathsf{GL}_{\alpha_v}(\mathbb{C}).$$

An element of this group, g, has a natural action on $\mathsf{Rep}_\alpha Q$:

$$W := (W_a)_{a\in A},\ W^g := (g_{t(a)} W_a g_{s(a)}^{-1})_{a\in A}$$

Definition 2.1. Denote the algebraic quotient of $\mathsf{Rep}_\alpha Q$ by GL_α with $\mathsf{iss}_\alpha Q$. The points of this space are the closed GL_α-orbits in $\mathsf{Rep}_\alpha Q$. The coordinate ring of this variety is the ring of GL_α-invariant polynomial functions on $\mathsf{Rep}_\alpha Q$.

$$\mathbb{C}[\mathsf{iss}_\alpha Q] := \mathbb{C}[\mathsf{Rep}_\alpha Q]^{\mathsf{GL}_\alpha}$$

For more details of this construction see [10].

Another way of looking at this problem comes from the representation theoretic point of view. Two representations in $\mathsf{Rep}_\alpha Q$ are called equivalent, if they belong to the same orbit under the action of GL_α.

A representation W is called *simple* if the only collections of subspaces $(\mathsf{V}_v)_{v\in V}$, $\mathsf{V}_v \subseteq \mathbb{C}^{\alpha_v}$ having the property

$$\forall a \in A : W_a \mathsf{V}_{s(a)} \subset \mathsf{V}_{t(a)}$$

are the trivial ones (i.e. the collection of zero-dimensional subspaces and $(\mathbb{C}^{\alpha_v})_{v\in V}$).

The direct sum $W \oplus W'$ of two representations W, W' has as dimension vector the sum of the two dimension vectors and as matrices $(W \oplus W')_a := W_a \oplus W'_a$. A representation equivalent to a direct sum of simple representations is called *semisimple*.

In [2] it is proven that an orbit of a representation is closed if and only if this representation is semisimple. So one can also consider $\mathsf{iss}_\alpha Q$ as the space classifying all semisimple α-dimensional representation classes.

In order to study $\mathsf{iss}_\alpha Q$ more closely, we recall some of the results of [5], which studies the local structure of the invariant ring $\mathbb{C}[\mathsf{iss}_\alpha Q]$.

A sequence of arrows $a_1 \dots a_p$ in a quiver Q is called a *path of length p* if $s(a_i) = t(a_{i+1})$, this path is called a cycle if $s(a_p) = t(a_1)$. Every cycle defines a GL_α-invariant polynomial function

$$f_c : \mathsf{Rep}_\alpha Q \to \mathbb{C} : W \mapsto \mathsf{Tr}(W_{a_1} \cdots W_{a_p})$$

We will call a cycle *quasi-primitive* for a dimension vector α if the vertices that are ran through more than once have dimension bigger than 1. By cyclically permuting a cycle and splitting the trace of a product of two 1×1 matrices into a product of traces, we can always decompose an f_c into a product of traces of quasi-primitive cycles. We now have the following result

Theorem 2.1 (Le Bruyn-Procesi)**.** $\mathbb{C}[\mathsf{iss}_\alpha Q]$ is generated by all f_c where c is a quasi-primitive cycle of length smaller than $|\alpha|^2 + 1$.

3. Changing the structure of Quivers

In this section we define the reduction steps in the proof. More information can be found in [3] and [4].

Two vertices v and w are said to be *strongly connected* if there is a path from v to w and vice versa. It is easy to check that this relation is an equivalence so we can divide the set of vertices into equivalence classes V_i. The subquiver Q_i having V_i as set of vertices, and as arrows all arrows between vertices of V_i is called a *strongly connected component* of Q.

Lemma 3.1. If (Q, α) is a quiver setting then

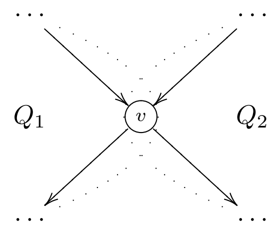
$$\mathbb{C}[\mathsf{iss}_\alpha Q] := \bigotimes_i \mathbb{C}[\mathsf{iss}_{\alpha_i} Q_i]$$

where $Q_i = (V_i, A_i, s_i, t_i)$ are the strongly connected components of Q and $\alpha_i := \alpha|_{V_i}$.

Proof. This follows from the fact that that every quasi-primitive cycle is contained in just one strongly connected component. See [3]. □

Definition 3.1. A quiver $Q = (V, A, s, t)$ is said to be the *connected sum* of 2 subquivers $Q_1 = (V_1, A_1, s_1, t_1)$ and $Q_1 = (V_2, A_2, s_2, t_2)$ at the vertex v, if the two subquivers make up the whole quiver and only intersect in the vertex v. So in symbols $V = V_1 \cup V_2$, $A = A_1 \cup A_2$, $V_1 \cap V_2 = \{v\}$ and $A_1 \cap A_2 = \emptyset$.

$Q_1 \underset{v}{\#} Q_2 :=$



If we connect three or more components we write $Q_1 \underset{v}{\#} Q_2 \underset{w}{\#} Q_3$ instead of $(Q_1 \underset{v}{\#} Q_2) \underset{w}{\#} Q_3$ for sake of simplicity.

Lemma 3.2. Suppose $Q = Q_1 \underset{v}{\#} Q_2$ and $\alpha_v = 1$ then

$$\mathbb{C}[\mathsf{iss}_\alpha Q] := \mathbb{C}[\mathsf{iss}_{\alpha_1} Q_1] \otimes \mathbb{C}[\mathsf{iss}_{\alpha_2} Q_2]$$

where $\alpha_i := \alpha|_{Q_i}$.

Proof. See [3], similar to lemma 3.1. □

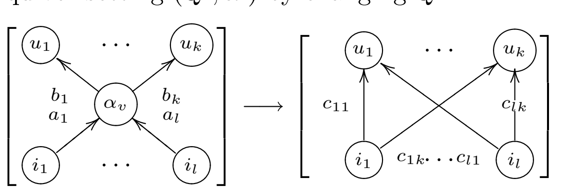
The foregoing lemmas tell us that the ring of invariants of a quiver setting can be written as the tensor product of rings of invariants of strongly connected quiver settings that cannot be split into smaller pieces, by cutting it at a vertex with dimension 1. Sincere quiver settings with this property will be called *prime*.

Apart from cutting we can also perform other operations on quivers, which remove vertices or arrows. Such operations will be called *reductions*.

Lemma 3.3 (Reduction $\mathcal{R}_I$: Removing vertices)**.** Suppose (Q, α) is a quiver setting and v is a vertex without loops such that

$$\chi_Q(\alpha, \epsilon_v) \geq 0 \text{ or } \chi_Q(\epsilon_v, \alpha) \geq 0.$$

Construct a new quiver setting (Q', α') by changing Q:

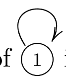


(Some of the top and bottom vertices in the picture may be the same.) Those two quiver settings have isomorphic rings of invariants.

Proof. See [4] □

Lemma 3.4 (Reduction $\mathcal{R}_{II}$: Removing loops of dimension 1)**.** Suppose that (Q, α) is a quiver setting and v a vertex with k loops and $\alpha_v = 1$. Take Q' the corresponding quiver without loops, then the following identity hold

$$\mathbb{C}[\mathsf{iss}_\alpha Q] \cong \mathbb{C}[\mathsf{iss}_\alpha Q'] \otimes \mathbb{C}[X_1, \cdots, X_k]$$

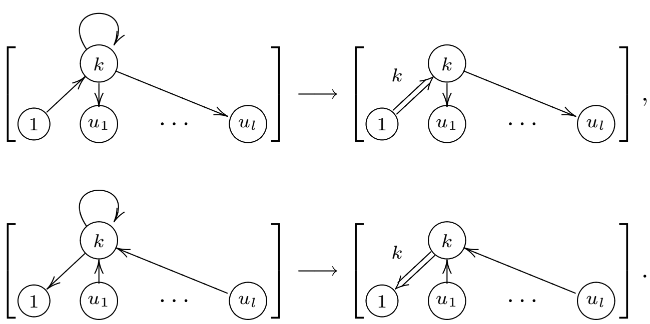
Proof. This follows easily from 3.2 and the fact that the ring of invariants of 

is $\mathbb{C}[X]$. □

Lemma 3.5 (Reduction $\mathcal{R}_{III}$: Removing a loop of higher dimension)**.** Suppose (Q, α) is a quiver setting and v is a vertex of dimension $k \geq 2$ with one loop such that

$$\chi_Q(\alpha, \epsilon_v) = -1 \text{ or } \chi_Q(\epsilon_v, \alpha) = -1.$$

Construct a new quiver setting (Q', α') by changing (Q, α):



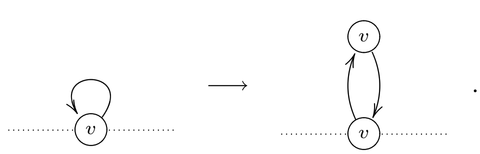
We have the following identity:

$$\mathbb{C}[\mathsf{iss}_\alpha Q] \cong \mathbb{C}[\mathsf{iss}_{\alpha'} Q'] \otimes \mathbb{C}[X_1, \ldots, X_k]$$

Proof. See [4]. □

The lemmas show us that these reductions can be used to simplify the structure of the quiver setting while keeping the ring of invariants intact up to a tensor product with a polynomial ring. Strongly connected sincere quiver settings to which we cannot apply any of the three reduction steps will be called *reduced*.

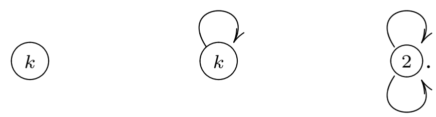
In some cases we want to avoid loops in the quivers we investigate. This can be done easily using the inverse of the first reduction, as is shown in the diagram:



A reduced quiver setting where we have changed every loop as in the diagram will be called *reduced without loops*.

The characterization of coregular quivers, provided in [4], can be stated as:

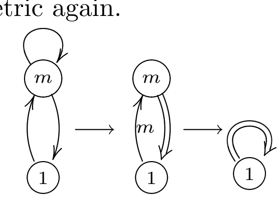
Theorem 3.6. The only reduced quiver settings that are coregular are:



4. Complete intersections

The conditions of primeness and reducedness make also sense to investigate the C.I.-property. Because a tensor product of rings is a C.I. if and only if all the factors are C.I.'s, all the reductions keep the C.I.-property intact and the connected sum of two quiver settings is C.I. if and only if those settings both are C.I.

However if there are no extra demands on the structure of the quivers, the task of classifying all such quiver settings seems a hopeless task, therefore we restrict to the case of symmetric quivers. If (Q,α) is a symmetric quiver setting, application of the moves $\mathcal{R}_I$ and $\mathcal{R}_{II}$ will give us again a symmetric setting. If we apply the $\mathcal{R}_{III}$-move this is not true anymore but in this case we always can apply an $\mathcal{R}_I$ afterwards to make it symmetric again.



Therefore the notion of being reduced or reduced without loops makes sense in the symmetric case. Now we will classify the symmetric prime reduced quiver settings that have a quotient space which is a C.I. For sake of simplicity we will call those quiver settings C.I. as well.

The outline of the proof uses the technique of local quivers. If we want to check whether a certain $\mathsf{iss}_\alpha Q$ is a C.I., we have to check that $\mathsf{iss}_\alpha Q$ is a C.I. in the neighborhood of every point. Take a point $p \in \mathsf{iss}_\alpha Q$, this point will correspond to the isomorphism class of a semisimple representation $V \in \mathsf{Rep}_\alpha Q$, which can be decomposed as a direct sum of simple representations:

$$V = S_1^{\oplus a_1} \oplus \cdots \oplus S_k^{\oplus a_k}.$$

A theorem by Le Bruyn and Procesi [5, Theorem 5] states that we can build a new quiver setting with a similar quotient space, but having a simpler structure.

Theorem 4.1 (Le Bruyn-Procesi)**.** For a point $p \in \mathsf{iss}_\alpha Q$ corresponding to a semisimple representation $V = S_1^{\oplus a_1} \oplus \cdots \oplus S_k^{\oplus a_k}$, there is a quiver setting (Q_p, α_p) such that we have an étale isomorphism between an open neighborhood of the zero representation in $\mathsf{iss}_{\alpha_p} Q_p$ and an open neighborhood of p.

(Q_p, α_p) is called the *local quiver setting* and has the following structure: Q_p is has k vertices corresponding to the set $\{S_i\}$ of simple factors of V and between S_i and S_j the number of arrows equals

$$\delta_{ij} - \chi_Q(\beta_i, \beta_j)$$

where β_i is the dimension vector of the simple component S_i and χ_Q is the Euler form of the quiver Q. The dimension vector α_p is defined to be $(a_1, \ldots, a_k)$, where the a_i are the multiplicities of the simple components in V.

Because being a complete intersection is preserved by étale isomorphisms, see [7], the local quiver setting will be C.I. as soon as the original setting is C.I. This must hold for every point so we have to check all possible points p. If we find a p for which the local quiver is not C.I., we know that the original quotient space is not a C.I.

The structure of the local quiver setting only depends on the dimension vectors of the simple components. Therefore one has to look at decompositions of α into dimension vectors β_i

$$\alpha = a_1\beta_1 + \cdots + a_k\beta_k \text{ (the } \beta_i \text{ need not to be different).}$$

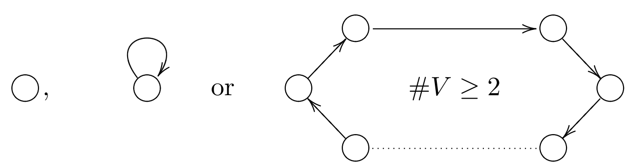
rather than to explicit direct sums of simple representations. One can ask whether there is a semisimple representation corresponding to such a decomposition. The answer to this question will be positive whenever for all the β_i there exist simple representations of that dimension vector and if there are two or more β_i equal, there are at least as many different simple representation classes with dimension vector

β_i (otherwise you cannot make a direct sum with different simple representations having the same dimension vector).

To check the above conditions we must also have a characterization of the dimension vectors for which a quiver has simple representations. We recall a result from Le Bruyn and Procesi [5, Theorem 4].

Theorem 4.2. Let (Q, α) be a sincere quiver setting. There exist simple representations of dimension vector α if and only if

- If Q is of the form



  and $\alpha = \mathbf{1}$ (this is the constant map from the vertices to 1).
- Q is not of the form above, but strongly connected and

$$\forall v \in V : \chi_Q(\alpha, \epsilon_v) \leq 0 \text{ and } \chi_Q(\epsilon_v, \alpha) \leq 0.$$

In both cases the dimension of $\mathsf{iss}_\alpha Q$ is given by $1 - \chi_Q(\alpha, \alpha)$. In all cases except for the one vertex without loops this dimension is bigger then 0, so then there are infinite classes of simples with that dimension vector. In the case of the one vertex v without loops, there is one unique simple representation S_v.

If (Q, α) is not sincere, the simple representations classes are in bijective correspondence to the simple representations classes of the sincere quiver setting obtained by deleting all vertices with dimension zero.

To check whether a given local quiver setting is not C.I. it suffices to look at a subquiver.

Definition 4.1. Define a partial ordering on the set of quivers as follows. A quiver $Q' = (V', A', s', t')$ is smaller than $Q = (V, A, s, t)$ if (up to isomorphism)

$$V' \subseteq V,\ A' \subseteq A,\ s' = s|_{A'} \text{ and } t' = t|_{A'},$$

Q' is called a *subquiver* of Q.

Lemma 4.3. If $\mathsf{iss}_\alpha Q$ is a C.I. and $Q' \leq Q$ then $\mathsf{iss}_{\alpha'} Q'$ is also a C.I., where $\alpha' := \alpha|_{V'}$

Proof. We have an embedding

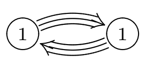
$$\mathsf{Rep}_{\alpha'} Q' \hookrightarrow \mathsf{Rep}_\alpha Q$$

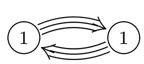
by assigning to the additional arrows in Q zero matrices. So

$$\mathbb{C}[\mathsf{Rep}_\alpha Q] \twoheadrightarrow \mathbb{C}[\mathsf{Rep}_{\alpha'} Q'] \;\Rightarrow\; \mathbb{C}[\mathsf{Rep}_\alpha Q]^{\mathsf{GL}_\alpha} \twoheadrightarrow \mathbb{C}[\mathsf{Rep}_{\alpha'} Q']^{\mathsf{GL}_\alpha}\ .$$

Because the action of GL_α on $\mathsf{Rep}_{\alpha'} Q'$ reduces to that of $\mathsf{GL}_{\alpha'}$, $\mathbb{C}[\mathsf{iss}_{\alpha'} Q']$ is a quotient ring of $\mathbb{C}[\mathsf{iss}_\alpha Q]$. The only relations that we have to divide out are the generators that correspond to a cycle containing one of the additional arrows we put zero, so $\mathbb{C}[\mathsf{iss}_{\alpha'} Q']$, therefore $\mathsf{iss}_{\alpha'} Q'$ is also a C.I. □

5. Proof of the main theorem

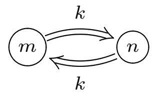
The proof of the main theorem now proceeds as follows. First of all we prove that the quiver setting  is not a complete intersection (lemma 5.1). After that we look at two special quivers settings that are variants of cases II_b and III_b, (we denote these variants with a prime). We prove that these variants are C.I. if and only if they are of the form II_b and III_b (lemmas 5.2-5.3).

In the second part of the proof we will study in depth all possible prime reduced quiver settings and prove that if they are C.I. if they are of the forms $I - VI, II_b'$ or III_b' (lemmas 5.4-5.7). This is done by finding decompositions in simples that give rise to a local quiver that contains . In the proofs we will only state the two components of the decomposition that are important and we will leave the calculation that these components correspond to simples and that $-\chi_Q(\alpha_1, \alpha_2) \geq 3$ to the reader. This calculation can be done visually using the rule that the number of arrows between two components equals

$$\sum_{(w) \leftarrow (v) \in A} \alpha_1(v)\alpha_2(w) - \sum_{v \in V} \alpha_1(v)\alpha_2(v).$$

Showing that I_a, I_b, III_a, IV are indeed C.I. in combination with lemmas 5.2 and 5.3 finishes our proof.

Lemma 5.1. The quiver setting

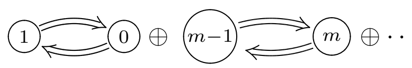


$$, \; m \leq n, k > 1$$

has a quotient space which is a complete intersection if and only if $m = 1$ and $n \geq k - 1$.

Proof. If $m = 1$, we can see $\mathsf{Rep}_\alpha Q$ as the space of couples of a $k \times n$- and a $n \times k$-matrix, and the action reduces to the natural base change action in the n-dimensional space. In [10] it is proven that this is in fact the same as the space of $k \times k$-matrices with rank smaller than n. It is well known that this space is a complete intersection if and only if $k \geq n - 1$, see [6].

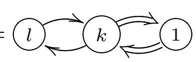
If both n, m are bigger than 1, we can make a decomposition



with $-\chi_Q(\alpha_1, \alpha_2) = 2m - m + 1 = m + 1 \geq 3$. □

Lemma 5.2. The quiver setting

$$II'_b =$$



$$, \; k \geq 1$$

is a complete intersection if and only if $k = 2$.

Proof. Because we only have to check this for reduced settings without loops we can suppose that $l = k, k-1$ (otherwise we can apply $\mathcal{R}_I$ to the middle vertex). For every representation of (Q, α) we define A to be the $2 \times k$-matrix corresponding to the double arrow to the left. B is the $k \times 2$ matrix coming from the double arrow to the right and C is the $k \times k$-matrix coming from the cycle starting in k running through l.

Using 2.1 we can check that $\mathbb{C}[\mathsf{iss}_\alpha Q]$ is generated by the following $5k$ invariants:

$$X_i := \mathsf{Tr} C^i, \; 1 \leq i \leq k \text{ and } Y_j^{st} := (AC^jB)_{st}, \; 0 \leq j \leq k-1, s, t = 1, 2.$$

Using 4.2 we can check that the dimension of $\mathsf{iss}_\alpha Q$ is

$$1 - \chi_Q(\alpha, \alpha) = \begin{cases} 4k & \text{if } l = k, \\ 4k & \text{if } l = k - 1. \end{cases}$$

So if $\mathsf{iss}_\alpha Q$ is a C.I. there must be exactly k relations if $k = l$ and $k + 1$ if $l = k - 1$.

- Case 1: $k = 2$. Suppose first that $l = 2$. We can deduce 2 simple relations

  (i) $\mathsf{Tr}(ABACB) = \mathsf{Tr}((BA)^2C) = Tr(AB)\mathsf{Tr}(ACB) - \det(AB)\mathsf{Tr}(C)$ implies that

$$\sum_{st} Y_0^{st} Y_1^{ts} = \sum_{st} Y_0^{ss} Y_1^{tt} - X_1(Y_0^{11}Y_0^{22} - Y_0^{12}Y_0^{21})$$

  (ii) $\det(ACB) = \det C \det(AB)$ implies that

$$Y_1^{11}Y_1^{22} - Y_1^{12}Y_1^{21} = \frac{1}{2}(X_1^2 - X_2)(Y_0^{11}Y_0^{22} - Y_0^{12}Y_0^{21})$$

  Using the Groebner base algorithm in Maple we can check that these relations generate the ideal of relations. We know that $\mathsf{Dim}\ \mathsf{iss}_\alpha Q = 8 = 10 - 2$ and hence (Q, α) is a complete intersection.

  For $k = 2$ and $l = 1$ the dimension of $\mathsf{Dim}\ \mathsf{iss}_\alpha Q$ is one lower and we have one extra relation $\det C = \frac{1}{2}(X_1^2 - X_2) = 0$. So if $k = 2$ $\mathsf{iss}_\alpha Q$ is always a complete intersection.

- Case 2: $k = 3$. If $k = 3$ we can produce relations similar to (i). Using the Cayley-Hamilton identity in 3 dimensions for the sum of 2 matrices

$$(M+N)^3 - \mathsf{Tr}(M+N)X^2 + \frac{1}{2}((\mathsf{Tr}(M+N))^2 - \mathsf{Tr}(M+N)^2))$$
$$+ \frac{1}{6}(\mathsf{Tr}(M+N)^3 - 3(\mathsf{Tr}(M+N))^3 + 2\mathsf{Tr}(M+N)\mathsf{Tr}(M+N)^2) = 0,$$

  we can express a MNM in terms of all other products of M and N with degree smaller or equal than 3. With this expression we can produce relations of the following forms

$$\sum_{s,t} Y_1^{ts} Y_1^{st} = \mathsf{Tr}(A(CBAC)B) = \dots$$

take $M = C$ and $N = BA$ and substitute MNM,

$$\sum_{s,t} Y_1^{ts} Y_2^{st} = \mathsf{Tr}(A(CBAC)CB) = \dots$$

take $M = C$ and $N = BA$ and substitute MNM,

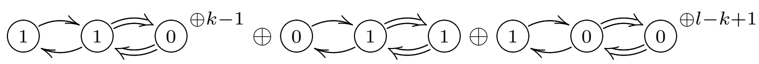
$$\sum_{s,t} Y_2^{ts} Y_2^{st} = \mathsf{Tr}(AC(CBAC)CB) = \dots$$

take $M = C$ and $N = BA$ and substitute MNM,

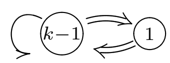
$$\sum_{s,t} Y_2^{ts} Y_2^{st} = \mathsf{Tr}(A(C^2BAC^2)B) = \dots$$

take $M = C^2$ and $N = BA$ and substitute MNM, Using Maple we can check that none of these relations is generated by the others. This, together with the fact that there are no relations of smaller degree, shows us that for $k = l = 3$, $\mathsf{iss}_\alpha Q$ is not a complete intersection. The same can be deduced for $l = 2$.

- Case 3: $k > 3$. We can construct a decomposition of the form



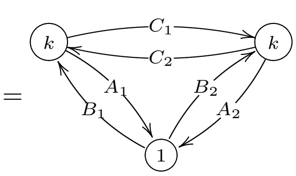
The local quiver contains a subquiver of the form



So if $\mathsf{iss}_\alpha Q$ is a C.I. for k then it is a C.I. for $k - 1$. Continuing this construction we can reduce to the case $k = 3$. For $k = 3$, $\mathsf{iss}_\alpha Q$ is not a C.I. and therefore neither for $k > 3$.

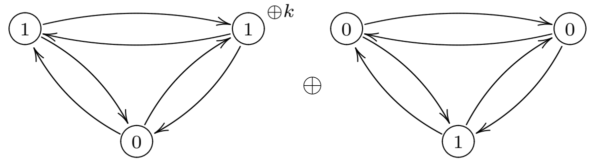
□

Lemma 5.3. The quiver setting

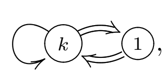
$III'_b =$ 

is a complete intersection if and only if $k \leq 2$.

Proof. If $k > 2$ then $\mathsf{iss}_\alpha Q$ cannot be a complete intersection because we can make the decomposition



of which the local quiver is

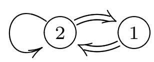


which is not a C.I. by lemma 5.2.

If $k = 2$ the ring of invariants is generated by the folowing traces

$$\begin{array}{rl} \mathsf{Tr} C_1 C_2 & \mathsf{Tr}(C_1 C_2)^2 \\ \mathsf{Tr} A_1 B_1 & \mathsf{Tr} A_1 C_1 C_2 B_1 \\ \mathsf{Tr} A_2 B_2 & \mathsf{Tr} A_2 C_1 C_2 B_2 \\ \mathsf{Tr} A_1 C_1 B_2 & \mathsf{Tr} A_1 C_1 C_2 C_1 B_2 \\ \mathsf{Tr} A_2 C_2 B_1 & \mathsf{Tr} A_2 C_2 C_1 C_2 B_1, \end{array}$$

where the matrices A_i, B_i and C_i correspond to the arrows drawn in the quiver setting. Consider the points in the open subset $U \subset \mathsf{iss}_\alpha Q$ for which $\det C_1 C_2 \neq 0$. These points correspond to orbits that contain a representation for which C_1 is a unit matrix. Simplifying the traces, by setting $C_1 = 1$ and $C_2 = C$, we can consider $\mathbb{C}[\mathsf{iss}_\alpha Q]$ as a subring of the ring of invariants of the quiver setting



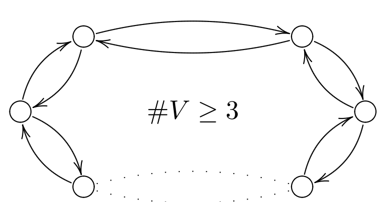
generated by

$$\begin{array}{l} X_1, X_2, Y_0^{11}, Y_1^{11} \\ Y_0^{12}, Y_1^{12}, Y_0^{21}, Y_1^{21} \\ Y_1^{22} \text{ and } X_1 Y_1^{22} - \frac{1}{2}(X_1^2 - X_2) Y_0^{22}. \end{array}$$

Using this fact for a Groebner base computation in Maple allows us again to prove that $\mathsf{iss}_\alpha Q$ is a C.I. □

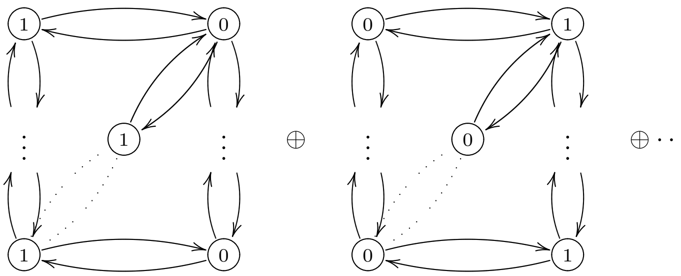
In the following lemmas we examine all possible structure elements that a symmetric quiver setting can contain. In each of these lemma we assume that (Q, α) is a reduced prime symmetric quiver setting without loops for which $\mathsf{iss}_\alpha Q$ is a C.I. Using this information we show that Q is of the forms I-IV.

Lemma 5.4. If Q contains a subquiver of the form



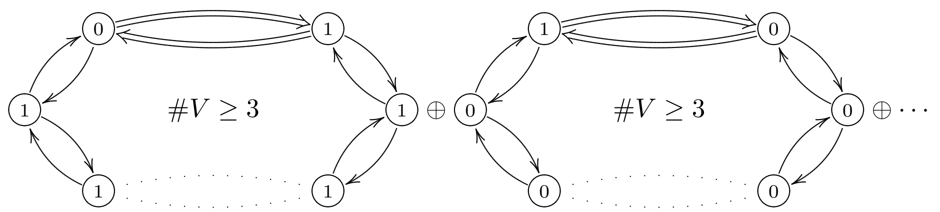
then (Q, α) is of the form III_a or III_b.

Proof. If 2 vertices of the cycle in Q are connected to each other by a third way we can make a decomposition



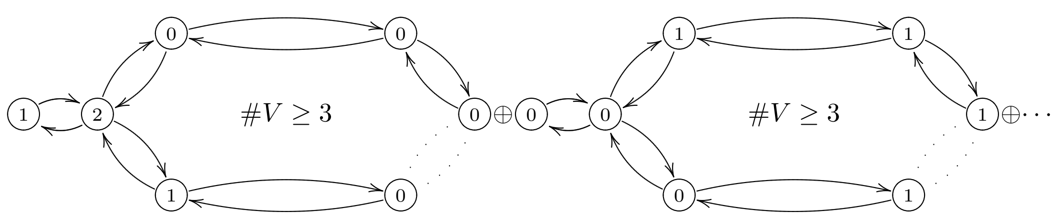
of which the locale quiver is not C.I., so this is not possible. Note that the lower right vertex and the central vertex may be identified, without changing the argument.

Also a situation where there are two vertices connected by a double arrow in both directions is impossible. At least one of those two vertices must have dimension one according to lemma 5.1, suppose the left one. Make a decomposition of the form



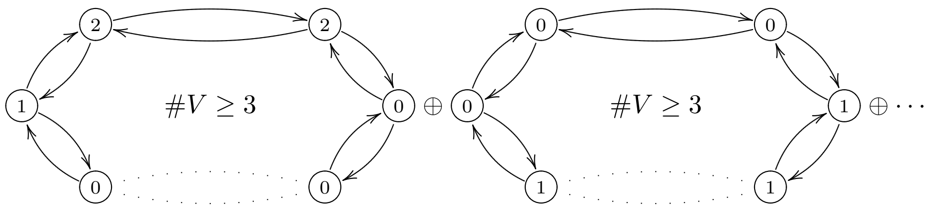
We remark that one of the vertices of the cycle must have dimension one otherwise we would have a decompositon containing two simples with dimension vector $\mathbb{1}$. Between those 2 components there are $\#A - \#V = \#V > 2$ arrows.

There cannot be any branching vertex (i.e. a vertex connected to at least 3 other vertices) with dimension $k > 1$ in the cycle. If there were, we could make a decomposition



This decomposition is always possible because if the second component contains only one vertex (i.e. in the case that the cycle has 3 vertices) the previous remark allows us to assume that this vertex has dimension 1 in our quiver setting.

If the cycle contains more than 3 vertices, the dimension vector must be $\mathbb{1}$. Otherwise, because of the reducedness, there must be two consecutive vertices with dimension 2 or more and we can make a decomposition of the form

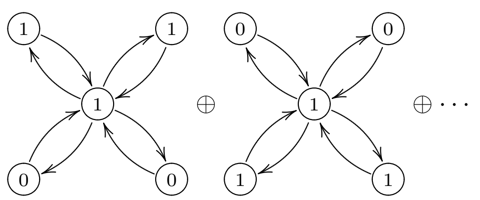


So if there are more than 3 vertices in the cycle we are in case III_a.

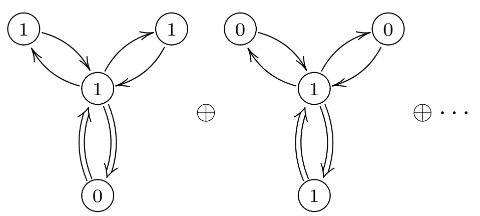
If there are 3 vertices and we are not in III_a, one of them must have dimension 1 the other must both have the same dimension because of the reducedness, so this is case III_b' and by the lemma 5.3 more precise case III_b. □

Lemma 5.5. If Q contains a branching vertex we are in case IV.

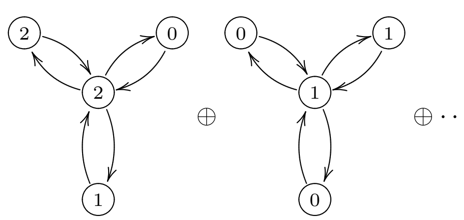
Proof. It is impossible that there are four vertices connected with the branching vertex. Because of 5.4 the quiver is a tree (i.e. all primitve cycles are of length 2) and by the primeness of (Q,α), the branching vertex must have dimension 2 or more. Make a subdecomposition of the form



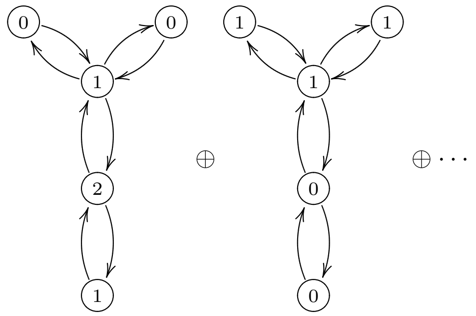
Also there is only one arrow to each vertex connected with the branching vertex because of the decomposition



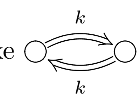
Suppose there are only 3 vertices connected with the branching vertex and its dimension is 3. Because (Q,α) is reduced, one of the adjecent vertices must have dimension bigger than 1. We can make a decomposition of the form



If the branching vertex has dimension 2 then Q is of the form IV. If there would be another vertex connected to one of the outer vertices, this vertex would have dimension 2 or more because of the primeness. Therefore we would have a decomposition like

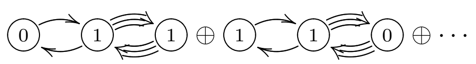


□

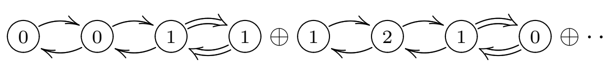
Lemma 5.6. If Q contains a quiver like  then we care in cases II_a or II_b

Proof. By lemma 5.1 we already know that one of the 2 vertices must have dimension one, and is because of the primeness at the end of the quiver.

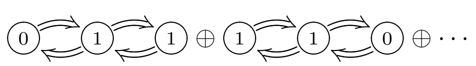
If $k > 2$ we can make a decomposition like



Suppose now that $k = 2$ then we are in situation II_a or II_b because if there are 4 vertices in Q we can make a decomposition like



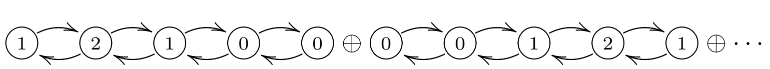
and if there are 3 vertices, only one double arrow can leave a vertex, otherwise we can make a decomposition like



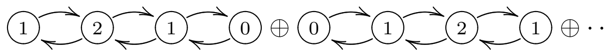
Together with 5.1 this implies that if Q has 2 vertices we are in case II_a and if Q has 3 vertices we are in II_b. □

Lemma 5.7. If Q is linear, not containing double arrows then we are in cases I_a, I_b or (Q, α) is coregular.

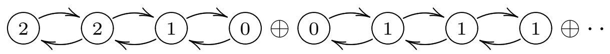
Proof. If Q is linear and prime, Q cannot have five vertices. Otherwise we would have a subdecomposition like



Take Q with 4 vertices and consider the vertex with the highest dimension to be the second from the left (this is always possible because otherwise we could reduce the dimension of the outer vertex). If this dimension is bigger than 2 than we can make a decomposition like

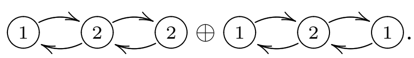


or



Depending on whether the first vertex has dimension 1 or higher. So if Q has 4 vertices we are in case I_b.

If Q has 3 vertices the central vertex has the highest dimension. In order to be reduced, none of the vertices can have dimension 1 (otherwise we could apply move $\mathcal{R}_{III}$). If this dimension is bigger than 3 we can make a decomposition like



So in this case, if the dimension of the central vertex is 3 we are in I_a, if it is 2 then (Q, α) is coregular.

If Q has 2 vertices (Q, α) is coregular. □

Finally we only have to prove that the listed quiver settings have indeed a quotient space that is a C.I.

For II_a this is already done in lemma 5.1, for II_b and III_b in lemmas 5.2 and 5.3.

In [8] it is proven that the ring of invariants of the quiver situation IV where $l, n, m \geq 2$, can be seen as a free module of rank two over the subalgebra generated by the invariants

$$\mathsf{Tr}X_i, \mathsf{Tr}X_iX_j,\ 1 \leq i \leq j \leq 3$$

In this notation X_i stand for the matrix coming from the path that runs through one of the three branches.

This subalgebra is a polynomial ring and the element of rank two is $\mathsf{Tr}X_1X_2X_3$ and satisfies the equation

$$(\mathsf{Tr}X_1X_2X_3)^2 + A(\mathsf{Tr}X_1X_2X_3) + B = 0\ (\dagger)$$

where

$$A = \mathsf{Tr}X_1\mathsf{Tr}X_2X_3 + \mathsf{Tr}X_2\mathsf{Tr}X_3X_1 + \mathsf{Tr}X_3\mathsf{Tr}X_1X_2 - \mathsf{Tr}X_1\mathsf{Tr}X_2\mathsf{Tr}X_3$$

$$\begin{aligned} B = {} & \det X_1(\mathsf{Tr}X_2X_3)^2 + \det X_2(\mathsf{Tr}X_3X_1)^2 + \det X_3(\mathsf{Tr}X_1X_2)^2 \\ & - \mathsf{Tr}X_1\mathsf{Tr}X_2\mathsf{Tr}X_1X_2 \det X_3 - \mathsf{Tr}X_2\mathsf{Tr}X_3\mathsf{Tr}X_2X_3 \det X_1 \\ & - \mathsf{Tr}X_3\mathsf{Tr}X_1\mathsf{Tr}X_3X_1 \det X_2 \\ & + (\mathsf{Tr}X_1)^2 \det X_2 \det X_3 + (\mathsf{Tr}X_2)^2 \det X_3 \det X_1 + (\mathsf{Tr}X_3)^2 \det X_1 \det X_2 \\ & - 4 \det X_1 \det X_2 \det X_3 + \mathsf{Tr}X_1X_2\mathsf{Tr}X_2X_3\mathsf{Tr}X_3X_1 \end{aligned}$$

and $\det X_i$ stands for $\frac{1}{2}((\mathsf{Tr}X_i)^2 - \mathsf{Tr}X_i^2)$. So for $l, m, n \geq 2$, situation VI is definitely a complete intersection. For the dimension vectors where l, m or n equal 1, the only extra relations we have to divide out are of the form $\mathsf{Tr}X_i^2 = (\mathsf{Tr}X_i)^2$, so for situation IV, $\mathsf{iss}_\alpha Q$ is always a complete intersection.

Using this explicit expression for $\mathbb{C}[\mathsf{iss}_\alpha Q]$ one can also easily deduce that the subring

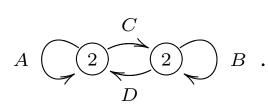
$$\mathbb{C}[\mathsf{Tr}X_i, \mathsf{Tr}X_iX_j,\ 1 \leq i \leq j \leq 3, (i,j) \neq (a,b)][\mathsf{Tr}X_1X_2X_3]$$

is a polynomial ring for every couple (a, b) (A relation in this ring combined with (†) would imply that either $\mathsf{Tr}X_1X_2X_3$ satisfies a linear equation or

$$\mathbb{C}[\mathsf{Tr}X_i, \mathsf{Tr}X_iX_j,\ 1 \leq i \leq j \leq 3]$$

is not polynomial. This fact can be used to prove that I_b is also a complete intersection.

Suppose first that m, n are bigger than 1, then we can modify setting I_b to the following situation



If $A, \ldots, D$ are the matrices that represent the corresponding arrows then we can make a list of generators:

$$\begin{array}{rl} \mathsf{Tr}A & \mathsf{Tr}A^2 \\ \mathsf{Tr}B & \mathsf{Tr}B^2 \\ \mathsf{Tr}CD & \mathsf{Tr}(CD)^2 \\ \mathsf{Tr}CAD & \mathsf{Tr}DBC \\ \mathsf{Tr}CADB & \mathsf{Tr}CADCDB. \end{array}$$

Every other trace of cycles can be written in function of those ten using the Cayley-Hamilton identity for A, B or CD.

The first nine traces generate a polynomial algebra. If there would be an algebraic relation between those traces, we could specialize this to $C = 1$ to obtain an algebraic relation in

$$\mathbb{C}[\mathsf{Tr}X_i, \mathsf{Tr}X_iX_j,\ 1 \leq i \leq j \leq 3, (i,j) \neq (1,2)][\mathsf{Tr}X_1X_2X_3]$$

where we set $X_1 = A$, $X_2 = B$, $X_3 = D$, which is impossible.

To find a quadratic relation for $\mathsf{Tr}CADCDB$ we use (†) and specialize this to $X_1 = CAD$, $X_2 = CD$ and $X_3 = B$. So again the ring of invariants is a rank 2 free module over a polynomial ring and hence I_b is a complete intersection when $m, n \geq 2$. For $m = 1$ or $n = 1$ we only have to divide out $\mathsf{Tr}A^2 = (\mathsf{Tr}A)^2$ or $\mathsf{Tr}B^2 = (\mathsf{Tr}B)^2$ so these cases we will also be C.I.

To prove that I_a is a complete intersection, we can use the same technique as for IV. In [13] it was proven that the ring of invariants of I_a can be seen as a rank 2 free module over the subalgebra generated by

$$\mathsf{Tr}X_1^j, \mathsf{Tr}X_2^j, \mathsf{Tr}X_1^sX_2^t,\ 1 \leq j \leq 3, 1 \leq s,t \leq 2.$$

where X_1 and X_2 are similar to the previous situation. This ring is again polynomial for all possible m, n.

Finally case III_a is a C.I. because its dimension is $\#V + 1$ and its quotient ring is

$$\mathbb{C}[X_1, \ldots, X_k, Y_+, Y_-]/(X_1 \cdots X_k - Y_+Y_-)$$

Where the X_i stand for traces of the small cycles between 2 vertices and Y_+, Y_- are the big cycles clock- and anti clockwise.

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