Vive la Difference!

We will describe a new, different, intriguing and surprisingly simple type of problem that is encountered in various applications of mathematics. First, we provide a few examples that produce this new type of problem; then we develop a mathematical procedure for solving these problems. Finally, we look at a number of other applications of the same ideas, some based on interesting mathematical problems but others taken from population models, economics, and so on. As we shall see, the underlying mathematics involves nothing that is technically demanding — indeed, only a passing acquaintance with AS-level mathematics (e.g. P1) is necessary (and, of course, the willingness to try something new).

1. Background

Let us start by thinking about what is quite natural: the birds and the bees — and the bees, in particular. A bee colony comprises drones (males) and the queen (female); further, a queen has a mother and a father, but a drone has only a mother. We will now trace the ancestry of a drone:

Here, we have represented a drone by \( \square \) and a queen by \( \bigstar \).

On counting back, we observe that the number of bees in this family tree, at each stage, is

\[ 1, 1, 2, 3, 5, 8, \text{ and so on.} \]

This set of numbers is called a Fibonacci sequence (named after an Italian mathematician: Fibonacci — literally ‘son of Bonaccia’, who was a merchant). He is often referred to as Leonardo of Pisa, and is regarded as the foremost mathematician of his time. He was born in about 1180 and died in 1250. (To help you fix these dates, Thomas à Becket was murdered in 1170, the Magna Carta was signed in 1215 and the first meeting of Simon de Montfort’s parliament was in 1265.) He wrote a number of books on mathematics, and one in particular that has come down to us is Liber Abaci
(The Book of the Abacus, 1202) which was concerned, mainly, with algebraic and commercial problems. It is in this book that a very famous problem (related to our bee ancestry above) was first stated:

‘How many pairs of rabbits will be produced in a year, beginning with a single pair, if in every month each pair bears a new pair which becomes productive from the next month?’ (The answer is 144.)

The Fibonacci numbers also arise in phyllotaxis (which means literally ‘leaf arrangement’); this word is used to describe the distribution of leaves along a twig, the arrangement of petals and of the florets in some flowers, the pattern of cells on the surface of a pineapple and of the blades in a pine cone (and many other naturally growing things). Two examples are given below: the head of a daisy and a pine cone.

The florets exhibit a pattern that has 21 spirals one way, and 34 the other (below, left); the pine cone, correspondingly, has 8 and 13 spirals (below, right).
You may wish to examine the surface of a pineapple, when you next get the opportunity; you should be able to identify patterns of 5, 8 and 13 spirals; these are called concho-spirals.) The sunflower has a pattern that contains 55, 89 and 144 spirals.

2. The mathematical problem

The typical problem is to find the answer to questions such as: what is the 25th number (or the 125th number) in the Fibonacci sequence? Of course, we could just work our way through the ‘construction’ – at least, if we can identify the relation between successive numbers. But if we can do that, then we might be able to find a direct method of solution. Have you noticed that any number in the sequence is obtained by adding the two preceding numbers? This was first observed – as far as we know – by Johannes Kepler (1571-1630, a German mathematician, astronomer and physicist, who discovered the laws of planetary motion). There is no evidence that Fibonacci ever realised this! Let any number in the sequence be \( x_n \), then the rule can be written

\[
x_n = x_{n-1} + x_{n-2} \quad \text{for} \quad n \geq 2,
\]

and we will start the process from the simplest pair of initial numbers: \( x_0 = 0, \ x_1 = 1 \).

(The most general Fibonacci sequence obeys the same rule, but starts with any pair of numbers, not both zero.) This type of equation is called a difference equation. In the rest of this work, we will show how to solve this equation, and others like it, and also see how such equations arise in other applications.

The method of solution was discovered by Jacques P.M. Binet (1786-1856, a French mathematician) in 1843. To see how his method works, let us first consider a simpler difference equation:

\[
x_n = r x_{n-1}, \quad n \geq 1,
\]

where \( r \) is a given number, and we have the start number, \( x_0 \). In this case, we may easily find the ‘general’ solution by successive calculation. Thus

\[
x_n = r x_{n-1} = r(r x_{n-2}) = r^2 x_{n-2} = r^2 (r x_{n-3}) = r^3 x_{n-3} = \ldots = r^n x_0,
\]

and so the general solution of this equation is \( x_n = r^n x_0 \), for any \( r \), any choice of \( n \) and any start value, \( x_0 \). The question we now pose is: can we find this solution directly, without working through step-by-step?

The form of the answer above suggests that we try a solution of the form

\[
x_n = A b^n,
\]

where \( A \) and \( b \) are numbers to be found.

We substitute this into the equation, to give

\[
A b^n = A r b^{n-1} \quad \text{and for} \quad A \neq 0, \ b \neq 0, \ \text{we obtain} \quad b = r.
\]
Thus the solution is \( x_n = Ar^n \), for any \( A \); but \( x_0 \) is given, so we evaluate on \( n = 0 \) to obtain \( x_0 = A \) (because \( r^0 = 1 \) (\( r \neq 0 \))). Thus the solution is exactly as obtained earlier: \( x_n = r^n x_0 \).

Now let us apply this method to a slightly more complicated difference equation:

\[
x_n = 3x_{n-1} - 2x_{n-2} \quad \text{for} \quad n \geq 2 \quad \text{with} \quad x_0 = 0, \ x_1 = 1.
\]

(The sequence here is 0, 1, 3, 7, 15, 31, …, and you might be able to guess the general solution – but that would not be mathematics!) With our knowledge of Binet’s method, it seems reasonable to try \( x_n = Ab^n \), which gives

\[
Ab^n = 3Ab^{n-1} - 2Ab^{n-2}
\]

and for \( A \neq 0 \) and \( b \neq 0 \), then

\[
b^2 - 3b + 2 = 0 \quad \text{or} \quad (b - 2)(b - 1) = 0 \quad \text{so} \quad b = 1 \quad \text{or} \quad b = 2.
\]

We have two values for \( b \)! What do we do now? Let us be really brave and use both values at the same time – or at least see what happens when we try this. To test this, we write

\[
x_n = A_1 1^n + A_2 2^n
\]

where \( A_1, \ A_2 \) allows two different choices for these constants. The original equation then gives

\[
A_1 + A_2 2^n = 3(A_1 + A_2 2^{n-1}) - 2(A_1 + A_2 2^{n-2}) = A_1 + (3 \times 2^{n-1} - 2^{n-1})A_2
\]

which is true for any pair of numbers \( A_1, \ A_2 \). When we impose the requirement to accommodate the two start values, we obtain

\[
0 = A_1 + A_2 \quad \text{and} \quad 1 = A_1 + 2A_2,
\]

which gives \( A_1 = -1, \ A_2 = 1 \), and so the solution of this difference equation is \( x_n = 2^n - 1^n = 2^n - 1 \).

You may like to try your hand at solving some similar problems; see the exercises at the end of these notes, and in particular Q1(a), and also Q1 on the exercise sheet.

3. The Fibonacci sequence

Let us now apply these ideas to the solution of the difference equation that describes the Fibonacci sequence:

\[
x_n = x_{n-1} + x_{n-2}, \ n \geq 2,
\]
with \( x_0 \) and \( x_1 \) given. We seek a solution of the form \( x_n = Ab^n \), which gives

\[
Ab^n = Ab^{n-1} + Ab^{n-2} \quad \text{or} \quad b^2 - b - 1 = 0 \quad \text{(for} \; A \neq 0, \; b \neq 0)\).
\]

This quadratic equation has the solution \( b = \frac{1}{2}(1 \pm \sqrt{5}) \) and so the most general solution is

\[
x_n = A_1 \left[ \frac{1}{2} \left(1 + \sqrt{5}\right)^n\right] + A_2 \left[ \frac{1}{2} \left(1 - \sqrt{5}\right)^n\right]
\]

which, with the start values (for the most usual sequence): \( x_0 = 0, \; x_1 = 1 \), gives

\[
0 = A_1 + A_2 \quad \text{and} \quad 1 = \frac{1}{2} A_1 (1 + \sqrt{5}) + \frac{1}{2} A_2 (1 - \sqrt{5})
\]

These are easily solved for \( A_1 \) and \( A_2 \), so that \( A_1 = -A_2 = \frac{1}{\sqrt{5}} \); the final solution is therefore

\[
x_n = \frac{1}{2^n} \frac{1}{\sqrt{5}} \left[ (1 + \sqrt{5})^n - (1 - \sqrt{5})^n \right], \quad n \geq 0.
\]

A real surprise here is that this answer is constructed from the irrational number \( \sqrt{5} \), yet the result must be an integer for every \( n \) ! The confirmation of this is left as an investigation for you (see also Q2 on the exercise sheet); if you know proof by induction, use this as a neat way to prove that, for all positive integers \( n \), the result is always an integer. (You might ask your teacher for more guidance on this.)

**4. Other difference equations**

It is now possible to invent many other difference equations – you can easily do so yourself – and then attempt to find solutions. In what follows, we write down a few more, and give information about how they can be solved. We present a few examples of various types, but generally avoid discussing general cases or very general methods of solution.

**Problem 1**

Here we consider the difference equation \( x_n = nx_{n-1} \) (\( n \geq 1 \)), with \( x_0 \) given; we proceed by direct construction:

\[
x_n = nx_{n-1} = n(n-1)x_{n-2} = n(n-1)(n-2)x_{n-3} = \ldots n(n-1)(n-2)\ldots 1x_0,
\]

and this is usually written using the factorial notation as \( x_n = n!x_0 \). It is quite easy to extend this type of solution when the equation takes the form \( x_n = f(n)x_{n-1} \).
Problem 2

Sometimes the difference equation is not simply a linear combination of terms such as \( x_n, x_{n-1} \) and so on; it may involve other powers of the terms, for example

\[
x_n = \sqrt{x_{n-1}} \quad \text{(where a square root has been used)}.
\]

In this situation, the neatest way forward is to take logs, which produces here

\[
\log x_n = \frac{1}{2} \log x_{n-1} \quad \text{and then we write} \quad y_n = \log x_n \quad \text{i.e.} \quad y_n = \frac{1}{2} y_{n-1}
\]

which is of the type discussed in Problem 1. We see, therefore, that

\[
y_n = \frac{1}{2} y_{n-1} = \frac{1}{2} \frac{1}{2} y_{n-2} = \ldots = \left(\frac{1}{2}\right)^n y_0,
\]

and so we obtain \( \log x_n = \left(\frac{1}{2}\right)^n \log x_0 \) or \( x_n = x_0^{2^{-n}} \) (a rather complicated structure to the solution). For a related example, see Q1(c) (and also Q1(b)) in the exercises at the end of these notes.

Problem 3

Now we return, apparently, to our first type of problem:

\[
x_n = 4x_{n-1} - 4x_{n-2} \quad (n \geq 2) \quad \text{with} \ x_0 \ \text{and} \ x_1 \ \text{given}.
\]

We seek a solution exactly as before, by writing \( x_n = Ab^n \), which produces

\[
Ab^n = 4Ab^{n-1} - 4Ab^{n-2} \quad \text{i.e.} \quad b^2 - 4b + 4 = 0 \quad \text{(for} \ A \neq 0, \ b \neq 0\text{)}.
\]

The quadratic for \( b \) is easily seen to be equivalent to \( (b - 2)^2 = 0 \), so \( b = 2 \) (as a repeated root); thus we have

\[
x_n = A2^n,
\]

but this can never satisfy the two conditions (given by the values of both \( x_0 \) and \( x_1 \)). We must conclude, therefore, that we have not obtained the general solution (which, presumably, should contain two arbitrary constants). The only change in this problem is the appearance of the repeated root. This, it can be shown by a suitable general argument, gives rise to an extension of the solution just found. Let us (on the basis of this discussion, which will not be developed here) try a solution of the form

\[
x_n = (A + Bn)2^n,
\]

which now incorporates the additional term in \( B \). This is substituted into the difference equation to produce
\[(A + Bn)2^n = 4[A + B(n-1)]2^{n-1} - 4[A + B(n-2)]2^{n-2}\]

which becomes

\[A(2^n - 2.2^n + 2^n) + B[n2^n - 2n2^n + n2^n + 2^n - 2.2^n] = 0.\]

But this equation is true for all possible choices of \(A\) and \(B\) – they both cancel identically – so another solution is indeed

\[x_n = (A + Bn)2^n,\]

which is the most general solution, and then we can determine the two constants to be \(A = x_0, B = \left(\frac{1}{2} x_1 - x_0\right)\).

**Problem 4**

Now we consider a difference equation that has a forcing term (so that zero for all \(n\) can never be a solution). Let us examine the equation

\[x_n = x_{n-1} + 1 \text{ for } n \geq 1 \text{ and } x_0 \text{ given.}\]

Rather than solve this successively (which is certainly the way forward if you have no clue as to what might happen – this is left for you to investigate) we will go directly to a more general approach. We will first see if we can find a particular and special solution which takes a very simple form: \(x_n = a + bn\), where \(a\) and \(b\) are fixed numbers to be determined. This is substituted into our equation, to give

\[a + bn = a + b(n-1) + 1\]

in which we observe that \(a\) and \(bn\) cancel identically, leaving \(b = 1\). Thus we have found a particular solution: \(x_n = n\) (where we have used the value of \(b\), but set \(a = 0\) because it does not contribute – remember that we are seeking nothing more than a special solution). Now let us write \(x_n = X_n + n\), then our equation becomes

\[X_n + n = X_{n-1} + (n-1) + 1 \text{ or } X_n = X_{n-1},\]

which is a very simple version of Problem 1. Indeed, we see immediately that the solution is just \(X_n = X_{n-1} = X_{n-2} = \ldots = X_0 = A\) (say), and thus the complete solution for \(x_n\) is

\[x_n = A + n \text{ and so } x_n = x_0 + n.\]

For a related example, see Q1(e) in the exercises at the end of these notes.
Problem 5

An extension of Problem 4 is provided by the equation

\[ x_n = x_{n-1} + n \text{ for } n \geq 1 \text{ and } x_0 \text{ given.} \]

This time we write a particular solution as \( x_n = an + bn + cn^2 \), for fixed numbers \( a, b \) and \( c \). (Note the beginning of a systematic approach here, by comparing the equations and particular solutions in this problem and the previous one.) Here we obtain

\[ a + bn + cn^2 = a + b(n - 1) + c(n - 1)^2 + n \]

which simplifies to

\[ 0 = -b + c(-2n + 1) + n. \]

This equation is true for all values of \( n \) provided that we choose \( c = 1/2 \) and \( b = c = 1/2 \) (and, as before, we may set \( a = 0 \)). We now seek a solution by writing

\[ x_n = X_n + \frac{1}{2} (n + n^2) \]

and so \( X_n = X_{n-1} \)

which shows, on noting the answer in Problem 4, that the complete solution is

\[ x_n = A + \frac{1}{2} (n + n^2) = x_0 + \frac{1}{2} (n + n^2). \]

Two related examples are Q1(d) and Q1(f) in the exercises at the end of these notes.

5. Applications

In this final section, we describe a few applications of difference equations that go far beyond the simple ideas of family trees and patterns of growth in plants.

Application 1 - mathematical

The first one we look at is a standard type of problem in elementary mathematics – indeed, you may have already met it. Let us find the sum of the first \( n \) integers i.e.

\[ \sum_{n=1}^{N} n = 1 + 2 + 3 + \ldots + N. \]

The way to proceed is to introduce \( s_N = \sum_{n=1}^{N} n \), then

\[ s_N = \sum_{n=1}^{N} n = \sum_{n=1}^{N-1} n + N = s_{N-1} + N. \]
This is a standard problem (see Problem 5), which we solve by writing

\[ s_N = a + b N + c N^2 + S_N, \]

which gives \( s_N = \frac{1}{2} N + \frac{1}{2} N^2 + A \). However, \( s_1 = 1 \) and so \( A = 0 \) i.e.

\[ \sum_{n=1}^{N} n = \frac{1}{2} N(1 + N). \]

Two similar calculations are given in Q2 in the exercises at the end of these notes.

**Application 2 – earning interest**

Now we look at the example of money that earns interest. First, we consider simple interest, in which the fixed rate, \( r\% \), is applied at fixed time (perhaps quarterly or yearly) to the amount initially deposited. If the amount in the account at one time point is \( m_n \), then the relation between this and the amount at the previous time point is

\[ m_n = m_{n-1} + \left( \frac{r}{100} \right) m_0, \]

where \( m_0 \) is the amount initially deposited. (Note here that the term \( rm_0/100 \) in the equation is a fixed number.) This is essentially Problem 4, so the solution is

\[ m_n = A + \left( \frac{rm_0}{100} \right) n \text{ with } m_0 \text{ given (at } n = 0), \]

so we finally have \( m_n = \left(1 + \frac{r}{100} n\right)m_0 \) as the amount at the end of \( n \) time intervals.

In the case of compound interest, the interest is earned on the amount on deposit at the time the interest is calculated. Thus the equation for the amount now becomes

\[ m_n = m_{n-1} + \left( \frac{r}{100} \right) m_{n-1} = \left(1 + \frac{r}{100}\right)m_{n-1}, \]

which is like the problem encountered in Problem 2 (which, in turn, is related to Problem 1). The solution is therefore

\[ m_n = \left(1 + \frac{r}{100}\right)^n A, \]

and the constant \( A \) is selected to be the amount put in the account at the start: \( m_0 \) at \( n = 0 \), so

\[ m_n = \left(1 + \frac{r}{100}\right)^n m_0. \]

A related problem is given in Q3 in the exercises at the end of these notes.
**Application 3 – wage negotiations**

In this application, we take a simple model of a wage negotiation between employers and employees. The premise that we take is that the workers ask for a wage of $w_0$, initially, and the employers offer $p_0$, which is less than $w_0$. The negotiations then proceed by the workers making a new request which is *less* than the previous one, but – in this simple model – the reduction is at a fixed fraction of the difference in the previous request and the counter offer from the employers. The employers follow the same procedure, but they *increase* the offer by an amount that is, again, a fixed fraction of the difference. Thus at any stage in the negotiations, we have

\[
    w_n = w_{n-1} - \alpha(w_{n-1} - p_{n-1}) \text{ and } \quad p_n = p_{n-1} + \beta(w_{n-1} - p_{n-1}),
\]

where $\alpha$ and $\beta$ are the fixed fractions (and each are numbers between 0 and 1). Here we have *two* equations, but we may eliminate one or other of the unknowns to obtain one equation. First we write

\[
    \beta w_{n-1} = p_n - p_{n-1} + \beta p_{n-1} \quad \text{and so} \quad \beta w_n = p_{n+1} - p_n + \beta p_n,
\]

which allows us to substitute into the first equation (after multiplication by $\beta$):

\[
    p_{n+1} - p_n + \beta p_n = (1-\alpha)(p_n - p_{n-1} + \beta p_{n-1}) + \alpha \beta p_{n-1}.
\]

This can be simplified to give

\[
    p_{n+1} - (2-\alpha - \beta)p_n + (1-\alpha - \beta)p_{n-1} = 0,
\]

which has the general solution (see our earlier discussion) obtained by first writing $p_n = Ab^n$:

\[
    b^2 - (2-\alpha - \beta)b + (1-\alpha - \beta) = 0 \quad \text{so} \quad b = 1, 1-\alpha - \beta.
\]

Thus

\[
    p_n = A1^n + B(1-\alpha - \beta)^n = A + B(1-\alpha - \beta)^n
\]

where $A$ and $B$ are the arbitrary constants. Now we may obtain the corresponding solution for $w_n$ directly from e.g. $w_n = \frac{1}{\beta}(p_{n+1} - p_n) + p_n$; the constants are then fixed by ensuring that the initial values, $w_0$ and $p_0$, are recovered for $n = 0$. (This is left as a simple exercise.)

The important observation we make, based on our solution, is that if the process of negotiation is to lead directly and steadily towards a resolution, then $\alpha$ and $\beta$ together must satisfy: $0 \leq \alpha + \beta \leq 1$ (which is in addition to each being numbers between 0 and 1).
Application 4 – population model

In the simplest modelling of this type, we consider a single-species population (so interaction with other species – predators, for example – is ignored). Further, we assume that each generation produces the next, and that the life spans do not overlap; this is true of some insect populations. Then the number in the next generation ($N_n$) is taken to depend on the number in the current generation ($N_{n-1}$), so

$$N_n = f(N_{n-1}) ,$$

where the function $f$ describes the relation between the two generations. More usually, this is expressed in the form

$$N_n = \alpha(N_{n-1})N_{n-1} ,$$

where $\alpha$ is a measure of the birth rate, but this itself depends on the size of the current population. The issue in such a model is: what are reasonable choices for $\alpha$?

When account is taken of environmental factors – food supply, for example – then typical models will allow an initial large birth rate, but this reduces dramatically for large populations. Thus we might use

$$\alpha(N) = a - bN \quad \text{or} \quad \alpha(N) = \frac{a}{1 + bN} ,$$

where $a$ and $b$ are positive constants. The first choice gives rise to a famous equation – called the logistic equation – but this cannot be completely solved. The second is a slightly less convenient one (because the population will always grow, but at an ever decreasing rate as the population increases). However, for us this is a good one to look at: we can solve this problem!

Let us be given

$$N_n = \frac{aN_{n-1}}{1 + bN_{n-1}} \text{, for } n \geq 1 ,$$

and $N_0$ known – we do need some population to start the process. This equation looks really difficult, but as with many problems in mathematics, it is sometimes possible to transform a new equation into a known, standard equation. In this case, let us write

$$N_n = 1/x_n ,$$

then our equation becomes

$$\frac{1}{x_n} = \frac{a/x_{n-1}}{1 + b/x_{n-1}} = \frac{a}{b + x_{n-1}} \quad \text{or} \quad ax_n = x_{n-1} + b \quad \text{or} \quad x_n = \frac{1}{a}x_{n-1} + \frac{b}{a} .$$

We may solve this equation by using the ideas in Problems 2 and 4; in particular, let us write
\[ x_n = Ad^n + c \]

then we obtain \[ Ad^n + c = \frac{1}{a} \left( Ad^{n-1} + c \right) + \frac{b}{a} \] which requires \( d = 1/a \) and \( c = b/(a - 1) \), for any \( A \) (provided that \( a \neq 1 \); if \( a = 1 \) we get a different solution – it is left as an exercise to find this). Thus

\[ x_n = Aa^{-n} + \frac{b}{a - 1} \]

and so \( N_n = \frac{1}{Aa^{-n} + (b/(a - 1))} \),

where \( A \) is chosen so that \( N_0 \) is recovered at \( n = 0 \) i.e. \( A = \frac{1}{N_0} - \frac{b}{a - 1} \).

An application involving the migration of populations can be found in Q5 in the exercises at the end of these notes.

**Application 5 – red blood cells**

The red blood cells transport oxygen around the body, but these cells are continuously being removed (by the spleen) and being created (in the bone marrow). We will assume that the spleen removes a fraction \((\alpha, 0 \leq \alpha < 1)\) of the blood cells each day, and that the marrow replaces a fraction \((\beta, 0 < \beta \leq 1)\) of those removed the previous day. Let the number of red blood cells at the start of a day be \( r_n \), then we have

\[ r_n = r_{n-1} - \alpha r_{n-1} + \alpha \beta r_{n-2} \]

which has a solution of the form \( r_n = Ab^n \) i.e. \( b^2 - (1 - \alpha)b - \alpha \beta = 0 \) (for any \( A \)). The solution of this quadratic equation is

\[ b = \frac{1}{2} \left( 1 - \alpha \pm \sqrt{(1 - \alpha)^2 + 4\alpha \beta} \right), \]

and because both \( \alpha \) and \( \beta \) are positive, the roots must satisfy the conditions: one is positive and one is negative. Let these two roots be \( b_+ > 0 \) and \( b_- < 0 \), then the general solution of the difference equation for \( r_n \) is

\[ r_n = A_1 b_+^n + A_2 b_-^n, \]

where \( A_1 \) and \( A_2 \) are the two arbitrary constants (given by the blood count on two successive days, for example). This result can be used to deduce an important property of a healthy body: *homeostasis*. This means that the number of blood cells remains essentially constant or, at least, this is the condition approached over a period of time. This requires that one of the two terms in the solution does not change and that the other decreases as \( n \) increases. The larger magnitude of the two values for \( b \) is \( b_+ \), so let us impose the requirement: \( b_+ = 1 \) i.e.
\[ 1 = \frac{1}{2} \left( 1 - \alpha + \sqrt{(1 - \alpha)^2 + 4\alpha} \right) \text{ or } 1 + \alpha = \sqrt{(1 - \alpha)^2 + 4\alpha}. \]

When squared out, this gives
\[ \alpha^2 + 2\alpha + 1 = \alpha^2 - 2\alpha + 1 + 4\beta, \]
and so either \( \alpha = 0 \) (which we reject – the properties of the blood exclude this possibility) or \( \beta = 1 \) i.e. the production rate is unity. Then the other value of \( b \) becomes
\[ \frac{1}{2} \left( 1 - \alpha - \sqrt{(1 - \alpha)^2 + 4\alpha} \right) = \frac{1}{2} \left( 1 - \alpha - (1 + \alpha) \right) = -\alpha, \]
so the term in \( b \) does indeed decrease as \( n \) increases (and then \( r_n \to A_1 \)).

An application that involves the modelling of the breathing process is discussed in Q4 in the exercises at the end of these notes.

**Exercises**

Other examples can be found on the Exercise Sheet that was available for discussion during the session on this topic.

1. Solve these difference equations (and for each \( n \geq 0 \)):
   - (a) \( x_{n+2} = 7x_{n+1} - 12x_n \) with \( x_0 = 0 \) and \( x_1 = 1 \);
   - (b) \( x_{n+1}^2 = \left( \frac{n+1}{n} \right) x_n^2 \) with \( x_0 = 1 \) [Hint: introduce \( y_n = x_n^2/n \).
   - (c) \( x_{n+1} = x_n^2 \) with \( x_0 \) given;
   - (d) \( x_{n+1} = 2 + n - x_n \) with \( x_0 = 2 \);
   - (e) \( x_{n+2} = 5x_{n+1} - 6x_n + 36 \) with \( x_0 = 5 \) and \( x_1 = 1 \);
   - (f) \( 2x_{n+2} = 5x_{n+1} - 2x_n + n^2 - n + 1 \) with \( x_0 = x_1 = 0 \).

2. Find formulae for these sums by formulating in terms of difference equations:
   - (a) \( \sum_{n=1}^{N} n^2 \); (b) \( \sum_{n=1}^{N} n^3 \).

3. Money is accrued according to compound interest, with a rate \( r\% \) in each (equal) time interval, but after the new total is determined an additional constant sum, \( s \), is added each time. Find the difference equation that describes this process and then solve it (given that the original amount first put in the account is \( x_0 \)).
4. The basic metabolic process produces carbon dioxide, which is breathed out. We assume that breathing is at a constant rate, and that the volume expelled in each breath is proportional to the concentration, \( c_n \), of carbon dioxide accumulated during the previous time interval. Thus the concentration level changes from one breath to the next according to

\[ c_{n+1} - c_n = -\alpha v_n + \mu, \]

where \( \alpha v_n \) measures the amount of carbon dioxide lost and \( \mu \) the amount produced by the metabolic process; the volume exhaled is described by

\[ v_n = \beta c_{n-1}, \]

where \( \alpha \), \( \beta \) and \( \mu \) are positive constants in this simple model. Find the equation for \( c_n \) and solve it. Discuss what happens in the case \( 4\alpha\beta < 1 \).

5. A community consists of two separate populations: the \( x \)s and the \( y \)s. However, there is some migration between the two communities. We assume that the population in each group has the same birth rate, and that the \( x \) population supplies an essential service to the whole community e.g. they might be the farmers, and so there is an optimal \( x \) to ensure the viability of the community. We assume that this is a fixed fraction of the total population. The model we examine is one in which the yearly migration is proportional to the excess of \( x \)s over the optimal.

Let the populations in one year be \( x_n \) and \( y_n \), and so the total population is \( x_n + y_n \) and then the optimal is taken to be \( \alpha(x_n + y_n) \). The excess of the \( x \) population, over the optimal, is therefore \( x_n - \alpha(x_n + y_n) \). Let the birth rate be \( \beta \) and the migration rate be \( \mu \), then we have

\[ x_{n+1} = \beta x_n - \mu(x_n - \alpha(x_n + y_n)); \quad y_{n+1} = \beta y_n + \mu(x_n - \alpha(x_n + y_n)). \]

It is reasonable to assume that the three rates satisfy: \( \beta > 1, \ 0 < \alpha < 1, \ \mu > 0 \). Eliminate one of the unknowns, find the equation for the other and solve it. Discuss the difference between the cases: \( \alpha > \beta > 0 \) and \( \beta > \alpha > 0 \).