PARTNERS Summer School
Mathematics & Statistics
Sessions 8, 9: Probability and Statistics: Queues

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Stochastic processes and queueing systems

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- We use stochastic models for queues because features of the behaviour depend on the “randomness.”

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  - The waiting time, W_n, of customer number n.
  - The proportion of time, p, when the server is idle.

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   3.3 Priority systems.
   3.4 Parallel queues, for example supermarket checkout.

Arrival mechanism: Poisson process

A commonly used model for arrivals is called a *Poisson process*. This is a very simple but useful model. The basic idea is that arrivals occur "completely at random in time". Examples where we might use a Poisson process model include:

- "Clicks" on a Geiger counter.
- Vehicles passing on a (quiet) road.
- Arrivals of telephone calls at an exchange.
- Accidents.

We are often interested in $N(t)$, the number of arrivals occurring in a time interval of length t. Let $p_n(t) = \Pr\{N(t) = n\}$.

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2. There is a constant rate λ, representing the mean number of arrivals per unit time, such that for a short time interval of length δt

$$\begin{aligned} p_0(\delta t) &\approx 1 - \lambda \delta t \\ p_1(\delta t) &\approx \lambda \delta t \\ p_n(\delta t) &\approx 0 \end{aligned}$$

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3. The numbers of arrivals in non-overlapping time intervals are independent.

Arrival mechanism: Poisson process

As an approximation, imagine dividing time up into sections, each of length δt. Suppose that the probability that an arrival occurs in a particular section is $\lambda \delta t$. In the time interval $(0, t)$ there are $t/\delta t$ sections. The number of sections with events in them has a *binomial distribution* and the probability of j events in $(0, t)$ is

$$
\begin{aligned}
\tilde{p}_j &\approx \binom{t/\delta t}{j} (\lambda \delta t)^j (1 - \lambda \delta t)^{t/\delta t - j} \\
&\approx \binom{t/\delta t}{j} \left(\frac{\lambda \delta t}{1 - \lambda \delta t} \right)^j (1 - \lambda \delta t)^{t/\delta t} \\
&\approx \frac{\lambda^j}{j!} t(t - \delta t)(t - 2\delta t) \cdots (t - \{j-1\}\delta t)(1 - \lambda \delta t)^{t/\delta t - j}
\end{aligned}
$$

Arrival mechanism: Poisson process

It is easily shown that

$$\tilde{p}_j \to \frac{\exp(-\lambda t)(\lambda t)^j}{j!}$$

as $\delta t \to 0$.
Thus $N(t)$, the number of arrivals in the time interval $(0, t)$, has a *Poisson distribution* with mean λt.

Arrival mechanism: Poisson process

We can prove formally that the number of arrivals in a time interval of length t has a Poisson distribution with mean λt. If this number is $N(t)$ then

$$\Pr\{N(t) = j\} = \frac{e^{-\lambda t}(\lambda t)^j}{j!}$$

for $j = 0, 1, 2, \ldots$.

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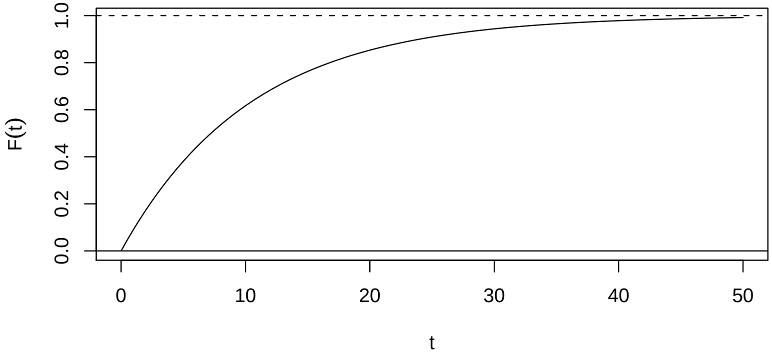
- This is the distribution function of an *exponential distribution* with parameter λ.

Distribution function of inter-arrival time



Probability density function

If a continuous random variable T has distribution function $F_T(t)$ then

$$f_T(t) = \frac{d}{dt} F_T(t)$$

is called the *probability density function* (or *pdf*) of T.
Note that

$$\int_{-\infty}^{\infty} f_T(t)\ dt = 1$$

— the total probability.
For our exponential distribution

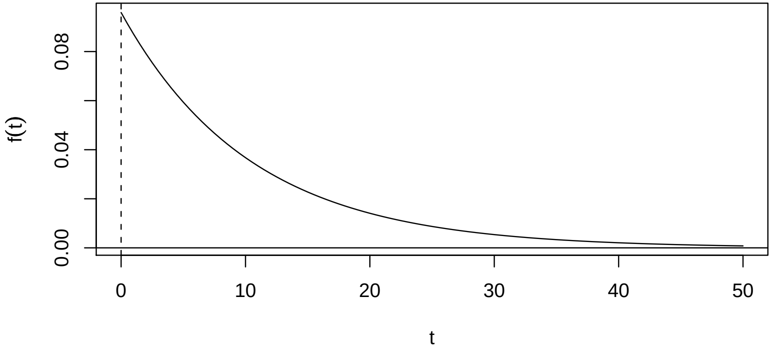
$$F_T(t) = 1 - \exp(-\lambda t)$$

so

$$f_T(t) = \lambda \exp(-\lambda t)$$

$(0 < t < \infty)$.

Probability density function of inter-arrival time



Example: Road vehicle headways

The times of arrivals of motor vehicles passing a point in Chester Road, Sunderland going East.

```
12.40   3, 6, 9,15,24,28,30
12.41   7,12,14,16,21,24,30,50
12.42   9,22,28,46,53
12.43  22,25,35,38,58
12.44   2, 5, 8,10,14,17,27,30,45
12.45   3,46
12.46  13,42,51
12.47   0, 9,11,18,23,26,39,51

 .............................

12.59   2,31,34,38,54,59
```

Example: Road vehicle headways

These can be converted to time intervals (in seconds)between vehicles. (The first value is the time till the first arrival, which has the same distribution under our model).

3	3	3	6	9	4	2	37	5	2	2
5	3	6	20	19	13	6	18	7	29	3
10	3	20	4	3	3	2	4	3	10	3
15	18	43	27	29	9	9	9	2	7	5
3	13	12	44	4	16	13	2	9	1	13
12	1	2	16	26	4	13	13	4	14	3
31	5	2	5	5	20	1	1	7	21	3
4	25	22	14	2	4	11	15	8	41	18
3	10	7	3	5	21	6	5	22	2	2
5	7	3	24	3	7	32	3	19	2	1
6	29	3	4	16	5					

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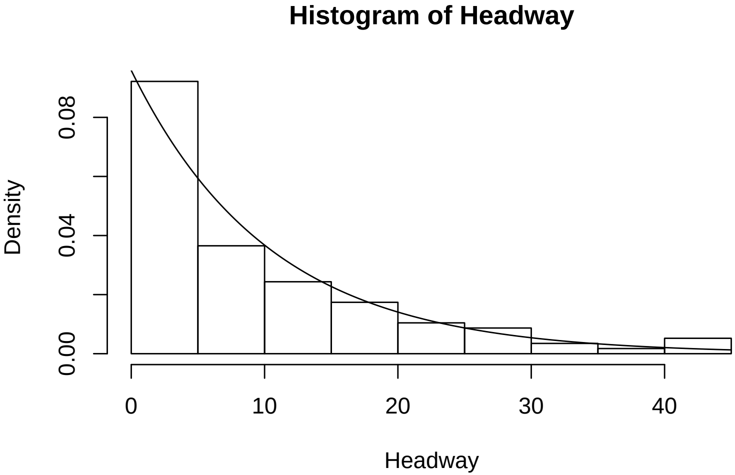

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- Hence the probability density function is

$$\begin{aligned} f_2(t) &= \frac{d}{dt} F_2(t) \\ &= \lambda(1 + \lambda t)e^{-\lambda t} - \lambda e^{-\lambda t} \\ &= \lambda^2 t e^{-\lambda t} \end{aligned}$$

Gamma distribution

If a continuous random variable T has probability density function

$$f_g(t) = \begin{cases} 0 & (t < 0) \\ \frac{\lambda^\alpha t^{\alpha-1} e^{-\lambda t}}{\Gamma(\alpha)} & (t \geq 0) \end{cases}$$

where α and λ are positive parameters, then we say that T has a *gamma*(α, λ) *distribution*.
Here $\Gamma(\alpha)$ denotes the *gamma function*. This has the property that, if x is a positive number, then

$$\Gamma(x) = (x-1)\Gamma(x-1).$$

If n is a positive integer, then

$$\Gamma(n) = (n-1)!.$$

Gamma distribution

Gamma pdf:

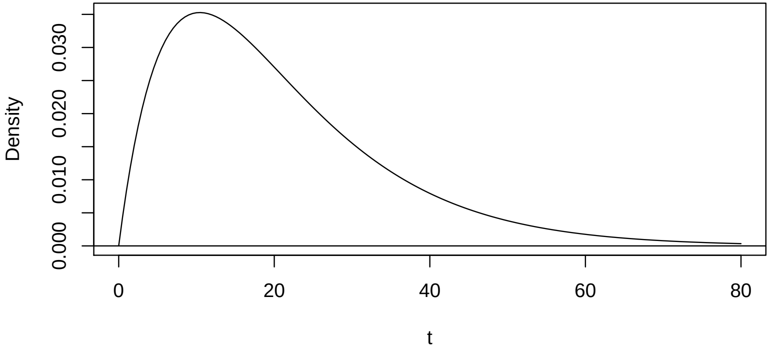
$$f_g(t) = \frac{\lambda^\alpha t^{\alpha-1} e^{-\lambda t}}{\Gamma(\alpha)}$$

The time to the second arrival has pdf

$$f_2(t) = \frac{\lambda^2 t^{2-1} e^{-\lambda t}}{1!} = \frac{\lambda^2 t^{2-1} e^{-\lambda t}}{\Gamma(2)}.$$

So this is a gamma$(2,\ \lambda)$ distribution.
Notice that an exponential(λ) distribution is a gamma$(1,\ \lambda)$ distribution.

Time to second arrival



Mean of a continuous random variable

A continuous random variable X has pdf $f_X(x)$.
Its *mean* or *expectation* (if it exists) is

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x \; f_X(x) \; dx.$$

Similarly, if $g(X)$ is some function of X, then the expectation of $g(X)$ is

$$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x) \; f_X(x) \; dx.$$

Means: Example 1

$$T \sim \text{exponential}(\lambda)$$

$$\begin{aligned}
\mathrm{E}(T) &= \int_0^\infty t\lambda e^{-\lambda t}\ dt \\
&= [-te^{-\lambda t}]_0^\infty + \int_0^\infty e^{-\lambda t}\ dt \\
&= \left[-\frac{1}{\lambda}e^{-\lambda t}\right]_0^\infty \\
&= \frac{1}{\lambda}
\end{aligned}$$

Means: Example 2

$$T \sim \text{gamma}(\alpha,\ \lambda)$$

$$\begin{aligned}
\mathrm{E}(T) &= \int_0^\infty \frac{t\lambda^\alpha t^{\alpha-1}e^{-\lambda t}}{\Gamma(\alpha)}\,dt \\
&= \int_0^\infty \frac{\lambda^\alpha t^{\alpha+1-1}e^{-\lambda t}}{\Gamma(\alpha)}\,dt \\
&= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}\frac{1}{\lambda}\int_0^\infty \frac{\lambda^{\alpha+1} t^{\alpha+1-1}e^{-\lambda t}}{\Gamma(\alpha+1)}\,dt \\
&= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}\frac{1}{\lambda} \\
&= \frac{\alpha}{\lambda}
\end{aligned}$$

Means: Example 2

$$T \sim \mathrm{gamma}(\alpha,\ \lambda)$$

$$\begin{aligned}
\mathrm{E}(T^2) &= \int_0^\infty \frac{t^2 \lambda^\alpha t^{\alpha-1} e^{-\lambda t}}{\Gamma(\alpha)}\, dt \\
&= \int_0^\infty \frac{\lambda^\alpha t^{\alpha+2-1} e^{-\lambda t}}{\Gamma(\alpha)}\, dt \\
&= \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} \frac{1}{\lambda^2} \int_0^\infty \frac{\lambda^{\alpha+2} t^{\alpha+2-1} e^{-\lambda t}}{\Gamma(\alpha+2)}\, dt \\
&= \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} \frac{1}{\lambda^2} \\
&= \frac{(\alpha+1)\alpha}{\lambda^2}
\end{aligned}$$

Variance of a random variable

The *variance* of a quantitative random variable X (if it exists) is

$$
\begin{aligned}
\mathrm{Var}(X) &= \mathrm{E}\left\{[X - \mathrm{E}(X)]^2\right\} \\
&= \mathrm{E}\left\{X^2 - 2X\mathrm{E}(X) + [\mathrm{E}(X)]^2\right\} \\
&= \mathrm{E}(X^2) - 2\mathrm{E}(X)\mathrm{E}(X) + [\mathrm{E}(X)]^2 \\
&= \mathrm{E}(X^2) - [\mathrm{E}(X)]^2
\end{aligned}
$$

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Variance: Example

$$T \sim \mathrm{gamma}(\alpha,\ \lambda)$$

- $$\mathrm{E}(T) = \frac{\alpha}{\lambda}$$
- $$\mathrm{E}(T^2) = \frac{(\alpha+1)\alpha}{\lambda^2}$$
- So
$$\mathrm{Var}(T) = \frac{(\alpha+1)\alpha}{\lambda^2} - \left\{\frac{\alpha}{\lambda}\right\}^2 = \frac{\alpha}{\lambda^2}$$

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The range of X is $-\infty < X < \infty$.
- The probability density function is

$$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \qquad (-\infty < x < \infty).$$

Normal distribution

- The *normal distribution*, sometimes called the *Gaussian distribution* is very important in probability and statistics. It is a continuous distribution.
- It has two parameters, the mean and the variance, often written $\mu,\ \sigma^2$. If X has a normal distribution with mean μ and variance σ^2 we write

$$X \sim N(\mu,\ \sigma^2$$

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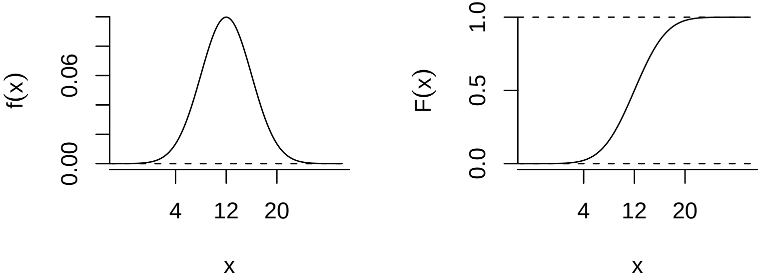
$$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \qquad (-\infty < x < \infty).$$

- The distribution function can not be written explicitly.

Normal distribution



Probability density function $f(x)$ and distribution function $F(x)$ for a normal distribution with mean $\mu = 12$ and variance $\sigma^2 = 16$.

The standard normal distribution

If $\mu = 0$ and $\sigma^2 = 1$ we have a *standard normal distribution*. If Z has a standard normal distribution we write $Z \sim N(0, 1)$.

- Probability density function:

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

- Distribution function:

$$\Phi(z) = \int_{-\infty}^{z} \phi(u)\ du$$

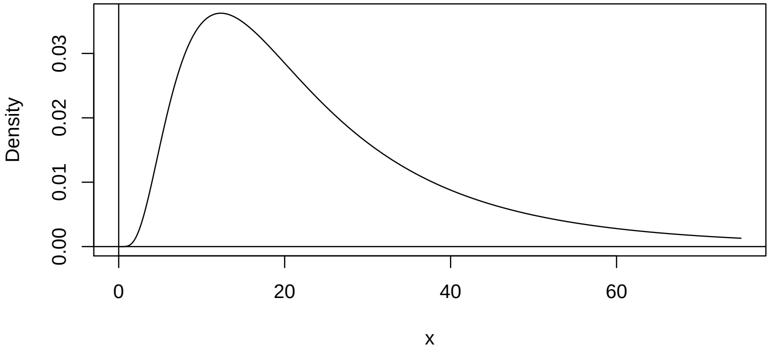
The lognormal distribution

- Sometimes a variable does not have a normal distribution but some *transformation* of the variable does.
- For example, if X must be positive, eg a service time, then perhaps $\ln(X)$ has a normal distribution.

$$0 < X < \infty \;\Leftrightarrow\; -\infty < \ln(X) < \infty$$

- If $Y \sim N(\mu,\ \sigma^2)$ and $X = e^Y$ (so $Y = \ln(X)$) then we say that X has a lognormal$(\mu,\ \sigma^2)$ distribution, $X \sim \mathrm{lognormal}(\mu,\ \sigma^2)$.

The lognormal distribution



Probability density function for a lognormal distribution with $\mu = 3$ and $\sigma^2 = 0.49$ ($\sigma = 0.7$).

The lognormal distribution

If $X \sim \mathrm{lognormal}(\mu,\ \sigma^2)$ what is the mean of X?
If $X \sim \mathrm{lognormal}(\mu,\ \sigma^2)$ and $Y = \ln(X)$ then $\mathrm{E}(X) = \mathrm{E}(e^Y)$.
Hence

$$\begin{aligned}
\mathrm{E}(X) = \mathrm{E}(e^Y) &= \int_{-\infty}^{\infty} e^y (2\pi\sigma^2)^{-1/2} \exp\left\{-\frac{1}{2\sigma^2}(y-\mu)^2\right\}\ dy \\
&= \int_{-\infty}^{\infty} (2\pi\sigma^2)^{-1/2} \exp\left\{y - \frac{1}{2\sigma^2}(y-\mu)^2\right\}\ dy
\end{aligned}$$

The lognormal distribution

$$
\begin{aligned}
y - \frac{1}{2\sigma^2}(y-\mu)^2 &= -\frac{1}{2\sigma^2}[y^2 - 2\mu y - 2\sigma^2 y + \mu^2] \\
&= -\frac{1}{2\sigma^2}[y^2 - 2(\mu+\sigma^2)y + (\mu+\sigma^2)^2 \\
&\quad -(\mu+\sigma^2)^2 + \mu^2] \\
&= -\frac{1}{2\sigma^2}[\{y - (\mu+\sigma^2)\}^2 - (\mu+\sigma^2)^2 + \mu^2] \\
&= -\frac{1}{2\sigma^2}[\{y - (\mu+\sigma^2)\}^2 - 2\mu\sigma^2 - \sigma^4] \\
&= -\frac{1}{2\sigma^2}(y-\theta)^2 + \mu + \frac{\sigma^2}{2}
\end{aligned}
$$

where $\theta = \mu + \sigma^2$.

The lognormal distribution

Hence

$$\begin{aligned}
\mathrm{E}(X) &= \int_{-\infty}^{\infty} (2\pi\sigma^2)^{-1/2} \exp\left\{-\frac{1}{2\sigma^2}(y-\theta)^2\right\} \, dy \;\; e^{\mu+\sigma^2/2} \\
&= e^{\mu+\sigma^2/2}
\end{aligned}$$

Estimation: Method of moments

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- We *estimate* μ with $\hat{\mu} = \bar{y} = 5.366$.

Estimation: Method of moments

- Sample variance:

$$
\begin{aligned}
s^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2 = \frac{1}{n-1}\left\{\sum_{i=1}^{n} y_i^2 - n\bar{y}^2\right\} \\
&= \frac{1}{49}\{1503.931 - 50 \times 5.366^2\} = 1.3109
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- We *estimate* σ^2 with $\hat{\sigma}^2 = s^2 = 1.3109$ and estimate the standard deviation σ with $s = \sqrt{s^2} = 1.1449$.

Estimation: Lognormal distribution

In the case of data from a lognormal distribution we can saimply take logs of the data and then proceed as for a normal distribution.
Eg: $X_1, \ldots, X_{50} \sim \mathrm{lognormal}(\mu,\ \sigma^2)$.
Data: $336.97,\ 52.46,\ 35.52, 148.41, \ldots$
Take logs: $5.82,\ 3.96,\ 3.57,\ 5.00, \ldots$ as before.

Simulation

- One way to investigate stochastic system behaviour is to *simulate*.
- Quicker and cheaper than experimenting on the real thing!
- Generate artificial *pseudo-random* inter-arrival times and service times.
- Build up the events and see what happens.
- Possibly repeat many times.
- Actually more useful in more complicated systems.

Simulation: Example

As an example, consider use of a computer network.

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- We assume that there is unlimited capacity.
- Let us start the simulation with an empty system.
- Generate inter-arrival and service times by computer. (In the case of lognormal service times we might generate normal random variables and then exponentiate).

Simulation: Inter-arrival and service times

User	Inter-arrival time	Arrival time	Service time	Service ends
1	15.7		44.2	
2	10.4		24.9	
3	3.9		83.0	
4	12.7		19.2	
5	7.8		41.8	
6	1.0		16.9	
7	28.2		37.7	
8	10.4		23.1	
9	13.4		83.8	
10	0.3		49.6	
11	3.5		11.3	
12	7.7		6.5	

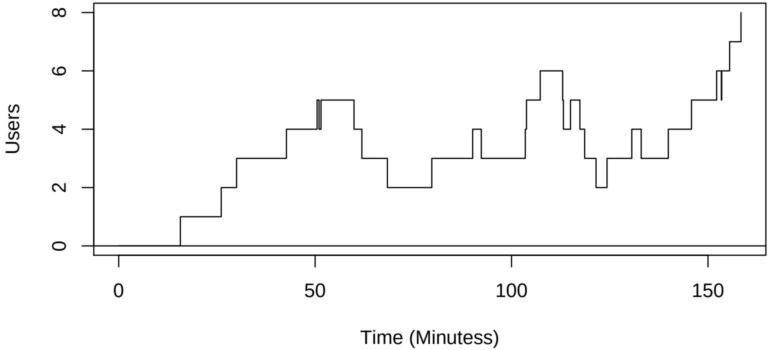
Simulation: Service start and end times

User	Inter-arrival time	Arrival time	Service time	Service ends
1	15.7	15.7	44.2	59.9
2	10.4	26.1	24.9	51.0
3	3.9	30.0	83.0	113.0
4	12.7	42.7	19.2	61.9
5	7.8	50.5	41.8	92.3
6	1.0	51.5	16.9	68.4
7	28.2	79.7	37.7	117.4
8	10.4	90.1	23.1	113.2
9	13.4	103.5	83.8	187.3
10	0.3	103.8	49.6	153.4
11	3.5	107.3	11.3	118.6
12	7.7	115.0	6.5	121.5

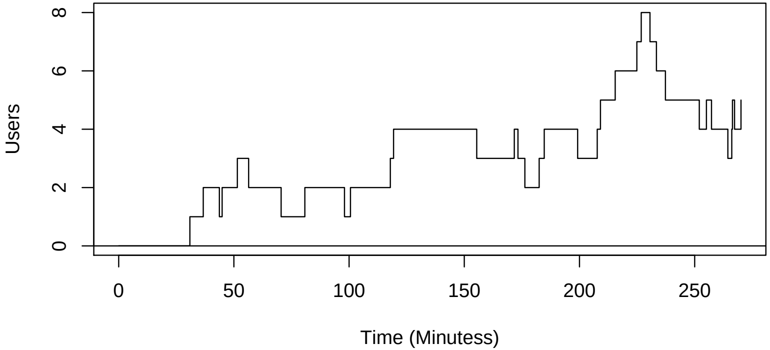
Simulation: Events and user numbers

Event	Time	Change	Number of users
	0.0	0	0
1	15.7	1	1
2	26.1	1	2
3	30.0	1	3
4	42.7	1	4
5	50.5	1	5
6	51.0	-1	4
7	51.5	1	5
8	59.9	-1	4
9	61.9	-1	3
10	68.4	-1	2
11	79.7	1	3

Simulation: Graph



Simulation: Second run



Simulation: Change to $\mu = 5$