NEWCASTLE UNIVERSITY

SCHOOL OF MATHEMATICS & STATISTICS

SEMESTER 1 2011/2012

MAS8303

Modern Bayesian Inference

Time allowed: 2 hours

Candidates should attempt all questions. Marks for each question are indicated. However you are advised that marks indicate the relative weight of individual questions, they do not correspond directly to marks on the University scale.

There are SEVEN questions on this paper.

Calculators may be used. Extracts from the WinBUGS manual are included in this paper.

- Beta $Beta(\alpha, \beta)$ distribution. It has density function

$$f(x|\alpha, \beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)},\ 0 < x < 1,\ \alpha > 0, \beta > 0$$

where $B(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha + \beta)$.
Also, $\mathrm{E}(X) = \alpha/(\alpha + \beta)$ and $\mathrm{Var}(X) = \alpha\beta/[(\alpha + \beta)^2(\alpha + \beta + 1)]$.

- Exponential $Exp(\lambda)$ distribution. It has density

$$f(x|\lambda) = \lambda e^{-\lambda x},\ x > 0,\ \lambda > 0.$$

Also, $\mathrm{E}(X) = 1/\lambda$ and $\mathrm{Var}(X) = 1/\lambda^2$.

- Gamma $Ga(\alpha, \lambda)$ distribution. It has density

$$f(x|\alpha, \lambda) = \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)},\ x > 0,\ \alpha > 0,\ \lambda > 0.$$

Also, $\mathrm{E}(X) = \alpha/\lambda$ and $\mathrm{Var}(X) = \alpha/\lambda^2$.

- Normal $N(\mu, \sigma^2)$ distribution. It has density

$$f(x|\mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\},\ -\infty < x < \infty,$$
$$-\infty < \mu < \infty,\ \sigma > 0.$$

Also, $\mathrm{E}(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$.

- Lognormal $LN(\mu, \sigma^2)$ distribution. It has density

$$f(x|\mu, \sigma) = \frac{1}{x\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(\log x-\mu)^2}{2\sigma^2}\right\},\ x > 0,\ -\infty < \mu < \infty,\ \sigma > 0.$$

Also, $\mathrm{E}(X) = \exp(\mu + \sigma^2/2)$ and $\mathrm{Var}(X) = \exp(2\mu + \sigma^2)[\exp(\sigma^2) - 1]$.

- Rayleigh $R(\theta)$ distribution. It has density

$$f(x|\theta) = 2x\theta e^{-\theta x^2},\quad x > 0,\quad \theta > 0.$$

Also, $\mathrm{E}(X) = \sqrt{\pi/(4\theta)}$ and $\mathrm{Var}(X) = (4 - \pi)/(4\theta)$.

- Poisson $Po(\lambda)$ distribution. It has probability function

$$Pr(X = j \mid \lambda) = \frac{e^{-\lambda}\lambda^j}{j!},\quad j = 0, 1, \ldots,\quad \lambda > 0.$$

Also,

$$\mathrm{E}(X) = \lambda,\qquad \mathrm{Var}(X) = \lambda.$$

- Dirichlet $D_d(a_1, \ldots, a_d)$ distribution. It has density

$$f(x_1, \ldots, x_n \mid a_1, \ldots, a_n) = \frac{\Gamma(A)}{\prod_{i=1}^{d} \Gamma(a_i)} \prod_{i-1}^{d} x_i^{a_i - 1}, \quad A = \sum_{i=1}^{d} a_i,$$

$$0 < a_i,\ 0 < x_i < 1,\ \sum_{i=1}^{d} x_i = 1.$$

Also

$$\mathrm{E}(X_i) = \frac{a_i}{A},\ \ \mathrm{Var}(X_i) = \frac{a_i(A - a_i)}{A^2(A+1)},\ \ \mathrm{Covar}(X_i, X_j) = -\frac{a_i a_j}{A^2(A+1)},\ (i \neq j).$$

BUGS functions

Function	Usage	Definition
Complementary	`cloglog(p)<-a+b*x`	$\log[-\log(1-p)] = a + bx$
log log	`y<-cloglog(p)`	$y = \log[-\log(1-p)]$
Logical equals	`y<-equals(x,z)`	$y = 1$ if $x = z$
		$y = 0$ if $x \neq z$
Exponential	`y<-exp(x)`	$y = e^x$
Inner product	`y<-inprod(a[],b[])`	$y = \sum_i a_i b_i$
Matrix inverse	`y[,]<-inverse(x[,])`	$y = x^{-1}$
		$y,\ x$ both $n \times n$ matrices
Natural logarithm	`log(lambda)<-a+b*x`	$\log(\lambda) = a + bx$
	`y<-log(x)`	$y = \log x$
Log determinant	`y<-logdet(x[,])`	$y = \log \lvert x \rvert$
		x is a $n \times n$ matrix
Log factorial	`y<-logfact(x)`	$y = \log(x!)$
Log(gamma function)	`y<-loggam(x)`	$y = \log[\Gamma(x)]$
Logit	`y<-logit(p)`	$y = \log[p/(1-p)]$
	`logit(p)<-a+b*x`	$\log[p/(1-p)] = a + bx$
Maximum	`c<-max(a,b)`	$c = \max(a, b)$
Mean	`x.bar<-mean(x[])`	$\bar{x} = \sum_i x_i/n$
Minimum	`c<-min(a,b)`	$c = \min(a, b)$
Standard Gaussian	`p<-phi(x)`	$p = \int_{-\infty}^{x} (2\pi)^{-1/2} e^{-t^2/2}\ dt$
distribution function		i.e. $p = \Phi(x)$
Power	`z<-pow(x,y)`	$z = x^y$
Probit	`y<-probit(p)`	$y = \Phi^{-1}(p)$
	`probit(p)<-a+b*x`	$\Phi^{-1}(p) = a + bx$
Standard deviation	`s<-sd(x[])`	$s = \sqrt{\sum_i (x_i - \bar{x})^2/n}$
Square root	`sigma<-sqrt(tau)`	$\sigma = \sqrt{\tau}$
Unit step	`y<-step(x)`	$y = 0$ if $x < 0$
		$y = 1$ if $x \geq 0$
Sum	`x.sum<-sum(x[])`	$x_{\text{sum}} = \sum_i x_i$

BUGS distributions

Distribution	Usage	Definition
Bernoulli	`r~dbern(p)`	$f(r \mid p) = p^r(1-p)^{1-r}$; $r = 0, 1$
beta	`p~dbeta(a,b)`	$f(p \mid a, b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} p^{a-1}(1-p)^{b-1}$; $0 < p < 1$
binomial	`r~dbin(p,n)`	$f(r \mid p, n) = \frac{n!}{r!(n-r)!} p^r (1-p)^{n-r}$; $r = 0, \dots, n$
categorical	`r~dcat(p[])`	$f(r \mid p_1, \dots, p_R) = p_r$; $r = 1, 2, \dots, R$ where $R = \dim(\mathbf{p})$
chi-squared	`x~dchisq(k)`	$f(x \mid k) = 2^{-k/2} x^{k/2-1} e^{-x/2} / \Gamma(\frac{k}{2})$; $x > 0$
double exponential	`x~ddexp(mu,tau)`	$f(x \mid \mu, \tau) = \frac{\tau}{2} e^{-\tau \lvert x-\mu \rvert}$; $-\infty < x < \infty$
Dirichlet	`p[]~ddirch(alpha[])`	$f(\mathbf{p} \mid \alpha) = \frac{\Gamma(\sum_i \alpha_i)}{\prod_i \Gamma(\alpha_i)} \prod_i p_i^{\alpha_i - 1}$; $0 < p_i < 1, \ \sum_i p_i = 1$
exponential	`x~dexp(lambda)`	$f(x \mid \lambda) = \lambda e^{-\lambda x}$; $x > 0$
gamma	`x~dgamma(r,mu)`	$f(x \mid r, \mu) = \mu^r x^{r-1} e^{-\mu x} / \Gamma(r)$; $x > 0$
lognormal	`x~dlnorm(mu,tau)`	$f(x \mid \mu, \tau) = \sqrt{\frac{\tau}{2\pi}} x^{-1} \exp[-\frac{\tau}{2}(\log x - \mu)^2]; \quad x > 0$
logistic	`x~dlogis(mu,tau)`	$f(x \mid \mu, \tau) = \tau e^{\tau(x-\mu)} / (1 + e^{\tau(x-\mu)})^2; \quad -\infty < x < \infty$
multivariate normal	`x[]~dmnorm(mu[],T[,])`	$f(\mathbf{x} \mid \mu, \mathbf{T}) = (2\pi)^{-N/2} \lvert\mathbf{T}\rvert^{1/2} \exp[-\frac{1}{2}(\mathbf{x} - \mu)' \mathbf{T} (\mathbf{x} - \mu)]$; $-\infty < x_i < \infty$
multinomial	`x[]~dmulti(p[],N)` $0 < p_i < 1, \ \sum_i p_i = 1$	$f(\mathbf{x} \mid \mathbf{p}, N) = \frac{(\sum_i x_i)!}{\prod_i x_i!} \prod_i p_i^{x_i}$; $\sum_i x_i = N$
negative binomial	`x~dnegbin(p,r)`	$f(x \mid p, r) = \frac{(x+r-1)!}{x!(r-1)!} p^r (1-p)^x$; $x = 0, 1, 2, \dots$
normal	`x~dnorm(mu,tau)`	$f(x \mid \mu, \tau) = \sqrt{\frac{\tau}{2\pi}} \exp[-\frac{\tau}{2}(x - \mu)^2]; \quad -\infty < x < \infty$
Pareto	`x~dpar(alpha,c)`	$f(x \mid \alpha, c) = \alpha c^\alpha x^{-(\alpha+1)}$; $x > c$

BUGS distributions continued

Distribution	Usage	Definition
Poisson	`r~dpois(lambda)`	$f(r \mid \lambda) = e^{-\lambda}\frac{\lambda^r}{r!}$; $r = 0, 1, \ldots$
Student's t	`x~dt(mu,tau,k)`	$f(x \mid \mu, \tau, k) = \frac{\Gamma([k+1]/2)}{\Gamma(k/2)}\sqrt{\frac{\tau}{k\pi}}[1 + \frac{\tau}{k}(x-\mu)^2]^{-(k+1)/2}$; $-\infty < x < \infty$
uniform	`x~dunif(a,b)`	$f(x \mid a, b) = 1/(b-a)$; $a < x < b$
Weibull	`x~dweib(v,lambda)`	$f(x \mid v, \lambda) = v\lambda x^{v-1}\exp(-\lambda x^v)$; $x > 0$
Wishart	`x[,]~dwish(R[,],k)`	$f(\mathbf{x} \mid \mathbf{R}, k) \propto \lvert\mathbf{R}\rvert^{k/2}\lvert\mathbf{x}\rvert^{(k-p-1)/2}\exp(-\frac{1}{2}\mathrm{tr}[\mathbf{Rx}])$; $\mathbf{x}$ symmetric and positive definite

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