MAS3301 Bayesian Statistics
Problems 3 and Solutions

Semester 2

2008-9

Problems 3

1. In a small survey, a random sample of 50 people from a large population is selected. Each person is asked a question to which the answer is either "Yes" or "No." Let the proportion in the population who would answer "Yes" be θ. Our prior distribution for θ is a beta$(1.5, 1.5)$ distribution. In the survey, 37 people answer "Yes."

   (a) Find the prior mean and prior standard deviation of θ.

   (b) Find the prior probability that $\theta < 0.6$.

   (c) Find the likelihood.

   (d) Find the posterior distribution of θ.

   (e) Find the posterior mean and posterior standard deviation of θ.

   (f) Plot a graph showing the prior and posterior probability density functions of θ on the same axes.

   (g) Find the posterior probability that $\theta < 0.6$.

   Notes:

   The probability density function of a beta(a, b) distribution is $f(x) = kx^{a-1}(1-x)^{b-1}$ where k is a constant.

   If $X \sim$ beta(a, b) then the mean of X is

   $$E(X) = \frac{a}{a+b}$$

   and the variance of X is

   $$\text{var}(X) = \frac{ab}{(a+b+1)(a+b)^2}.$$

   If $X \sim$ beta(a, b) then you can use a command such as the following in R to find $\Pr(X < c)$.

```
pbeta(c,a,b)
```

   To plot the prior and posterior probability densities you may use R commands such as the following.

```
theta<-seq(0.01,0.99,0.01)
prior<-dbeta(theta,a,b)
posterior<-dbeta(theta,c,d)
plot(theta,posterior,xlab=expression(theta),ylab="Density",type="l")
lines(theta,prior,lty=2)
```

2. The populations, n_i, and the number of cases, x_i, of a disease in a year in each of six districts are given in the table below.

Population n	Cases x
120342	2
235967	5
243745	3
197452	5
276935	3
157222	1

We suppose that the number X_i in a district with population n_i is a Poisson random variable with mean $n_i\lambda/100000$. The number in each district is independent of the numbers in other districts, given the value of λ. Our prior distribution for λ is a gamma distribution with mean 3.0 and standard deviation 2.0.

(a) Find the parameters of the prior distribution.

(b) Find the prior probability that $\lambda < 2.0$.

(c) Find the likelihood.

(d) Find the posterior distribution of λ.

(e) Find the posterior mean and posterior standard deviation of λ.

(f) Plot a graph showing the prior and posterior probability density functions of λ on the same axes.

(g) Find the posterior probability that $\lambda < 2.0$.

Notes:

The probability density function of a gamma(a, b) distribution is $f(x) = kx^{a-1}\exp(-bx)$ where k is a constant.

If $X \sim$ gamma(a, b) then the mean of X is $\mathrm{E}(X) = a/b$ and the variance of X is $\mathrm{var}(X) = a/(b^2)$.

If $X \sim$ gamma(a, b) then you can use a command such as the following in R to find $\Pr(X < c)$.

```
pgamma(c,a,b)
```

To plot the prior and posterior probability densities you may use R commands such as the following.

```
lambda<-seq(0.00,5.00,0.01)
prior<-dgamma(lambda,a,b)
posterior<-dgamma(lambda,c,d)
plot(lambda,posterior,xlab=expression(lambda),ylab="Density",type="l")
lines(lambda,prior,lty=2)
```

3. Geologists note the type of rock at fixed vertical intervals of six inches up a quarry face. At this quarry there are four types of rock. The following model is adopted.

The conditional probability that the next rock type is j given that the present type is i and given whatever has gone before is p_{ij}. Clearly $\sum_{j=1}^{4} p_{ij} = 1$ for all i.

The following table gives the observed (upwards) transition frequencies.

		To rock			
		1	2	3	4
From rock	1	56	13	24	4
	2	15	93	22	35
	3	20	25	153	11
	4	6	35	11	44

Our prior distribution for the transition probabilities is as follows. For each i we have a uniform distribution over the space of possible values of $p_{i1}, \dots, p_{i4}$. The prior distribution of $p_{i1}, \dots, p_{i4}$ is independent of that for $p_{k1}, \dots, p_{k4}$ for $i \neq k$.

Find the matrix of posterior expectations of the transition probabilities.

Note that the integral of $x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4}$ over the region such that $x_j > 0$ for $j = 1, \dots, 4$ and $\sum_{j=1}^{4} x_j = 1$, where $n_1, \dots, n_4$ are positive is

$$\int_0^1 x_1^{n_1} \int_0^{1-x_1} x_2^{n_2} \int_0^{1-x_1-x_2} x_3^{n_3}(1 - x_1 - x_2 - x_3)^{n_4}.dx_3.dx_2.dx_1$$

$$= \frac{\Gamma(n_1+1)\Gamma(n_2+1)\Gamma(n_3+1)\Gamma(n_4+1)}{\Gamma(n_1+n_2+n_3+n_4+4)}$$

4. A biologist is interested in the proportion, θ, of badgers in a particular area which carry the infection responsible for bovine tuberculosis. The biologist's prior distribution for θ is a beta$(1, 19)$ distribution.

   (a) i. Find the biologist's prior mean and prior standard deviation for θ.

      ii. Find the cumulative distribution function of the biologist's prior distribution and hence find values $\theta_1,\ \theta_2$ such that, in the biologist's prior distribution, $\Pr(\theta < \theta_1) = \Pr(\theta > \theta_2) = 0.05$.

   (b) The biologist captures twenty badgers and tests them for the infection. Assume that, given θ, the number, X, of these carrying the infection has a binomial$(20, \theta)$ distribution. The observed number carrying the infection is $x = 2$.

      i. Find the likelihood function.

      ii. Find the biologist's posterior distribution for θ.

      iii. Find the biologist's posterior mean and posterior standard deviation for θ.

      iv. Use R to plot a graph showing the biologist's prior and posterior probability density functions for θ.

5. A factory produces large numbers of packets of nuts. As part of the quality control process, samples of the packets are taken and weighed to check whether they are underweight. Let the true proportion of packets which are underweight be θ and assume that, given θ, the packets are independent and each has probability θ of being underweight. A beta$(1, 9)$ prior distribution for θ is used.

   (a) The procedure consists of selecting packets until either an underweight packet is found, in which case we stop and note the number X of packets examined, or $m = 10$ packets are examined and none is underweight, in which we case we stop and note this fact.

      i. Find the posterior distribution for θ when $X = 7$ is observed.

      ii. Find the posterior distribution for θ when no underweight packets are found out of $m = 10$.

   (b) Now consider varying the value of m. Use R to find the posterior probability that $\theta < 0.02$ when no underweight packets are found out of

      i. $m = 10$,

      ii. $m = 20$,

iii. $m = 30$.

6. The numbers of patients arriving at a minor injuries clinic in 10 half-hour intervals are recorded. It is supposed that, given the value of a parameter λ, the number X_j arriving in interval j has a Poisson distribution $X_j \sim \text{Poisson}(\lambda)$ and X_j is independent of X_k for $j \neq k$. The prior distribution for λ is a gamma(a, b) distribution. The prior mean is 10 and the prior standard deviation is 5.

   (a) i. Find the values of a and b.
   ii. Let $W \sim \chi^2_{2a}$. Find values w_1, w_2 such that $\Pr(W < w_1) = \Pr(W > w_2) = 0.025$. Hence find values l_1, l_2 such that, in the prior distribution, $\Pr(\lambda < l_1) = \Pr(\lambda > l_2) = 0.025$.
   iii. Using R (or otherwise) find a 95% prior highest probability density interval for λ.
   iv. Compare these two intervals.

   (b) The data are as follows.

   ```
   9  12  16  12  16  11  18  13  12  19
   ```

   i. Find the posterior distribution of λ.
   ii. Using R (or otherwise) find a 95% posterior highest probability density interval for λ.

7. The numbers of sales of a particular item from an Internet retail site in each of 20 weeks are recorded. Assume that, given the value of a parameter λ, these numbers are independent observations from the Poisson(λ) distribution.

   Our prior distribution for λ is a gamma(a, b) distribution.

   (a) Our prior mean and standard deviation for λ are 16 and 8 respectively. Find the values of a and b.

   (b) The observed numbers of sales are as follows.

   ```
   14 19 14 21 22 33 15 13 16 19 27 22 27 21 16 25 14 23 22 17
   ```

   Find the posterior distribution of λ.

   (c) Using R or otherwise, plot a graph showing both the prior and posterior probability density functions of λ.

   (d) Using R or otherwise, find a 95% posterior hpd interval for λ. *(Note: The R function* `hpdgamma` *is available from the Module Web Page).*

8. In a medical experiment, patients with a chronic condition are asked to say which of two treatments, A, B, they prefer. (You may assume for the purpose of this question that every patient will express a preference one way or the other). Let the population proportion who prefer A be θ. We observe a sample of n patients. Given θ, the n responses are independent and the probability that a particular patient prefers A is θ.

   Our prior distribution for θ is a beta(a, a) distribution with a standard deviation of 0.25.

   (a) Find the value of a.

   (b) We observe $n = 30$ patients of whom 21 prefer treatment A. Find the posterior distribution of θ.

   (c) Find the posterior mean and standard deviation of θ.

   (d) Using R or otherwise, plot a graph showing both the prior and posterior probability density functions of θ.

   (e) Using R or otherwise, find a symmetric 95% posterior probability interval for θ. *(Hint: The R command* `qbeta(0.025,a,b)` *will give the 2.5% point of a beta(a, b) distribution).*

9. The survival times, in months, of patients diagnosed with a severe form of a terminal illness are thought to be well modelled by an exponential(λ) distribution. We observe the survival times of n such patients. Our prior distribution for λ is a gamma(a, b) distribution.

   (a) Prior beliefs are expressed in terms of the median lifetime, m. Find an expression for m in terms of λ.

   (b) In the prior distribution, the lower 5% point for m is 6.0 and the upper 5% point is 46.2. Find the corresponding lower and upper 5% points for λ. Let these be $k_1,\ k_2$ respectively.

   (c) Let $k_2/k_1 = r$. Find, to the nearest integer, the value of ν such that, in a χ^2_ν distribution, the 95% point divided by the 5% point is r and hence deduce the value of a.

   (d) Using your value of a and one of the percentage points for λ, find the value of b.

   (e) We observe $n = 25$ patients and the sum of the lifetimes is 502. Find the posterior distribution of λ.

   (f) Using the relationship of the gamma distribution to the χ^2 distribution, or otherwise, find a symmetric 95% posterior interval for λ.

   Note: The R command `qchisq(0.025,nu)` *will give the lower 2.5% point of a* χ^2 *distribution on* `nu` *degrees of freedom.*

Homework 3

Solutions to Questions **7, 8, 9** *of Problems 3 are to be submitted in the Homework Letterbox no later than 4.00pm on Monday March 9th.*

Solutions

1. (a) In the prior $a = 1.5$ and $b = 1.5$. So the mean is

$$\frac{a}{a+b} = \frac{1.5}{3.0} = \underline{0.5}.$$

The variance is

$$\frac{ab}{(a+b)^2(a+b+1)} = \frac{1.5 \times 1.5}{3^2 \times 4} = \frac{1}{16}$$

so the standard deviation is

$$\frac{1}{4} = \underline{0.25}.$$

(b) Using R the prior probability that $\theta < 0.6$ is $\underline{0.62647}$.

```
> pbeta(0.6,1.5,1.5)
[1] 0.62647
```

(c) The likelihood is

$$\begin{pmatrix} 50 \\ 37 \end{pmatrix} \theta^{37}(1-\theta)^{13}.$$

(d) The prior density is proportional to $\theta^{1.5-1}(1-\theta)^{1.5-1}$
The likelihood is proportional to $\theta^{37}(1-\theta)^{13}$
Hence the posterior density is proportional to $\theta^{38.5-1}(1-\theta)^{14.5-1}$
The posterior distribution is $\underline{\text{beta}(38.5,\ 14.5)}$.

(e) In the posterior $a = 38.5$ and $b = 14.5$. So the mean is

$$\frac{a}{a+b} = \frac{38.5}{53.0} = \underline{0.7264}.$$

The variance is

$$\frac{ab}{(a+b)^2(a+b+1)} = \frac{38.5 \times 14.5}{53^2 \times 54} = 3.6803 \times 10^{-3}$$

so the standard deviation is $\underline{0.06067}$.

(f) See Figure 1.

```
> theta<-seq(0.01,0.99,0.01)
> prior<-dbeta(theta,1.5,1.5)
> posterior<-dbeta(theta,38.5,14.5)
> plot(theta,posterior,xlab=expression(theta),ylab="Density",type="l")
> lines(theta,prior,lty=2)
```

(g) Using R the posterior probability that $\theta < 0.6$ is $\underline{0.02490528}$.

```
> pbeta(0.6,38.5,14.5)
[1] 0.02490528
```

2. (a) The mean is $a/b = 3$ and the variance is $a/b^2 = 4$. So

$$\frac{9}{4} = \frac{a^2/b^2}{a/b^2} = a,$$

giving $a = \underline{2.25}$ and

$$b = \frac{2.25}{3} = \underline{0.75}.$$

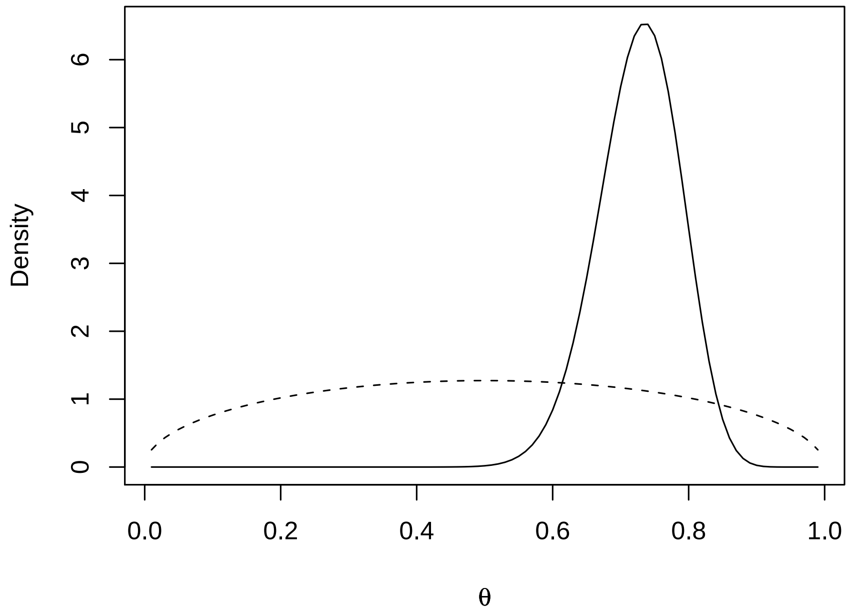

Figure 1: Prior (dashes) and posterior (solid) pdfs for Question 1.

(b) Using R the prior probability that $\lambda < 2.0$ is $\underline{0.3672305}$.

```
> pgamma(2,2.25,0.75)
[1] 0.3672305
```

(c) The likelihood is

$$\prod_{i=1}^{n} \frac{e^{-\lambda_i}\lambda_i^{x_i}}{x_i!} = \frac{e^{-\sum \lambda_i}\prod \lambda_i^{x_i}}{\prod x_i!}$$
$$= e^{-\lambda n/100000}\lambda^{S}\frac{\prod (n_i/100000)^{x_i}}{\prod x_i!}$$

where $n = \sum n_i = 1231663$ and $S = \sum x_i = 19$.
This is proportional to

$$e^{-12.31663\lambda}\lambda^{19}.$$

(d) The prior density is proportional to $\lambda^{2.25-1}e^{-0.75\lambda}$
The likelihood is proportional to $\lambda^{19}e^{-12.316663\lambda}$
Hence the posterior density is proportional to $\lambda^{21.25-1}e^{-13.06663\lambda}$
The posterior distribution is $\underline{\text{gamma}(21.25,\ 13.06663)}$.

(e) In the posterior $a = 21.25$ and $b = 13.06663$. So the mean is

$$\frac{a}{b} = \frac{21.25}{13.06663} = \underline{1.6262}.$$

The standard deviation is

$$\frac{\sqrt{a}}{b} = \frac{\sqrt{21.25}}{13.06663} = \underline{0.3528}.$$

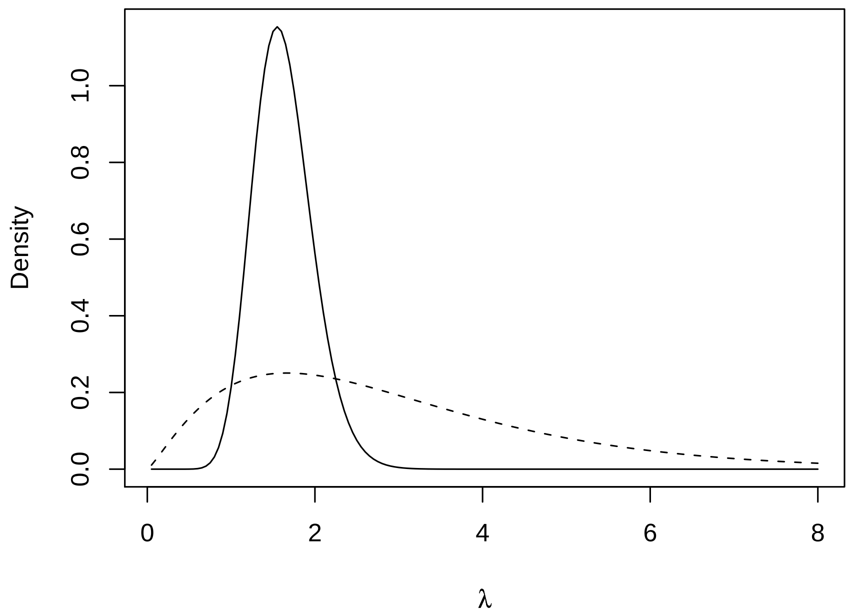

Figure 2: Prior (dashes) and posterior (solid) pdfs for Question 2.

(f) See Figure 2.

```
> lambda<-seq(0.05,8.0,0.05)
> prior<-dgamma(lambda,2.25,0.75)
> posterior<-dgamma(lambda,21.25,13.06663)
> plot(lambda,posterior,xlab=expression(lambda),ylab="Density",type="l")
> lines(lambda,prior,lty=2)
```

(g) Using R the posterior probability that $\lambda < 2.0$ is 0.8551274.

```
> pgamma(2,21.25,13.06663)
[1] 0.8551274
```

3. Since the prior distribution is uniform the prior density is a constant. Therefore the posterior density is proportional to the likelihood. the likelihood is

$$L = \prod_{i=1}^{4} \prod_{j=1}^{4} p_{ij}^{n_{ij}}$$

where n_{ij} is the observed number of transitions from rock i to rock j.

The posterior density is therefore

$$\prod_{i=1}^{4} f_i^{(1)}(p_{i1}, p_{i2}, p_{i3}, p_{i4})$$

where

$$f_i^{(1)}(p_{i1}, p_{i2}, p_{i3}, p_{i4}) = k_{1i} p_{i1}^{n_{i1}} p_{i2}^{n_{i2}} p_{i3}^{n_{i3}} p_{i4}^{n_{i4}}$$

is the posterior density of $p_{i1}, p_{i2}, p_{i3}, p_{i4}$.

Since

$$\int\int\int_R f_i^{(1)}(p_{i1}, p_{i2}, p_{i3}, p_{i4})\ dp_{i1}\ dp_{i2}\ dp_{i3} = 1,$$

we must have

$$\begin{aligned} k_{1i}^{-1} &= \int\int\int_R p_{i1}^{n_{i1}} p_{i2}^{n_{i2}} p_{i3}^{n_{i3}} p_{i4}^{n_{i4}}\ dp_{i1}\ dp_{i2}\ dp_{i3} \\ &= \frac{\Gamma(n_{i1}+1)\Gamma(n_{i2}+1)\Gamma(n_{i3}+1)\Gamma(n_{i4}+1)}{\Gamma(N_i+4)} \end{aligned}$$

where $N_i = n_{i1} + n_{i2} + n_{i3} + n_{i4}$ and the integrals are taken over the region R in which $(p_{i1}, p_{i2}, p_{i3}, p_{i4})$ must lie and $p_{i4} = 1 - p_{i1} - p_{i2} - p_{i3}$.

Now, the posterior mean of p_{i1}, for example, is

$$\begin{aligned} \mathrm{E}^{(1)}(p_{i1}) &= \int\int\int_R p_{i1} f_i^{(1)}(p_{i1}, p_{i2}, p_{i3}, p_{i4})\ dp_{i1}\ dp_{i2}\ dp_{i3} \\ &= \int\int\int_R k_{i1} p_{i1}^{n_{i1}+1} p_{i2}^{n_{i2}} p_{i3}^{n_{i3}} p_{i4}^{n_{i4}}\ dp_{i1}\ dp_{i2}\ dp_{i3} \\ &= k_{1i}k_{2i} \end{aligned}$$

where

$$k_{2i} = \frac{\Gamma(n_{i1}+2)\Gamma(n_{i2}+1)\Gamma(n_{i3}+1)\Gamma(n_{i4}+1)}{\Gamma(N_i+5)}.$$

So

$$\begin{aligned} \mathrm{E}^{(1)}(p_{i1}) &= \frac{\Gamma(n_{i1}+2)\Gamma(n_{i2}+1)\Gamma(n_{i3}+1)\Gamma(n_{i4}+1)}{\Gamma(n_{i1}+1)\Gamma(n_{i2}+1)\Gamma(n_{i3}+1)\Gamma(n_{i4}+1)} \frac{\Gamma(N_i+4)}{\Gamma(N_i+5)} \\ &= \frac{(n_{i1}+1)\Gamma(n_{i1}+1)}{\Gamma(n_{i1}+1)} \frac{\Gamma(N_i+4)}{(N_i+4)\Gamma(N_i+4)} \\ &= \frac{n_{i1}+1}{N_i+4} \end{aligned}$$

In general

$$\mathrm{E}^{(1)}(p_{ij}) = \frac{n_{ij}+1}{N_i+4}.$$

The table of posterior means is as follows.

		To rock			
		1	2	3	4
From rock	1	0.5644	0.1386	0.2475	0.0495
	2	0.0947	0.5562	0.1361	0.2130
	3	0.0986	0.1221	0.7230	0.0563
	4	0.0700	0.3600	0.1200	0.4500

4.

5.

6.

7. (a) Prior mean:

$$\frac{a}{b} = 16,$$

Prior variance:

$$\frac{a}{b^2} = 64.$$

Hence $\underline{a = 4}$ and $\underline{b = 0.25}$.

(1 mark)

(b) From the data $s = \sum_{i=1}^{20} x_i = 400$.
Prior density proportional to

$$\lambda^{4-1} e^{-0.25\lambda}$$

Likelihood proportional to

$$\prod_{i=1}^{20} e^{-\lambda} \lambda^{x_i} = e^{-20\lambda} \lambda^{s} = \lambda^{400} e^{-20\lambda}$$

Hence posterior density proportional to

$$\lambda^{404-1} e^{-20.25\lambda}$$

This is a $\underline{\text{ gamma(404, 20.25) }}$ distribution.

(1 mark)

(c) R commands:

```
> sales<-c(14,19,14,21,22,33,15,13,16,19,27,22,27,21,16,25,14,23,22,17)> lambda<-seq(10,25,0.05
> prior<-dgamma(lambda,4,0.25)
> post<-dgamma(lambda,404,20.25)
> pdf("probs309q7.pdf",height=5)
> plot(lambda,post,type="l",xlab=expression(lambda),ylab="Density")
> lines(lambda,prior,lty=2)
> abline(0,0)
> dev.off()
```

The graph is shown in Figure 3.

(2 marks)

(d) R commands and result:

```
> hpdgamma(0.95,404,20.25)
"   Lower   Upper   Difference"
1 9.17326 21.6109 1
2 13.7599 21.6109 1
3 16.0532 21.611 0.998601
4 17.1999 21.6305 0.86782
5 17.7732 21.7447 0.402608
6 18.0599 21.951 -0.0807799
7 17.9165 21.8224 0.189252
8 17.9882 21.878 0.062461
9 18.024 21.9119 -0.00689586
10 18.0061 21.8944 0.0283184
11 18.0151 21.903 0.0108487
12 18.0195 21.9074 0.00201123
13 18.0218 21.9096 -0.00243356
```

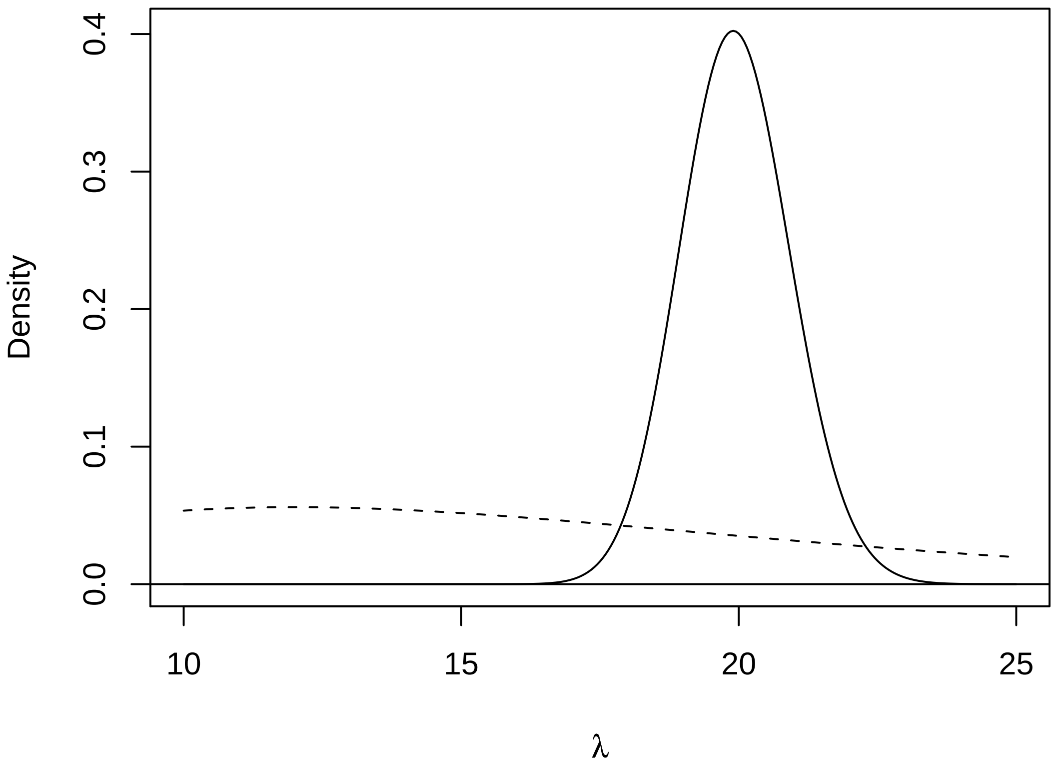

Figure 3: Prior (dashes) and posterior (solid line) density functions for λ (Question 7).

```
14 18.0207 21.9085 -0.00020898
15 18.0201 21.908 0.000901669
16 18.0204 21.9082 0.000346481
17 18.0205 21.9084 6.87842e-05
[1] 18.02052 21.90838
>
```

The 95% hpd interval is $\underline{18.02 < \lambda < 21.91}$.

(2 marks)

8. (a) Variance of beta(a, b):

$$\frac{ab}{(a+b+1)(a+b)^2}$$

Variance of beta(a, a):

$$\frac{a^2}{(2a+1)(2a)^2} = \frac{1}{4(2a+1)}$$

$$\frac{1}{4(2a+1)} = \frac{1}{4^2} \quad \Rightarrow \quad (2a+1) = 4 \quad \Rightarrow \quad \underline{a = 1.5}$$

(1 mark)

(b) Prior: beta(1.5, 1.5)
Likelihood: $\theta^{21}(1-\theta)^9$
Posterior: $\underline{\text{beta}(22.5,\ 10.5)}$

(1 mark)

(c) Posterior mean:

$$\frac{22.5}{22.5+10.5} = \frac{22.5}{33} = \underline{0.6818}$$

Posterior variance:

$$\frac{22.5 \times 10.5}{34 \times 33^2} = 0.006381$$

Posterior std.dev.:

$$\sqrt{0.006381} = \underline{0.0799}$$

(1 mark)

(d) R commands:

```
> theta<-seq(0,1,0.005)
> prior<-dbeta(theta,1.5,1.5)
> post<-dbeta(theta,22.5,10.5)
> pdf("probs309q8.pdf",height=5)
> plot(theta,post,type="l",xlab=expression(theta),ylab="Density")
> lines(theta,prior,lty=2)
> abline(0,0)
> dev.off()
```

The graph is shown in Figure 4.

(2 marks)

(e) R commands and results:

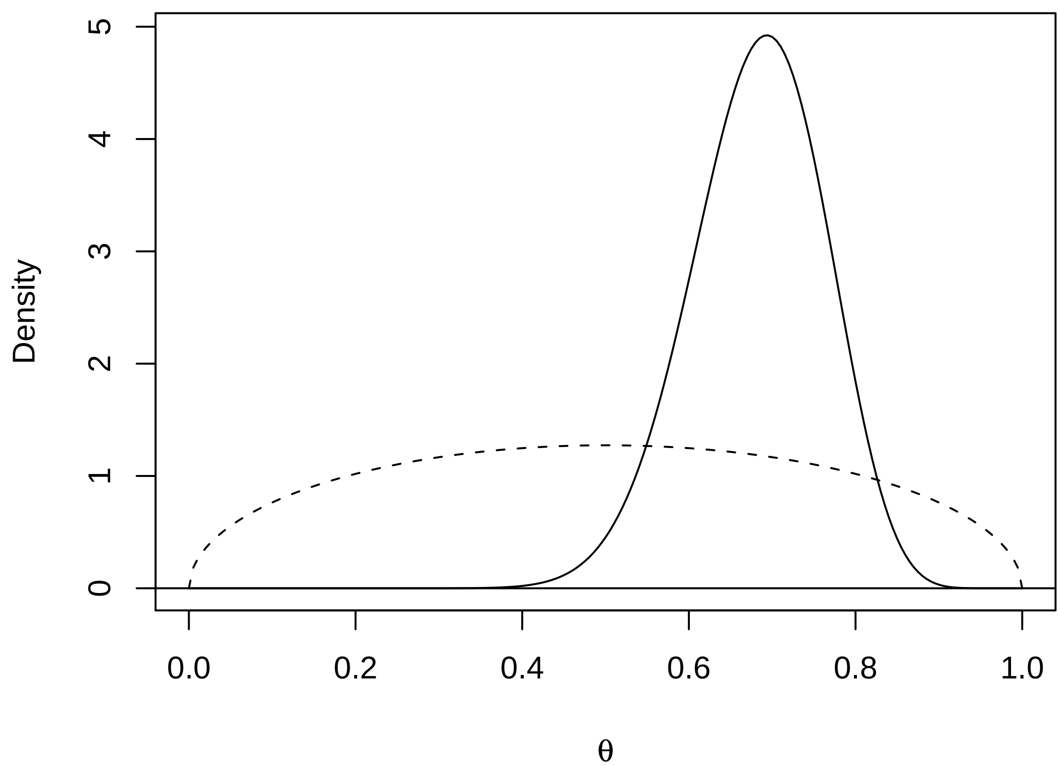

Figure 4: Prior (dashes) and posterior (solid line) density functions for θ (Question 8).

```
> qbeta(0.025,22.5,10.5)
[1] 0.5161281
> qbeta(0.975,22.5,10.5)
[1] 0.8266448
```

The 95% symmetric interval is $\underline{0.516 < \theta < 0.827}$.

(1 mark)

9. (a) Median

$$e^{-\lambda m} = \frac{1}{2} \quad \text{so} \quad \lambda m = \log 2 \quad \text{so} \quad m = \frac{\log 2}{\lambda}.$$

(1 mark)

(b) We have $\lambda = (\log 2)/m$ so

$$\begin{aligned} k_1 = \frac{\log 2}{46.2} &= \underline{0.0150} \\ k_2 = \frac{\log 2}{6.0} &= \underline{0.1155} \end{aligned}$$

(1 mark)

(c) Find r:

$$r = \frac{k_2}{k_1} = 7.7.$$

This is satisfied by $\nu = 6$. See R:

```
> nu=5
> qchisq(0.95,nu)/qchisq(0.05,nu)
[1] 9.664537
> nu=6
> qchisq(0.95,nu)/qchisq(0.05,nu)
[1] 7.699473
```

Hence $a = \nu/2 = \underline{3}$.

(1 mark)

(d) Lower 5% point of χ^2_6 (i.e. gamma(3, 1/2)) is 1.635383.

```
> qchisq(0.05,6)
[1] 1.635383
```

So

$$\frac{b}{1/2} = \frac{1.635383}{0.0150} = 109.00 \quad \text{so} \quad \underline{b = 54.5}.$$

Prior distribution is gamma(3, 54.5).

(2 marks)

Prior density proportional to

$$\lambda^{3-1}e^{-54.5\lambda}$$

Likelihood:

$$\prod_{i=1}^{25} \lambda e^{-\lambda t_i} = \lambda^{25} e^{-\lambda \sum t_i} = \lambda^{25} e^{-502\lambda}$$

Posterior density proportional to

$$\lambda^{28-1}e^{-556.5\lambda}$$

This is a $\underline{\text{gamma}(28,\ 556.5)}$ distribution.

(1 mark)

(e) Using the relationship with the χ^2 distribution:

$$2 \times 556.5\lambda \sim \chi^2_{56}$$

95% interval:

$$37.21159 < \chi^2_{56} < 78.56716$$

$$\frac{37.21559}{2 \times 556.5} < \lambda < \frac{78.56716}{2 \times 556.5}$$

$$\underline{0.03343 < \lambda < 0.07059}$$

(2 marks)

```
> qchisq(0.025,56)
[1] 37.21159
> qchisq(0.975,56)
[1] 78.56716
```