MAS3301 Bayesian Statistics
Problems 2 and Solutions

Semester 2

2008-9

Problems 2

Useful integrals: In solving these problems you might find the following useful.

- Gamma functions: Let a and b be positive. Then

$$\int_0^\infty x^{a-1}e^{-bx}\ dx = \frac{\Gamma(a)}{b^a}$$

where

$$\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}\ dx = (a-1)\Gamma(a-1).$$

If a is a positive integer then $\Gamma(a) = (a-1)!$.

- Beta functions: Let a and b be positive. Then

$$\int_0^1 x^{a-1}(1-x)^{b-1}\ dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

1. We are interested in the mean, λ, of a Poisson distribution. We have a prior distribution for λ with density

$$f^{(0)}(\lambda) = \begin{cases} 0 & (\lambda \le 0) \\ k_0(1+\lambda)e^{-\lambda} & (\lambda > 0) \end{cases}.$$

(a) i. Find the value of k_0.
ii. Find the prior mean of λ.
iii. Find the prior standard deviation of λ.

(b) We observe data $x_1, \ldots, x_n$ where, given λ, these are independent observations from the Poisson(λ) distribution.
i. Find the likelihood.
ii. Find the posterior density of λ.
iii. Find the posterior mean of λ.

2. We are interested in the parameter, θ, of a binomial(n, θ) distribution. We have a prior distribution for θ with density

$$f^{(0)}(\theta) = \begin{cases} k_0\{\theta^2(1-\theta) + \theta(1-\theta)^2\} & (0 < \theta < 1) \\ 0 & (\text{otherwise}) \end{cases}.$$

(a) i. Find the value of k_0.
ii. Find the prior mean of θ.
iii. Find the prior standard deviation of θ.

(b) We observe x, an observation from the binomial(n, θ) distribution.

i. Find the likelihood.
ii. Find the posterior density of θ.
iii. Find the posterior mean of θ.

3. We are interested in the parameter θ, of a binomial(n,θ) distribution. We have a prior distribution for θ with density

$$f^{(0)}(\theta) = \begin{cases} k_0\theta^2(1-\theta)^3 & (0<\theta<1) \\ 0 & (\text{otherwise}) \end{cases}.$$

(a) i. Find the value of k_0.
ii. Find the prior mean of θ.
iii. Find the prior standard deviation of θ.

(b) We observe x, an observation from the binomial(n,θ) distribution.
i. Find the likelihood.
ii. Find the posterior density of θ.
iii. Find the posterior mean of θ.

4. In a manufacturing process packages are made to a nominal weight of 1kg. All underweight packages are rejected but the remaining packages may be slightly overweight. It is believed that the excess weight X, in g, has a continuous uniform distribution on $(0,\theta)$ but the value of θ is unknown. Our prior density for θ is

$$f^{(0)}(\theta) = \begin{cases} 0 & (\theta<0) \\ k_0/100 & (0\le\theta<10) \\ k_0\theta^{-2} & (10\le\theta<\infty) \end{cases}.$$

(a) i. Find the value of k_0.
ii. Find the prior median of θ.

(b) We observe 10 packages and their excess weights, in g, are as follows.

3.8 2.1 4.9 1.8 1.7 2.1 1.4 3.6 4.1 0.8

Assume that these are independent observations, given θ.
i. Find the likelihood.
ii. Find a function $h(\theta)$ such that the posterior density of θ is $f^{(1)}(\theta) = k_1h(\theta)$, where k_1 is a constant.
iii. Evaluate the constant k_1. *(Note that it is a very large number but you should be able to do the evaluation using a calculator).*

5. Repeat the analysis of the Chester Road example in section 6.3, using the same likelihood but with the following prior density.

$$f^{(0)}(\lambda) = \begin{cases} k_0[1+(8\lambda)^2]^{-1} & (0<\lambda<\infty) \\ 0 & (\text{otherwise}) \end{cases}$$

(a) Find the value of k_0.
(b) Use numerical methods in R to do the following.
i. Find the posterior density and plot a graph showing both the prior and posterior densities.
ii. Find the posterior mean and standard deviation.

Note: For the numerical calculations and the plot in part (b) I suggest that you use a range $0.0 \le \lambda \le 0.2$. When plotting the graph, it is easiest to plot the posterior first as this will determine the length of the vertical axis. The value of k_0 can be found analytically. If you do use numerical integration to find it, you will need a much wider range of values of λ.

6. We are interested in the parameter λ of a Poisson(λ) distribution. We have a prior distribution for λ with density

$$f^{(0)}(\lambda) = \begin{cases} 0 & (\lambda < 0) \\ k_0 \lambda^3 e^{-\lambda} & (\lambda \geq 0) \end{cases}.$$

   (a) i. Find the value of k_0.
      ii. Find the prior mean of λ.
      iii. Find the prior standard deviation of λ.

   (b) We observe $x_1, \ldots, x_n$ which are independent observations from the Poisson(λ) distribution.
      i. Find the likelihood function.
      ii. Find the posterior density of λ.
      iii. Find the posterior mean of λ.

7. In a fruit packaging factory apples are examined to see whether they are blemished. A sample of n apples is examined and, given the value of a parameter θ, representing the proportion of aples which are blemished, we regard x, the number of blemished apples in the sample, as an obervation from the binomial($n,\ \theta$) distribution. The value of θ is unknown.

   Our prior density for θ is

$$f^{(0)}(\theta) = \begin{cases} k_0(20\theta(1-\theta)^3 + 1) & (0 \leq \theta \leq 1) \\ 0 & (\text{otherwise}) \end{cases}.$$

   (a) i. Show that, for $0 \leq \theta \leq 1$, the prior density can be written as

$$f^{(0)}(\theta) = \frac{1}{2}\left\{ \frac{\Gamma(6)}{\Gamma(2)\Gamma(4)}\theta^{2-1}(1-\theta)^{4-1} + \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)}\theta^{1-1}(1-\theta)^{1-1} \right\}.$$

      ii. Find the prior mean of θ.
      iii. Find the prior standard deviation of θ.

   (b) We observe $n = 10$ apples and $x = 4$.
      i. Find the likelihood function.
      ii. Find the posterior density of θ.
      iii. Find the posterior mean of θ.
      iv. Use R to plot a graph showing both the prior and posterior densities of θ. (*Hint: It is easier to get the vertical axis right if you plot the posterior density and then superimpose the prior density, rather than the other way round.*)

Homework 2

Solutions to Questions **6, 7** *of Problems 2 are to be submitted in the Homework Letterbox no later than 4.00pm on Monday February 23rd.*

Solutions

1. (a) i.

$$\int_0^\infty f^{(0)}(\lambda)\ d\lambda = k_0\left\{\int_0^\infty e^{-\lambda}\ d\lambda + \int_0^\infty \lambda e^{-\lambda}\ d\lambda\right\}$$
$$= k_0\{1+1\} = 2k_0$$

Hence $k_0 = 1/2$.

ii.

$$\mathrm{E}_0(\lambda) = \int_0^\infty \lambda f^{(0)}(\lambda)\ d\lambda = \frac{1}{2}\left\{\int_0^\infty \lambda e^{-\lambda}\ d\lambda + \int_0^\infty \lambda^2 e^{-\lambda}\ d\lambda\right\}$$
$$= \frac{1}{2}\{1+2\} = \frac{3}{2} = 1.5$$

iii.

$$\mathrm{E}_0(\lambda^2) = \int_0^\infty \lambda^2 f^{(0)}(\lambda)\ d\lambda = \frac{1}{2}\left\{\int_0^\infty \lambda^2 e^{-\lambda}\ d\lambda + \int_0^\infty \lambda^3 e^{-\lambda}\ d\lambda\right\}$$
$$= \frac{1}{2}\{2+6\} = \frac{8}{2}$$

So

$$\mathrm{var}_0(\lambda) = \frac{8}{2} - \left(\frac{3}{2}\right)^2 = \frac{16-9}{4} = \frac{7}{4}$$

and

$$\mathrm{std.dev}_0(\lambda) = \sqrt{\frac{7}{4}} = \frac{\sqrt{7}}{2} = 1.323.$$

(b) i. Likelihood

$$L = \prod_{i=1}^n \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \frac{e^{-n\lambda}\lambda^{\sum x_i}}{\prod x_i!} = \frac{e^{-n\lambda}\lambda^S}{\prod x_i!}$$

where

$$S = \sum_{i=1}^n x_i.$$

ii. Posterior density proportional to

$$f^{(0)}(\lambda)L \propto (1+\lambda)e^{-\lambda}e^{-n\lambda}\lambda^S$$
$$\propto e^{-(n+1)\lambda}\lambda^S + e^{-(n+1)\lambda}\lambda^{S+1}$$

The posterior density is

$$f^{(1)}(\lambda) = k_1\left\{e^{-(n+1)\lambda}\lambda^S + e^{-(n+1)\lambda}\lambda^{S+1}\right\}$$

where

$$\int_0^\infty f^{(1)}(\lambda)\ d\lambda = 1 = k_1\left\{\frac{\Gamma(S+1)}{(n+1)^{S+1}} + \frac{\Gamma(S+2)}{(n+1)^{S+2}}\right\}.$$

Hence

$$k_1 = \left\{\frac{\Gamma(S+1)}{(n+1)^{S+1}} + \frac{\Gamma(S+2)}{(n+1)^{S+2}}\right\}^{-1}$$

and

$$f^{(1)}(\lambda) = \left\{\frac{\Gamma(S+1)}{(n+1)^{S+1}} + \frac{\Gamma(S+2)}{(n+1)^{S+2}}\right\}^{-1}\left\{e^{-(n+1)\lambda}\lambda^S + e^{-(n+1)\lambda}\lambda^{S+1}\right\}.$$

iii. Posterior mean

$$\begin{aligned} \mathrm{E}_1(\lambda) &= \int_0^\infty \lambda f^{(1)}(\lambda)\ d\lambda \\ &= k_1 \left\{ \int_0^\infty \lambda^{S+1} e^{-(n+1)\lambda}\ d\lambda + \int_0^\infty \lambda^{S+2} e^{-(n+1)\lambda}\ d\lambda \right\} \\ &= k_1 \left\{ \frac{\Gamma(S+2)}{(n+1)^{S+2}} + \frac{\Gamma(S+3)}{(n+1)^{S+3}} \right\} \\ &= \frac{\left\{ \frac{(S+1)}{(n+1)} + \frac{(S+1)(S+2)}{(n+1)^2} \right\}}{\left\{ 1 + \frac{(S+1)}{(n+1)} \right\}} \end{aligned}$$

2. (a) i.

$$\begin{aligned} \int_0^1 f^{(0)}(\theta)\ d\theta &= k_0 \left\{ \int_0^1 \theta^2(1-\theta)\ d\theta + \int_0^1 \theta(1-\theta)^2\ d\theta \right\} \\ &= k_0 \left\{ \frac{\Gamma(3)\Gamma(2)}{\Gamma(5)} + \frac{\Gamma(2)\Gamma(3)}{\Gamma(5)} \right\} \\ &= 2k_0 \frac{\Gamma(3)\Gamma(2)}{\Gamma(5)} = 2k_0 \frac{2!1!}{4!} = \frac{k_0}{6} \end{aligned}$$

Hence $k_0 = 6$.

ii. Prior mean

$$\begin{aligned} \mathrm{E}_0(\theta) &= \int_0^1 \theta f^{(0)}(\theta)\ d\theta = k_0 \left\{ \int_0^1 \theta^3(1-\theta)\ d\theta + \int_0^1 \theta^2(1-\theta)^2\ d\theta \right\} \\ &= k_0 \left\{ \frac{\Gamma(4)\Gamma(2)}{\Gamma(6)} + \frac{\Gamma(3)\Gamma(3)}{\Gamma(6)} \right\} \\ &= k_0 \left\{ \frac{3!1! + 2!2!}{5!} \right\} = k_0 \left\{ \frac{6+4}{120} \right\} = \frac{1}{2} \end{aligned}$$

iii.

$$\begin{aligned} \mathrm{E}_0(\theta^2) &= \int_0^1 \theta^2 f^{(0)}(\theta)\ d\theta \\ &= k_0 \left\{ \int_0^1 \theta^4(1-\theta)\ d\theta + \int_0^1 \theta^3(1-\theta)^2\ d\theta \right\} \\ &= k_0 \left\{ \frac{\Gamma(5)\Gamma(2)}{\Gamma(7)} + \frac{\Gamma(4)\Gamma(3)}{\Gamma(7)} \right\} \\ &= k_0 \left\{ \frac{4!1! + 3!2!}{6!} \right\} = \left\{ \frac{24+12}{5 \times 24} \right\} = \frac{3}{10} \end{aligned}$$

Hence

$$\mathrm{var}_0(\theta) = \frac{3}{10} - \left(\frac{1}{2} \right)^2 = \frac{6-5}{20} = \frac{1}{20}$$

and

$$\mathrm{std.dev}_0(\theta) = \frac{1}{\sqrt{20}} = 0.2236.$$

(b) i. Likelihood

$$L = \left(\begin{array}{c} n \\ x \end{array} \right) \theta^x (1-\theta)^{n-x}.$$

ii. Posterior density $f^{(1)}(\theta)$ proportional to $f^{(0)}(\theta)L$. Hence

$$\begin{aligned} f^{(1)}(\theta) &= k_1 \left\{ \theta^2(1-\theta) + \theta(1-\theta)^2 \right\} \theta^x (1-\theta)^{n-x} \\ &= k_1 \left\{ \theta^{x+2}(1-\theta)^{n-x+1} + \theta^{x+1}(1-\theta)^{n-x+2} \right\} \end{aligned}$$

Now

$$\int_0^1 f^{(1)}(\theta)\ d\theta = 1 \quad = \quad k_1 \left\{ \int_0^1 \theta^{x+2}(1-\theta)^{n-x+1}\ d\theta + \int_0^1 \theta^{x+1}(1-\theta)^{n-x+2}\ d\theta \right\}$$
$$= \quad k_1 \left\{ \frac{\Gamma(x+3)\Gamma(n-x+2)}{\Gamma(n+5)} + \frac{\Gamma(x+2)\Gamma(n-x+3)}{\Gamma(n+5)} \right\}.$$

Hence

$$k_1 = \left\{ \frac{\Gamma(x+3)\Gamma(n-x+2)}{\Gamma(n+5)} + \frac{\Gamma(x+2)\Gamma(n-x+3)}{\Gamma(n+5)} \right\}^{-1}$$

and

$$\begin{aligned} f^{(1)}(\theta) \quad &= \quad \left\{ \frac{\Gamma(x+3)\Gamma(n-x+2)}{\Gamma(n+5)} + \frac{\Gamma(x+2)\Gamma(n-x+3)}{\Gamma(n+5)} \right\}^{-1} \\ &\quad \times \left\{ \theta^{x+2}(1-\theta)^{n-x+1} + \theta^{x+1}(1-\theta)^{n-x+2} \right\}. \end{aligned}$$

iii. Posterior mean

$$\begin{aligned} \mathrm{E}_0(\theta) \quad &= \quad \int_0^1 \theta f^{(1)}(\theta)\ d\theta \\ &= \quad k_1 \left\{ \int_0^1 \theta^{x+3}(1-\theta)^{n-x+1}\ d\theta + \int_0^1 \theta^{x+2}(1-\theta)^{n-x+2}\ d\theta \right\} \\ &= \quad k_1 \left\{ \frac{\Gamma(x+4)\Gamma(n-x+2)}{\Gamma(n+6)} + \frac{\Gamma(x+3)\Gamma(n-x+3)}{\Gamma(n+6)} \right\} \\ &= \quad \frac{\left\{ \frac{\Gamma(x+4)\Gamma(n-x+2)}{\Gamma(n+6)} + \frac{\Gamma(x+3)\Gamma(n-x+3)}{\Gamma(n+6)} \right\}}{\left\{ \frac{\Gamma(x+3)\Gamma(n-x+2)}{\Gamma(n+5)} + \frac{\Gamma(x+2)\Gamma(n-x+3)}{\Gamma(n+5)} \right\}} \\ &= \quad \frac{1}{n+5} \left\{ \frac{\Gamma(x+4)\Gamma(n-x+2) + \Gamma(x+3)\Gamma(n-x+3)}{\Gamma(x+3)\Gamma(n-x+2) + \Gamma(x+2)\Gamma(n-x+3)} \right\} \\ &= \quad \frac{1}{n+5} \left\{ \frac{\Gamma(x+4) + \Gamma(x+3)(n-x+2)}{\Gamma(x+3) + \Gamma(x+2)(n-x+2)} \right\} \\ &= \quad \frac{1}{n+5} \left\{ \frac{(x+3)(x+2) + (x+2)(n-x+2)}{(x+2) + (n-x+2)} \right\} \\ &= \quad \left(\frac{x+2}{n+5} \right) \left\{ \frac{(x+3) + (n-x+2)}{(x+2) + (n-x+2)} \right\} \\ &= \quad \frac{x+2}{n+4} \end{aligned}$$

3. (a) i.

$$\int_0^1 f^{(0)}(\theta)\ d\theta = k_0 \int_0^1 \theta^2(1-\theta)^3\ d\theta = k_0 \frac{\Gamma(3)\Gamma(4)}{\Gamma(7)}.$$

Hence

$$k_0 = \frac{\Gamma(7)}{\Gamma(3)\Gamma(4)} = \frac{6!}{2!3!} = \frac{6\times 5\times 4}{2} = \underline{60}.$$

ii. Prior mean

$$\begin{aligned} \mathrm{E}_0(\theta) &= \int_0^1 \theta f^{(0)}(\theta)\ d\theta = k_0 \int_0^1 \theta^3(1-\theta)^3\ d\theta \\ &= \frac{\Gamma(4)\Gamma(4)}{\Gamma(8)} \frac{\Gamma(7)}{\Gamma(3)\Gamma(4)} = \frac{3}{7} = \underline{0.4286}. \end{aligned}$$

iii.

$$\begin{aligned} \mathrm{E}_0(\theta^2) &= \int_0^1 \theta^2 f^{(0)}(\theta)\ d\theta = k_0 \int_0^1 \theta^4(1-\theta)^3\ d\theta \\ &= \frac{\Gamma(5)\Gamma(4)}{\Gamma(9)} \frac{\Gamma(7)}{\Gamma(3)\Gamma(4)} = \frac{4\times 3}{8\times 7} = \frac{3}{14} \end{aligned}$$

Hence

$$\mathrm{var}_0(\theta) = \frac{3}{14} - \left(\frac{3}{7}\right)^2 = \frac{21-18}{98} = \frac{3}{98}$$

and

$$\mathrm{std.dev}_0(\theta) = \sqrt{\frac{3}{98}} = \underline{0.1750}.$$

(b) i. Likelihood

$$L(\theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x}.$$

ii. Posterior density

$$f^{(1)}(\theta) \propto f^{(0)}(\theta)L(\theta) = k_1 \theta^{x+2}(1-\theta)^{n-x+3}.$$

Now

$$\int_0^1 f^{(1)}(\theta)\ d\theta = 1 = k_1 \int_0^1 \theta^{x+2}(1-\theta)^{n-x+3}\ d\theta n = k_1 \frac{\Gamma(x+3)\Gamma(n-x+4)}{\Gamma(n+7)}.$$

Hence

$$k_1 = \frac{\Gamma(n+7)}{\Gamma(x+3)\Gamma(n-x+4)}$$

and

$$f^{(1)}(\theta) = \frac{\Gamma(n+7)}{\Gamma(x+3)\Gamma(n-x+4)} \theta^{x+2}(1-\theta)^{n-x+3} \qquad (0 < \theta < 1).$$

iii. Posterior mean

$$\begin{aligned} \mathrm{E}_1(\theta) &= \int_0^1 \theta f^{(1)}(\theta)\ d\theta = k_1 \int_0^1 \theta^{x+3}(1-\theta)^{n-x+3}\ d\theta \\ &= \frac{\Gamma(n+7)}{\Gamma(x+3)\Gamma(n-x+4)} \frac{\Gamma(x+4)\Gamma(n-x+4)}{\Gamma(n+8)} = \frac{x+3}{n+7} \end{aligned}$$

4.

5.

6. (a) i. Value of k_0 :

$$\int_0^\infty \lambda^3 e^{-\lambda}\, d\lambda = \int_0^\infty \lambda^{4-1} e^{\lambda}\, d\lambda = \Gamma(4) = 3! = 6$$

Hence

$$k_0 = \frac{1}{6}.$$

(1 mark)

ii. Prior mean:

$$\begin{aligned} \mathrm{E}_0(\lambda) &= \int_0^\infty \lambda k_0 \lambda^3 e^{-\lambda}\, d\lambda \\ &= k_0 \int_0^\infty \lambda^{5-1} e^{-\lambda}\, d\lambda \\ &= k_0 \Gamma(5) = \frac{4!}{3!} = \underline{4} \end{aligned}$$

(1 mark)

iii. Prior std.dev.:

$$\begin{aligned} \mathrm{E}_0(\lambda^2) &= \int_0^\infty \lambda^2 k_0 \lambda^3 e^{-\lambda}\, d\lambda \\ &= k_0 \int_0^\infty \lambda^{6-1} e^{-\lambda}\, d\lambda \\ &= k_0 \Gamma(6) = \frac{5!}{3!} = 20 \end{aligned}$$

Hence $\mathrm{var}_0(\lambda) = \mathrm{E}_0(\lambda^2) - [\mathrm{E}_0(\lambda)]^2 = 20 - 16 = 4$ and prior sd is

$$\sqrt{4} = \underline{2}.$$

(2 marks)

(b) i. Likelihood:

$$L = \prod_{i=1}^n \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \frac{e^{-n\lambda}\lambda^{\sum x_i}}{\prod x_i!}$$

(1 mark)

ii. Posterior density proportional to

$$\lambda^3 e^{-\lambda} \times e^{-n\lambda}\lambda^{\sum x_i}$$

That is

$$\lambda^{\sum x_i + 4 - 1} e^{-(n+1)\lambda}$$

Now

$$\int_0^\infty \lambda^{a-1} e^{-b\lambda}\, d\lambda = \frac{\Gamma(a)}{b^a}$$

Hence the posterior density is

$$\frac{(n+1)^{\sum x_i + 4}}{\Gamma(\sum x_i + 4)} \lambda^{\sum x_i + 4 - 1} e^{-(n+1)\lambda}$$

(2 marks)

iii. To find the posterior mean, increase the power of λ by 1 and integrate. Posterior mean is

$$\frac{(n+1)^{\sum x_i+4}}{\Gamma(\sum x_i+4)} \frac{\Gamma(\sum x_i+5)}{(n+1)^{\sum x_i+5}} = \frac{\sum x_i+4}{n+1}$$

(1 mark)

7. (a) i. The expression given is proportional to the prior density since

$$\begin{aligned} \frac{\Gamma(6)}{\Gamma(2)\Gamma(4)} &= \frac{5!}{3!} = 30 \\ \text{and} \quad \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)} &= 1 \end{aligned}$$

Now we only need to show that

$$\int_0^1 \frac{1}{2}\left\{\frac{\Gamma(6)}{\Gamma(2)\Gamma(4)}\theta^{2-1}(1-\theta)^{4-1} + \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)}\theta^{1-1}(1-\theta)^{1-1}\right\} d\theta = 1$$

and this follows since

$$\begin{aligned} \int_0^1 \theta^{2-1}(1-\theta)^{4-1}\ d\theta &= \frac{\Gamma(2)\Gamma(4)}{\Gamma(6)} \\ \text{and} \quad \int_0^1 \theta^{1-1}(1-\theta)^{1-1}\ d\theta &= \frac{\Gamma(1)\Gamma(1)}{\Gamma(2)}. \end{aligned}$$

(1 mark)

ii. Prior mean:

$$\begin{aligned} \mathrm{E}_0(\theta) &= \int_0^1 \theta f^{(0)}(\theta)\ d\theta \\ &= \frac{1}{2}\int_0^1 \left\{\frac{\Gamma(6)}{\Gamma(2)\Gamma(4)}\theta^{3-1}(1-\theta)^{4-1} + \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)}\theta^{2-1}(1-\theta)^{1-1}\right\} d\theta \\ &= \frac{1}{2}\left\{\frac{\Gamma(6)}{\Gamma(2)\Gamma(4)}\frac{\Gamma(3)\Gamma(4)}{\Gamma(7)} + \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)}\frac{\Gamma(2)\Gamma(1)}{\Gamma(3)}\right\} \\ &= \frac{1}{2}\left\{\frac{2}{6}+\frac{1}{2}\right\} = \frac{5}{12} = \underline{0.4167} \end{aligned}$$

(2 marks)

iii. Prior std. dev.:

$$\begin{aligned} \mathrm{E}_0(\theta^2) &= \int_0^1 \theta^2 f^{(0)}(\theta)\ d\theta \\ &= \frac{1}{2}\int_0^1 \left\{\frac{\Gamma(6)}{\Gamma(2)\Gamma(4)}\theta^{4-1}(1-\theta)^{4-1} + \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)}\theta^{3-1}(1-\theta)^{1-1}\right\} d\theta \\ &= \frac{1}{2}\left\{\frac{\Gamma(6)}{\Gamma(2)\Gamma(4)}\frac{\Gamma(4)\Gamma(4)}{\Gamma(8)} + \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)}\frac{\Gamma(3)\Gamma(1)}{\Gamma(4)}\right\} \\ &= \frac{1}{2}\left\{\frac{3\times 2}{7\times 6}+\frac{1}{3}\right\} = \frac{5}{21} \end{aligned}$$

Hence

$$\mathrm{var}_0(\theta) = \frac{5}{21} - \left(\frac{5}{12}\right)^2 = 0.06448$$

so the prior std. dev. is

$$\sqrt{0.06448} = \underline{0.2539}.$$

(2 marks)

(b) i. Likelihood:

$$L = \left(\begin{array}{c} 10 \\ 4 \end{array} \right) \theta^4 (1-\theta)^6$$

(1 mark)

ii. Posterior density:

$$\text{Posterior } propto\ Prior \ \times \text{ Likelihood}$$

$$f^{(1)}(\theta) = k_1 \left\{ \frac{\Gamma(6)}{\Gamma(2)\Gamma(4)} \theta^{6-1}(1-\theta)^{10-1} + \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)} \theta^{5-1}(1-\theta)^{7-1} \right\}$$

$$\begin{aligned}
\int_0^1 f^{(1)}(\theta)\ d\theta &= k_1 \left\{ \frac{\Gamma(6)}{\Gamma(2)\Gamma(4)} \frac{\Gamma(6)\Gamma(10)}{\Gamma(16)} + \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)} \frac{\Gamma(5)\Gamma(7)}{\Gamma(12)} \right\} \\
&= k_1 \left\{ \frac{5 \times 4 \times 5 \times 4 \times 3 \times 2}{15 \times 14 \times 13 \times 12 \times 11 \times 10} + \frac{4 \times 3 \times 2}{11 \times 10 \times 9 \times 8 \times 7} \right\} \\
&= k_1 \left\{ \frac{2}{7 \times 13 \times 3 \times 11} + \frac{1}{11 \times 10 \times 3 \times 7} \right\} \\
&= \frac{k_1}{3 \times 11 \times 7} \left\{ \frac{2}{13} + \frac{1}{10} \right\} \\
&= \frac{k_1}{3 \times 11 \times 7} \left\{ \frac{33}{130} \right\} = \frac{k_1}{7 \times 130}
\end{aligned}$$

Hence $k_1 = 7 \times 130 = 910$.
Posterior density:

$$f^{(1)}(\theta) = 910 \left\{ 20\theta^{6-1}(1-\theta)^{10-1} + \theta^{5-1}(1-\theta)^{7-1} \right\}$$

(2 marks)

iii. Posterior mean:

$$\begin{aligned}
\mathrm{E}_1(\theta) &= \int_0^1 \theta f^{(1)}(\theta)\ d\theta \\
&= 910 \int_0^1 \left\{ 20\theta^{7-1}(1-\theta)^{10-1} + \theta^{6-1}(1-\theta)^{7-1} \right\}\ d\theta \\
&= 910 \left\{ 20 \frac{\Gamma(7)\Gamma(10)}{\Gamma(17)} + \frac{\Gamma(6)\Gamma(7)}{\Gamma(13)} \right\} \\
&= 910 \left\{ \frac{20 \times 6 \times 5 \times 4 \times 3 \times 2}{16 \times 15 \times 14 \times 13 \times 12 \times 11 \times 10} + \frac{5 \times 4 \times 3 \times 2}{12 \times 11 \times 10 \times 9 \times 8 \times 7} \right\} \\
&= 910 \left\{ \frac{1}{14 \times 13 \times 11 \times 2} + \frac{1}{11 \times 9 \times 8 \times 7} \right\} = \underline{0.3914}
\end{aligned}$$

(2 marks)

iv. Plot: suitable R commands:

```
> theta<-seq(0,1,0.01)
> prior<-0.5*(20*theta*((1-theta)^3)+1)
> post<-910*(20*(theta^5)*((1-theta)^9)+(theta^4)*((1-theta)^6))
> plot(theta,post,type="l",xlab=expression(theta),ylab="Density")
> lines(theta,prior,lty=2)
```

The plot is shown in Figure 1.

(2 marks)

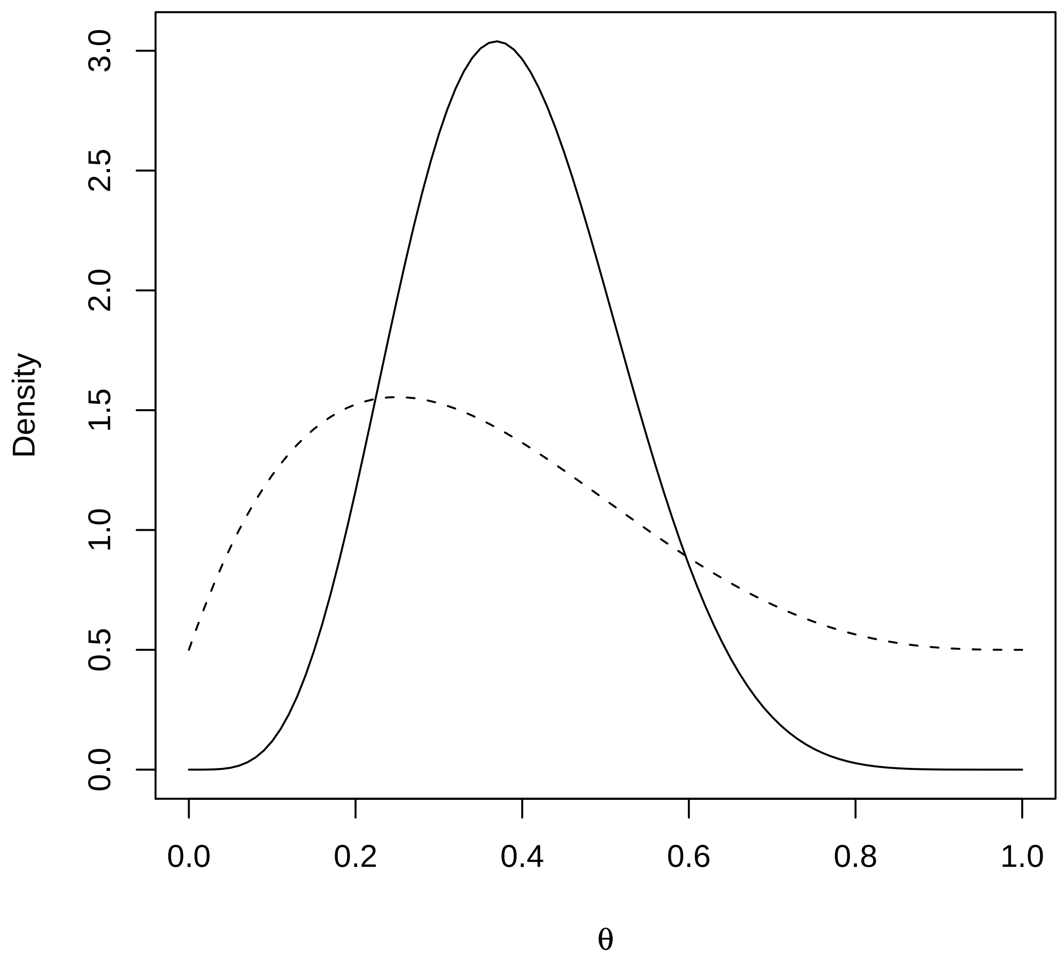

Figure 1: Prior (dashes) and posterior (solid line) density functions for θ.