MAS3301 Bayesian Statistics
Problems 1 and Solutions

Semester 2

2008-9

Problems 1

1. Let E_1, E_2, E_3 be events. Let I_1, I_2, I_3 be the corresponding indicators so that $I_1 = 1$ if E_1 occurs and $I_1 = 0$ otherwise.

   (a) Let $I_A = 1 - (1 - I_1)(1 - I_2)$. Verify that I_A is the indicat or for the event A where $A = (E_1 \vee E_2)$ (that is "E_1 or E_2") and show that

   $$\Pr(A) = \Pr(E_1) + \Pr(E_2) - \Pr(E_1 \wedge E_2)$$

   where $(E_1 \wedge E_2)$ is "E_1 and E_2".

   (b) Find a formula, in terms of I_1, I_2, I_3 for I_B, the indicator for the event B where $B = (E_1 \vee E_2 \vee E_3)$ and derive a formula for $\Pr(B)$ in terms of $\Pr(E_1), \Pr(E_2), \Pr(E_3), \Pr(E_1 \wedge E_2), \Pr(E_1 \wedge E_3), \Pr(E_2 \wedge E_3), \Pr(E_1 \wedge E_2 \wedge E_3)$.

2. In a certain place it rains on one third of the days. The local evening newspaper attempts to predict whether or not it will rain the following day. Three quarters of rainy days and three fifths of dry days are correctly predicted by the previous evening's paper. Given that this evening's paper predicts rain, what is the probability that it will actually rain tomorrow?

3. A machine is built to make mass-produced items. Each item made by the machine has a probability p of being defective. Given the value of p, the items are independent of each other. Because of the way in which the machines are made, p could take one of several values. In fact $p = X/100$ where X has a discrete uniform distribution on the interval $[0, 5]$. The machine is tested by counting the number of items made before a defective is produced. Find the conditional probability distribution of X given that the first defective item is the thirteenth to be made.

4. There are five machines in a factory. Of these machines, three are working

   properly and two are defective. Machines which are working properly produce articles each of which has independently a probability of 0.1 of being imperfect. For the defective machines this probability is 0.2.

   A machine is chosen at random and five articles produced by the machine are examined. What is the probability that the machine chosen is defective given that, of the five articles examined, two are imperfect and three are perfect?

5. A crime has been committed. Assume that the crime was committed by exactly one person, that there are 1000 people who could have committed the crime and that, in the absence of any evidence, these people are all equally likely to be guilty of the crime.

   A piece of evidence is found. It is judged that this evidence would have a probability of 0.99 of being observed if the crime were committed by a particular individual, A, but a probability of only 0.0001 of being observed if the crime were committed by any other individual.

   Find the probability, given the evidence, that A committed the crime.

6. In an experiment on extra-sensory perception (ESP) a person, A, sits in a sealed room and points at one of four cards, each of which shows a different picture. In another sealed room a second person, B, attempts to select, from an identical set of four cards, the card at which A is pointing. This experiment is repeated ten times and the correct card is selected four times.

   Suppose that we consider three possible states of nature, as follows.

   State 1 : There is no ESP and, whichever card A chooses, B is equally likely to select any one of the four cards. That is, subject B has a probability of 0.25 of selecting the correct card.

   Before the experiment we give this state a probability of 0.7.

   State 2 : Subject B has a probability of 0.50 of selecting the correct card.

   Before the experiment we give this state a probability of 0.2.

   State 3 : Subject B has a probability of 0.75 of selecting the correct card.

   Before the experiment we give this state a probability of 0.1.

   Assume that, given the true state of nature, the ten trials can be considered to be independent.

   Find our probabilities after the experiment for the three possible states of nature.

   Can you think of a reason, apart from ESP, why the probability of selecting the correct card might be greater than 0.25?

7. In a certain small town there are n taxis which are clearly numbered $1, 2, \ldots, n$. Before we visit the town we do not know the value of n but our probabilities for the possible values of n are as follows.

n	0	1	2	3	4
Probability	0.00	0.11	0.12	0.13	0.14
n	5	6	7	8	≥ 9
Probability	0.14	0.13	0.12	0.11	0.00

   On a visit to the town we take a taxi which we assume would be equally likely to be any of taxis $1, 2, \ldots, n$. It is taxi number 5. Find our new probabilities for the value of n.

8. A dishonest gambler has a box containing 10 dice which all look the same. However there are actually three types of dice.

   - There are 6 dice of type A which are fair dice with $\Pr(6 \mid A) = 1/6$ (where $\Pr(6 \mid A)$ is the probability of getting a 6 in a throw of a type A die).
   - There are 2 dice of type B which are biassed with $\Pr(6 \mid B) = 0.8$.
   - There are 2 dice of type C which are biassed with $\Pr(6 \mid C) = 0.04$.

   The gambler takes a die from the box at random and rolls it. Find the conditional probability that it is of type B given that it gives a 6.

9. In a forest area of Northern Europe there may be wild lynx. At a particular time the number X of lynx can be between 0 and 5 with

$$\Pr(X = x) = \binom{5}{x} 0.6^x 0.4^{5-x} \qquad (x = 0, \ldots, 5).$$

   A survey is made but the lynx is difficult to spot and, given that the number present is x, the number Y observed has a probability distribution with

$$\Pr(Y = y \mid X = x) = \begin{cases} \binom{x}{y} 0.3^y 0.7^{x-y} & (0 \le y \le x) \\ 0 & (x < y) \end{cases} .$$

Find the conditional probability distribution of X given that $Y = 2$.

(That is, find $\Pr(X = 0 \mid Y = 2), \ldots, \Pr(X = 5 \mid Y = 2)$).

10. A particular species of fish makes an annual migration up a river. On a particular day there is a probability of 0.4 that the migration will start. If it does then an observer will have to wait T minutes before seeing a fish, where T has an exponential distribution with mean 20 (i.e. an exponential(0.05) distribution). If the migration has not started then no fish will be seen.

    (a) Find the conditional probability that the migration has not started given that no fish has been seen after one hour.

    (b) How long does the observer have to wait without seeing a fish to be 90% sure that the migration has not started?

Homework 1

Solutions to Questions **8, 9, 10** *of Problems 1 are to be submitted in the Homework Letterbox no later than 4.00pm on Monday February 9th.*

Solutions

1. (a) We compare the truth table with the indicators.

Truth table

E_1	E_2	A
T	T	T
T	F	T
F	T	T
F	F	F

Indicators

I_1	I_2	I_A
1	1	1
1	0	1
0	1	1
0	0	0

Hence I_A is the indicator for A.
So,

$$\begin{aligned}\Pr(A) = \mathrm{E}(I_A) &= \mathrm{E}\{1-(1-I_1)(1-I_2)\}\\ &= \mathrm{E}\{1-1+I_1+I_2-I_1I_2\}\\ &= \mathrm{E}(I_1)+\mathrm{E}(I_2)-\mathrm{E}(I_1I_2)\\ &= \Pr(E_1)+\Pr(E_2)-\Pr(E_1\wedge E_2)\end{aligned}$$

(3 marks)

(b) The indicator for B is

$$I_B = 1-(1-I_1)(1-I_2)(1-I_3)$$

So

$$\begin{aligned}\Pr(B) = \mathrm{E}(I_B) &= \mathrm{E}\{1-(1-I_1)(1-I_2)(1-I_3)\}\\ &= \mathrm{E}\{1-1+I_1+I_2+I_3-I_1I_2-I_1I_3-I_2I_3+I_1I_2I_3\}\\ &= \mathrm{E}(I_1)+\mathrm{E}(I_2)+\mathrm{E}(I_3)-\mathrm{E}(I_1I_2)-\mathrm{E}(I_1I_3)-\mathrm{E}(I_2I_3)+\mathrm{E}(I_1I_2I_3)\\ &= \Pr(E_1)+\Pr(E_2)+\Pr(E_3)-\Pr(E_1\wedge E_2)-\Pr(E_1\wedge E_3)-\Pr(E_2\wedge E_3)\\ &\quad +\Pr(E_1\wedge E_2\wedge E_3)\end{aligned}$$

(3 marks)

2. Let R be "rain", $\bar{R}$ be "dry", P be "rain predicted" .
We require $\Pr(R \mid P)$. By Bayes' theorem, this is

$$\begin{aligned}\Pr(R\mid P) &= \frac{\Pr(R)\Pr(P\mid R)}{\Pr(R)\Pr(P\mid R)+\Pr(\bar{R})\Pr(P\mid \bar{R})}\\ &= \frac{\frac{1}{3}\times\frac{3}{4}}{\frac{1}{3}\times\frac{3}{4}+\frac{2}{3}\times\frac{2}{5}}\\ &= \frac{\frac{3}{4}}{\frac{3}{4}+\frac{4}{5}} = \frac{15}{15+16} = \frac{15}{31}\\ &= \underline{0.4839}\end{aligned}$$

(4 marks)

3. Let D be "1st defective item is 13th to be made."
We require $\Pr(X = i \mid D)$ for $i = 0, \ldots, 5$.
Now

$$\Pr(D \mid X = i) = \left(1-\frac{i}{100}\right)^{12}\left(\frac{i}{100}\right)$$

and
$$\Pr(X=i)=\frac{1}{6}.$$
By Bayes' theorem
$$\Pr(X=i\mid D)=\frac{\Pr(X=i)\Pr(D\mid X=i)}{\sum_{j=0}^{5}\Pr(X=j)\Pr(D\mid X=j)}$$
and, since $\Pr(X=i)=1/6$ for all i,
$$\Pr(X=i\mid D)=\frac{\Pr(D\mid X=i)}{\sum_{j=0}^{5}\Pr(D\mid X=j)}.$$
So we obtain the following.

i	p	$\Pr(D\mid X=i)$	$\Pr(X=i\mid D)$
0	0.00	0.0000	0.0000
1	0.01	0.0089	0.0915
2	0.02	0.0157	0.1620
3	0.03	0.0208	0.2148
4	0.04	0.0245	0.2529
5	0.05	0.0270	0.2788
		0.0969	

(6 marks)

4. Let D be "2 out of 5 imperfect." Let M be "machine defective" and let $\bar{M}$ be "machine not defective."

We require $\Pr(M\mid D)$.

Now
$$\Pr(D\mid M)=\binom{5}{2}0.2^2 0.8^3$$
and
$$\Pr(D\mid\bar{M})=\binom{5}{2}0.1^2 0.9^3.$$
By Bayes' theorem
$$\begin{aligned}\Pr(M\mid D) &= \frac{\Pr(M)\Pr(D\mid M)}{\Pr(M)\Pr(D\mid M)+\Pr(\bar{M})\Pr(D\mid\bar{M})}\\ &= \frac{\frac{2}{5}\binom{5}{2}0.2^2 0.8^3}{\frac{2}{5}\binom{5}{2}0.2^2 0.8^3+\frac{3}{5}\binom{5}{2}0.1^2 0.9^3}\\ &= \frac{2\times 0.2^2\times 0.8^3}{2\times 0.2^2\times 0.8^3+3\times 0.1^2\times 0.9^3}\\ &= \frac{0.04096}{0.04096+0.02187}\\ &= \underline{0.6519}\end{aligned}$$

(4 marks)

5.

6.

7.

8. Prior probabilities: $\Pr(A) = 0.6,\ \Pr(B) = 0.2,\ \Pr(C) = 0.2.$
   Likelihood: $\Pr(6 \mid A) = 1/6,\ \Pr(6 \mid B) = 0.8,\ \Pr(6 \mid C) = 0.04.$
   Prior $\times$ likelihood:

$$\begin{aligned} \Pr(A)\Pr(6 \mid A) = 0.6 \times 1/6 &= 0.1 \\ \Pr(B)\Pr(6 \mid B) = 0.2 \times 0.8 &= 0.16 \\ \Pr(C)\Pr(6 \mid C) = 0.2 \times 0.04 &= 0.008 \end{aligned}$$

$$\Pr(6) = 0.1 + 0.16 + 0.008 = 0.268.$$

$$\Pr(B \mid 6) = \frac{0.16}{0.268} = \underline{0.597}.$$

(4 marks)

9. Prior $\times$ likelihood:

$$\begin{aligned} \Pr(x)\Pr(y \mid x) &= \binom{5}{x} 0.6^x 0.4^{5-x} \binom{x}{y} 0.3^y 0.7^{x-y} \\ &= \frac{5!}{x!(5-x)!}\frac{x!}{y!(x-y)!}\left(\frac{0.6 \times 0.7}{0.4}\right)^x 0.6^6 0.4^5 \left(\frac{0.3}{0.7}\right)^y \\ &\propto 1.05^x \frac{1}{(5-x)!(x-y)!} \\ \Pr(x=0)\Pr(y=2 \mid x=0) &= 0 \\ \Pr(x=1)\Pr(y=2 \mid x=1) &= 0 \\ \Pr(x=2)\Pr(y=2 \mid x=2) &\propto 0.18375 \\ \Pr(x=3)\Pr(y=2 \mid x=3) &\propto 0.57881 \\ \Pr(x=4)\Pr(y=2 \mid x=4) &\propto 0.60775 \\ \Pr(x=5)\Pr(y=2 \mid x=5) &\propto \underline{0.21271} \\ \Pr(y=2) &\propto 1.58303 \end{aligned}$$

(5 marks)

$$\Pr(x=j \mid y=2) = \frac{\Pr(x=j)\Pr(y=2 \mid x=j)}{\Pr(y=2)}$$

j	$\Pr(x=j \mid y=2)$
0	0.0000
1	0.0000
2	0.1161
3	0.3656
4	0.3839
5	0.1344

(3 marks)

10. Notation:

M: Migration started
$\bar{M}$: Migration not started
W: No fish in 60 minutes

Prior: $\Pr(M) = 0.4,\ \Pr(\bar{M}) = 0.6$

(a) Likelihood:

$$\begin{aligned} \Pr(W \mid M) &= e^{-60/20} = e^{-3} = 0.04979 \\ \Pr(W \mid \bar{M}) &= 1 \end{aligned}$$

Prior $\times$ likelihood:

$$\begin{aligned} \Pr(M)\Pr(W \mid M) &= 0.4 \times 0.04979 = & 0.01991 \\ \Pr(\bar{M})\Pr(W \mid \bar{M}) &= 0.6 \times 1 = & \underline{0.6} \\ \Pr(W) &= 0.01991 + 0.6 = & 0.61991 \end{aligned}$$

$$\Pr(\bar{M} \mid W) = \frac{0.6}{0.61991} = \underline{0.9679}$$

(4 marks)

(b) We require

$$\begin{aligned} \frac{0.6}{0.4e^{-t/20} + 0.6} &= 0.9 \\ 0.6 &= 0.36e^{-t/20} + 0.6 \times 0.9 \\ 0.06 &= 0.36e^{-t/20} \\ e^{-t/20} &= 1/6 \\ t/20 &= \log 6 \\ t &= 20 \log 6 = \underline{35.8 \text{ minutes}} \end{aligned}$$

(4 marks)