7. (a) Predictive pdf:

$$f_{\text{pred}}(x) = \int_{-\infty}^{\infty} f_\theta(\theta) f_X(x \mid \theta)\; d\theta$$

(3)

(b) i. Predictive probability density function of T_1 :

$$\begin{aligned} f_{\text{pred}}(t) &= \int_0^\infty \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta} \theta e^{-\theta t}\; d\theta \\ &= \int_0^\infty \frac{b^a}{\Gamma(a)} \theta^{a+1-1} e^{-(b+t)\theta}\; d\theta \\ &= \frac{\Gamma(a+1)}{\Gamma(a)} \frac{b^a}{(b+t)^{a+1}} \\ &= \frac{a\; b^a}{(b+t)^{a+1}} \\ &= \frac{a}{b}\left(\frac{b}{b+t}\right)^{a+1} \qquad (0 < t < \infty) \end{aligned}$$

(3)

ii. Joint predictive probability density function of T_1 and T_2 :

$$\begin{aligned} f_{\text{pred}}(t_1, t_2) &= \int_0^\infty \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta} \theta^2 e^{-\theta(t_1+t_2)}\; d\theta \\ &= \int_0^\infty \frac{b^a}{\Gamma(a)} \theta^{a+2-1} e^{-(b+t_1+t_2)\theta}\; d\theta \\ &= \frac{\Gamma(a+2)}{\Gamma(a)} \frac{b^a}{(b+t_1+t_2)^{a+1}} \\ &= \frac{a(a+1)b^a}{(b+t_1+t_2)^{a+2}} \\ &= \frac{a(a+1)}{b^2}\left(\frac{b}{b+t_1+t_2}\right)^{a+2} \qquad (0 < t_1, t_2 < \infty) \end{aligned}$$

(4)

(c) i. Expectation of θ^2 :

$$\begin{aligned} \mathrm{E}(\theta^2) &= \int_0^1 \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \theta^{a+2-1} (1-\theta)^{b-1}\; d\theta \\ &= \frac{\Gamma(a+b)\Gamma(a+2)}{\Gamma(a+b+2)\Gamma(a)} = \frac{(a+1)a}{(a+b+1)(a+b)} \end{aligned}$$

(3)

ii. Probability density function:

$$\begin{aligned} f(\theta) &\propto k_1 \frac{\Gamma(a_1)\Gamma(b_1)}{\Gamma(a_1+b_1)} \frac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1)} \theta^{a_1-1}(1-\theta)^{b_1-1} \\ &\quad + k_2 \frac{\Gamma(a_2)\Gamma(b_2)}{\Gamma(a_2+b_2)} \frac{\Gamma(a_2+b_2)}{\Gamma(a_2)\Gamma(b_2)} \theta^{a_2-1}(1-\theta)^{b_2-1} \\ &\propto q_1 f_1(\theta) + q_2 f_2(\theta) \end{aligned}$$

where $q_1, q_2, f_1(\theta), f_2(\theta)$ are as specified.
Now

$$f(\theta) = k\{q_1 f_1(\theta) + q_2 f_2(\theta)\}$$

and

$$\int_0^1 f(\theta)\ d\theta = 1 = k\{q_1 + q_2\}$$

so

$$k = \{q_1 + q_2\}^{-1}$$

and

$$p_j = \frac{q_j}{q_1 + q_2}.$$

(3)

(d) i.

$$f^{(0)}(\theta) = 1 = k_0 \left[\theta - \frac{(1-\theta)^3}{3}\right]_0^1 = k_0 \left\{1 + \frac{1}{3}\right\} = \frac{4}{3} k_0$$

So

$$\underline{k_0 = \frac{3}{4}}.$$

(3)

ii. Density:

$$\begin{aligned} f^{(0)}(\theta) &= \frac{3}{4}[1 + (1-\theta)^2] \\ &= \frac{3}{4}\left[1 + \frac{\Gamma(1)\Gamma(3)}{\Gamma(4)} \frac{\Gamma(4)}{\Gamma(1)\Gamma(3)} \theta^{1-1}(1-\theta)^{3-1}\right] \\ &= \frac{3}{4} f_1^{(0)}(\theta) + \frac{1}{4} f_2^{(0)}(\theta) \end{aligned}$$

where:
$f_1^{(0)}(\theta)$ is the pdf of a $\underline{\text{beta}(1,1)}$ distribution,
$f_2^{(0)}(\theta)$ is the pdf of a $\underline{\text{beta}(1,3)}$ distribution,

$$p_1^{(0)} = \frac{3}{4} \qquad \text{and} \qquad p_2^{(0)} = \frac{1}{4}.$$

(4)

iii. Prior mean:

$$\mathrm{E}_0(\theta) = \frac{3}{4} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{4} = \frac{7}{16} = \underline{0.4375}$$

Prior variance:

$$\mathrm{E}_0(\theta^2) = \frac{3}{4} \times \frac{2 \times 1}{3 \times 2} + \frac{1}{4} \times \frac{2 \times 1}{5 \times 4} = \frac{1}{4} + \frac{1}{40} = \frac{11}{40} = 0.275$$

$$\mathrm{var}_0(\theta) = 0.275 - 0.4375^2 = \underline{0.0836}$$

(3)

iv. Likelihood proportional to $\theta^4(1-\theta)^6$.
Posterior density proportional to $\theta^4(1-\theta)^6 + \theta^4(1-\theta)^8$which is proportional to

$$\frac{\Gamma(5)\Gamma(7)}{\Gamma(12)} f_1^{(1)}(\theta) + \frac{\Gamma(5)\Gamma(9)}{\Gamma(14)} f_2^{(1)}(\theta)$$

where
$f_1^{(1)}(\theta)$ is the pdf of a $\underline{\text{beta}(5,7)}$ distribution,
$f_2^{(1)}(\theta)$ is the pdf of a $\underline{\text{beta}(5,9)}$ distribution.
If the posterior density is $f^{(1)}(\theta) = p_1^{(1)} f_1^{(1)}(\theta) + p_2^{(1)} f_2^{(1)}(\theta)$, then

$$\begin{aligned} p_1^{(1)} &= \left\{\frac{\Gamma(5)\Gamma(7)}{\Gamma(12)}\right\} \left\{\frac{\Gamma(5)\Gamma(7)}{\Gamma(12)} + \frac{\Gamma(5)\Gamma(9)}{\Gamma(14)}\right\}^{-1} \\ &= \left\{1 + \frac{8 \times 7}{13 \times 12}\right\}^{-1} \\ &= \{1 + 0.35897\}^{-1} \\ &= \underline{0.7358} \end{aligned}$$

and

$$p_2^{(1)} = 1 - p_1^{(1)} = \underline{0.2642}.$$

(4)