Probability Density Functions (pdfs)

The key features of pdfs are

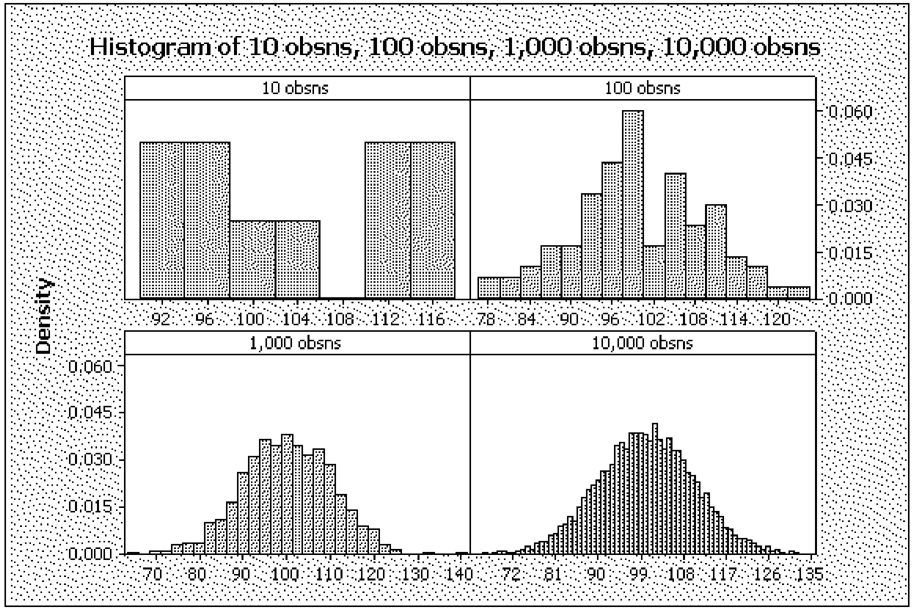
1. pdfs never take negative values

2. the area under a pdf is one: $P(-\infty < X < \infty) = 1$

3. areas under the curve correspond to probabilities

4. $P(X \leq x) = P(X < x)$ since $P(X = x) = 0$.

Continuous Data



Outline

- Probability Density Functions (pdfs)

- The Uniform Distribution

- The Exponential Distribution

Chapter 9

Continuous Probability Models

The Uniform Distribution

General case:

Random variable X with a uniform distribution on a to b has pdf

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

and probabilities can be calculated using the formula

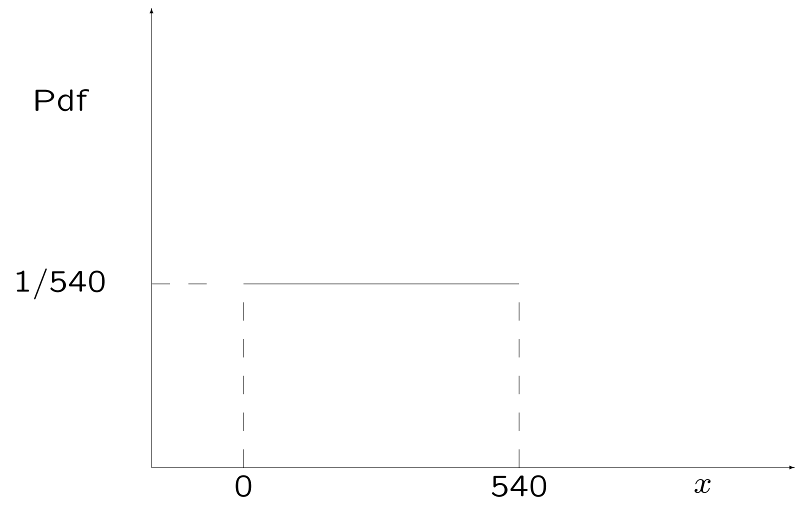
$$P(X \le x) = \begin{cases} 0 & \text{for } x < a \\ \dfrac{x-a}{b-a} & \text{for } a \le x \le b \\ 1 & \text{for } x > b. \end{cases}$$

Example: Environmental Health Officers visit local hotel at a “random” time during the working day (9.00 to 18.00)

- Let X be the time to their arrival at the hotel measured in terms of minutes from the start of the day.

- Then X is a random variable with a uniform distribution between 0 and 540, with pdf

$$f(x) = \begin{cases} \dfrac{1}{540} & \text{for } 0 \le x \le 540 \\ 0 & \text{otherwise.} \end{cases}$$



The Uniform Distribution

- Outcomes measured on a continuous scale.

- All outcomes equally likely.

The Exponential Distribution

General form of pdf

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

and probabilities can be calculated using

$$P(X \leq x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - e^{-\lambda x} & \text{for } x > 0. \end{cases}$$

The Uniform Distribution

Mean:

$$E(X) = \mu = \frac{(a+b)}{2}$$

Variance:

$$Var(X) = \sigma^2 = \frac{(b-a)^2}{12}$$

The Exponential Distribution

- used to model lifetimes of products and times between "random" events
- arrivals of orders, customers in a queueing system, ...
- has one (positive) parameter λ

Probability that the inspectors visit the hotel in the morning (within 180 minutes after 9am) is

$$P(X \leq 180) = \frac{180-0}{540-0} = \frac{1}{3}$$

Probability of a visit during the lunch hour (12.30 to 13.30) is

$$\begin{aligned} P(210 \leq X \leq 270) &= P(X \leq 270) - P(X < 210) \\ &= \frac{270-0}{540-0} - \frac{210-0}{540-0} \\ &= \frac{270-210}{540} \\ &= \frac{60}{540} \\ &= \frac{1}{9}. \end{aligned}$$

Probability that the gap is more than 10 minutes is

$$\begin{aligned} P(X > 10) &= 1 - P(X \leq 10) \\ &= 1 - \left(1 - e^{-0.3\times 10}\right) \\ &= e^{-0.3\times 10} \\ &= 0.050 \end{aligned}$$

Probability that the gap is between 5 and 10 minutes is

$$\begin{aligned} P(5 < X < 10) &= P(X < 10) - P(X \leq 5) \\ &= 0.950 - 0.777 \\ &= 0.173 \end{aligned}$$

The Exponential Distribution

- X = time between use of pay phone
- Model using exponential distribution with $\lambda = 0.3$

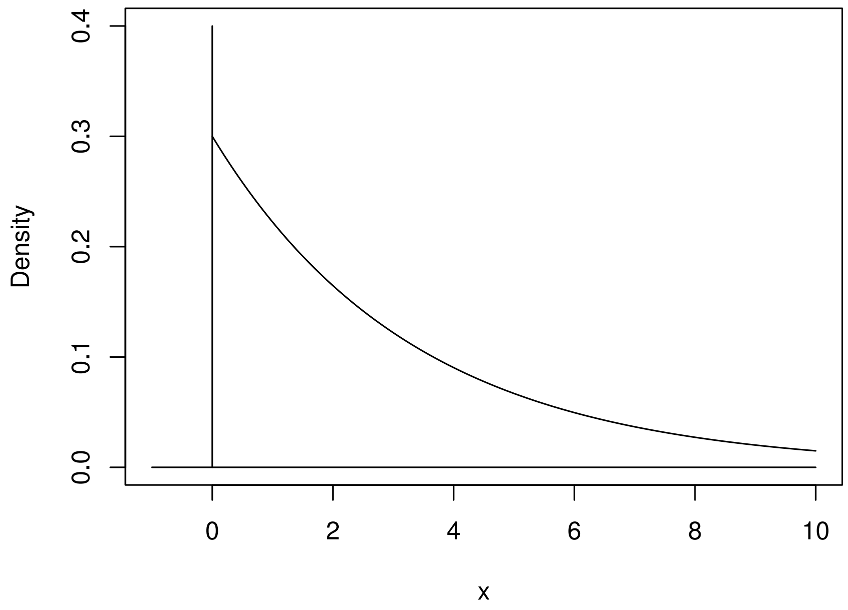
Probability of the gap between phone users being less than 5 minutes is

$$P(X < 5) = 1 - e^{-0.3\times 5} = 1 - 0.223 = 0.777.$$

The Exponential Distribution

Main features

1. it refers to positive quantities: $x > 0$
2. larger values of x are increasingly unlikely – exponential decay
3. the value of λ fixes the rate of decay – larger values of λ correspond to more rapid decay.



The Exponential Distribution

Mean:

$$E(X) = \mu = \frac{1}{\lambda}$$

Variance:

$$Var(X) = \sigma^2 = \frac{1}{\lambda^2}$$