Chapter 8

The Binomial and Poisson Distributions

Outline

- Binomial Distribution

- Poisson Distribution

Binomial Distribution

Scenario

- Each person/item has only two possible (exclusive) responses (Yes/No, Defective/Not defective etc)
  – each *trial* is a *success* or *failure*

- The survey/experiment is a random sample
  – the responses are independent.

- $P(\text{success}) = p$

- $X =$ total number of successes out of n trials

Binomial Distribution

- $X =$ total number of successes out of n trials

- Probability distribution

$$P(X = r) = {}^{n}\mathrm{C}_{r}\, p^{r}(1-p)^{n-r}, \quad r = 0, 1, \ldots, n$$

- $X \sim Bin(n, p)$

- Mean and variance are

$$E(X) = np, \qquad Var(X) = np(1-p)$$

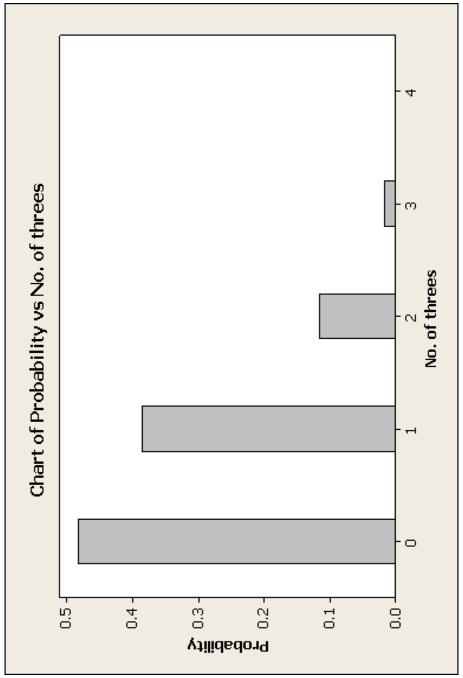
Example: die-rolling experiment

- Success $\leftrightarrow$ getting a three
- $X =$ number of successes out of 4 trials
- $X \sim Bin(4, 1/6)$
- Probability distribution

$$P(X = r) = {}^n\text{C}_r\, p^r (1-p)^{n-r}, \quad r = 0, 1, \ldots, n$$
$$= {}^4\text{C}_r \left(\frac{1}{6}\right)^r \left(1 - \frac{1}{6}\right)^{4-r}, \quad r = 0, 1, 2, 3, 4$$

so that

$$P(X = 0) = {}^4\text{C}_0 \left(\frac{1}{6}\right)^0 \left(1 - \frac{1}{6}\right)^4 = \left(\frac{5}{6}\right)^4 = 0.4823$$
$$P(X = 1) = {}^4\text{C}_1 \left(\frac{1}{6}\right)^1 \left(1 - \frac{1}{6}\right)^3 = 4 \times \frac{1}{6} \times \left(\frac{5}{6}\right)^3 = 0.3858$$

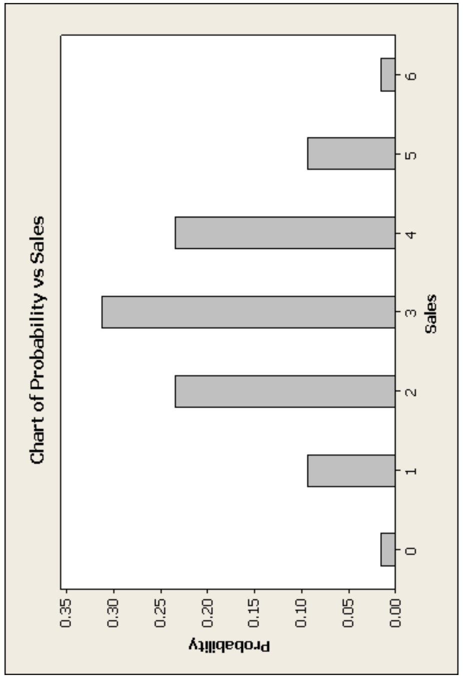


Another example:

A salesperson has a 50% chance of making a sale on a customer visit and she arranges 6 visits in a day. What are the probabilities of her making 0,1,2,3,4,5 and 6 sales?

Let $X =$ number of sales. Assuming the visits result in sales independently, $X \sim Bin(6, 0.5)$ and

No. of sales r	Probability $P(X = r)$	Cumulative Probability $P(X \leq r)$
0	0.015625	0.015625
1	0.093750	0.109375
2	0.234375	0.343750
3	0.312500	0.656250
4	0.234375	0.890625
5	0.093750	0.984375
6	0.015625	1.000000
sum	1.000000	



Poisson Distribution

- Counts of events occurring randomly in time
- $X =$ number of calls to an ISP
- Probability distribution

$$P(X = r) = \frac{\lambda^r \, e^{-\lambda}}{r!}, \quad r = 0, 1, 2, \ldots$$

- $X \sim Po(\lambda)$
- Mean and variance are

$$E(X) = \lambda, \qquad Var(X) = \lambda$$

$X \sim Po(5)$

$$P(X = r) = \frac{5^r \, e^{-5}}{r!}, \quad r = 0, 1, 2, \ldots$$

r	Probability $P(X = r)$	Cumulative Probability $P(X \leq r)$
0	0.0067	0.0067
1	0.0337	0.0404
2	0.0843	0.1247
3	0.1403	0.2650
4	0.1755	0.4405
5	0.1755	0.6160
6	0.1462	0.7622
7	0.1044	0.8666
8	0.0653	0.9319
9	0.0363	0.9682
10	0.0181	0.9863
⋮	⋮	⋮
sum	1.000000	

Example:

An Internet service provider (ISP) has thousands of subscribers, but each one will call with a very small probability. The ISP knows that on average 5 calls will be made in one minute.

Let $X =$ number of calls made in a minute.

Then $X \sim Po(5)$ and

$$\begin{aligned} P(X = r) &= \frac{\lambda^r \, e^{-\lambda}}{r!}, \quad r = 0, 1, 2, \ldots \\ &= \frac{5^r \, e^{-5}}{r!}, \quad r = 0, 1, 2, \ldots \end{aligned}$$

For example,

$$P(X = 4) = \frac{5^4 e^{-5}}{4!} = 0.1755$$