Chapter 7

Discrete Probability Models

Outline

- Counting arguments

- Probability Distributions

- Expectation and Variance

Counting

- How many ways can we list n events?
  - with replacement
  - without replacement

n events can be sequenced

in n^n ways (with replacement):

$$n \times n \times n \times \cdots \times n = n^n$$

eg. $n = 3 \quad \longrightarrow \quad 3^3 = 27$ ways

$$\begin{array}{ccc} A & A & A \\ A & A & B \\ A & A & C \\ A & B & A \\ A & B & B \\ A & B & B \\ \vdots & \vdots & \vdots \\ C & C & C \end{array}$$

n events can be sequenced

in $n!$ ways (without replacement):

$$n(n-1)(n-2)(n-3) \times \cdots \times 3 \times 2 \times 1 = n!$$

eg. $n = 3 \quad \longrightarrow \quad 3! = 3 \times 2 \times 1 = 6$ ways

$$\begin{array}{ccc} A & B & C \\ A & C & B \\ B & A & C \\ B & C & A \\ B & A & B \\ C & B & A \end{array}$$

Choosing r from n events

Order important: permutation

$r = 2$ from $n = 3$:

$$A,\ B \qquad A,\ C \qquad B,\ A \qquad B,\ C \qquad C,\ A \qquad C,\ B.$$

$${}^{n}\mathsf{P}_{r} = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)$$

$$= \frac{n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1}{(n-r) \times (n-r-1) \times \cdots \times 3 \times 2 \times 1}$$

$$= \frac{n!}{(n-r)!}.$$

Order not important: combination

$r = 2$ from $n = 3$:

$$A,\ B \qquad A,\ C \qquad B,\ C.$$

$$^{n}\mathsf{C}_r = \frac{\text{number of ordered samples of size } r}{\text{number of orderings of samples of size } r}$$

$$= \frac{^{n}\mathsf{P}_r}{^{r}\mathsf{P}_r}$$

$$= \frac{^{n}\mathsf{P}_r}{r!}$$

$$= \frac{n!}{r!(n-r)!}.$$

National Lottery

- 49 balls

- Match 3, 4, 5, 5 plus bonus ball, 6 balls

- Order not important

- Number of different outcomes is

$$^{49}C_6 = 13,983,816.$$

$$P(\text{Match exactly 3 balls}) = \frac{{}^{6}C_{3}\ {}^{43}C_{3}}{{}^{49}C_{6}}$$

$$= \frac{246,820}{13,983,816}$$

$$\simeq 0.0177.$$

Number of balls matched	Probability	Prize
6	0.00000007	£2.4M
5 plus bonus	0.0000004	£240K
5	0.00002	£3K
4	0.0001	£100
3	0.0177	£10
< 3	0.981	£0

$$\begin{aligned} EMV &= 2.4M \times \frac{1}{13{,}983{,}816} + 240K \times \frac{6}{13{,}983{,}816} \\ &\quad + \ldots + 10 \times \frac{246{,}820}{13{,}983{,}816} \\ &= 0.6176. \end{aligned}$$

Fair price: 62p + Good causes + Camelot profit

Probability Distributions

- Random variable X

  $X = \text{Voting intention}$

- Interested in $P(X = x)$

- Probability distribution – list of all values X can take and the probability they occur

  $P(X = \text{Labour}) = 0.4, \quad P(X = \text{Conservative}) = 0.3,$

  $P(X = \text{Lib Dem}) = 0.3$

- Random variable X – upper case X

- Observation x – lower case X

- Event $X = x$

$X =$ outcome of a roll of a die

x	$P(X = x)$
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6
sum	1

Expectation of a Discrete Random Variable

Recall

$$EMV = \sum P(\text{Event}) \times \text{ Monetary value of Event}$$

Expectation of a discrete random variable is

$$E(X) = \mu = \sum x\, P(X = x)$$

Die-rolling experiment:

$$E(X) = \sum x\, P(X = x) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + \ldots + 6 \times \frac{1}{6} = 3.5$$

Variance of a Discrete Random Variable

The variance of a discrete random variable is

$$Var(X) = \sigma^2 = E\left[(X-\mu)^2\right] = E\left(X^2\right) - \mu^2$$

Die-rolling experiment: $\mu = 3.5$

x	$P(X=x)$	$(x-\mu)^2$	$(x-\mu)^2 P(X=x)$
1	1/6	6.25	1.0417
2	1/6	2.25	0.3750
3	1/6	0.25	0.0417
4	1/6	0.25	0.0417
5	1/6	2.25	0.3750
6	1/6	6.25	1.0417
sum	1		2.9167

$\rightarrow Var(X) = 2.9167$ and $SD(X) = \sqrt{Var(X)} = 1.7078$.

Simulation of die-rolling using Minitab.

1. Select *Calc* $\rightarrow$ *Random Data* $\rightarrow$ *Integer*.

2. Enter the number of simulations (rows of data), e.g. 100.

3. Enter the minimum (1) and maximum (6).

4. Enter the column for the results (e.g. `c1`).

We can then, for example,

- Make a bar chart of the results.

- Look at how the average score develops.