Chapter 10

The Normal Distribution

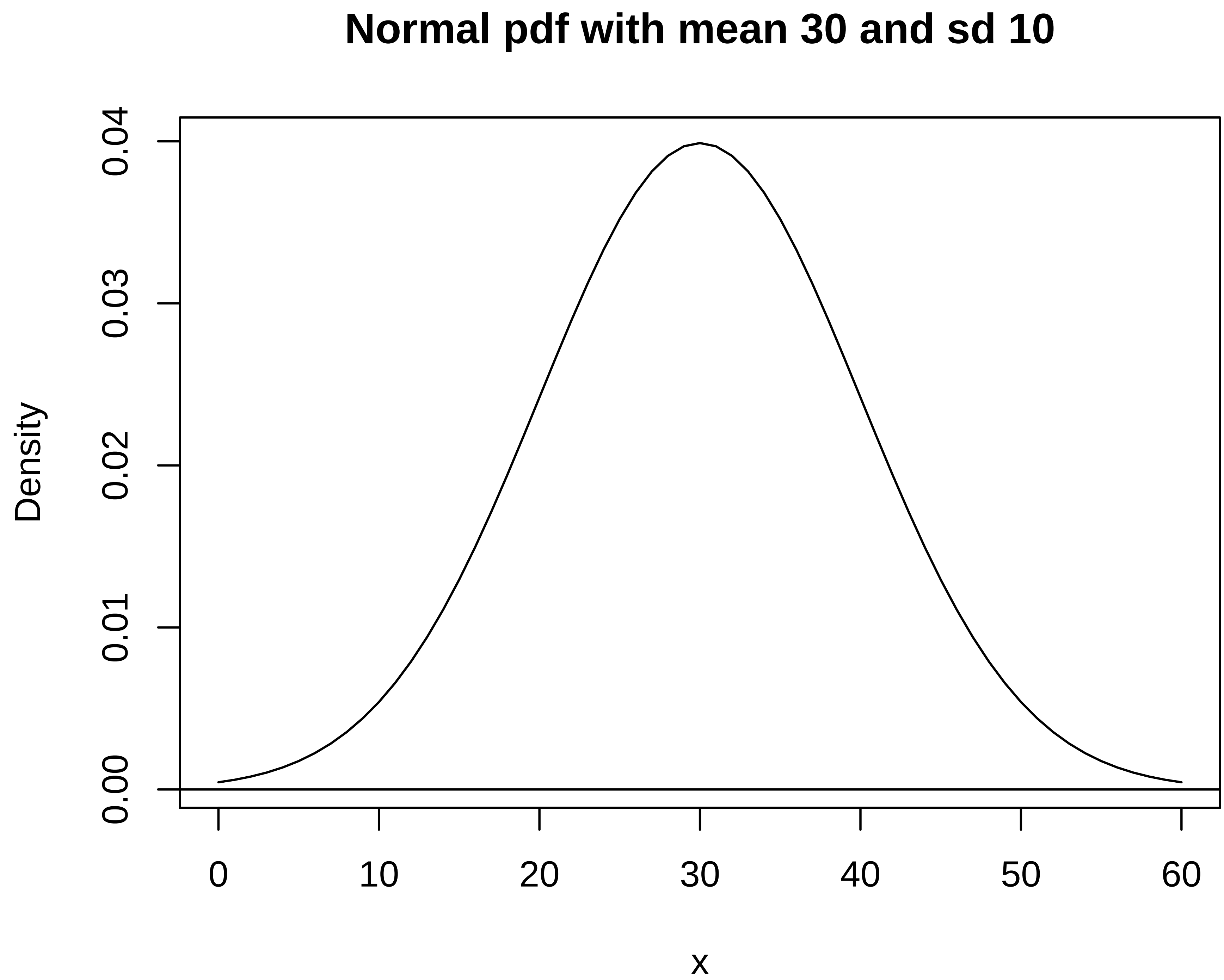
Outline

- The Normal Distribution.
- The Standard Normal Distribution.
- Calculating Probabilities

The Normal Distribution

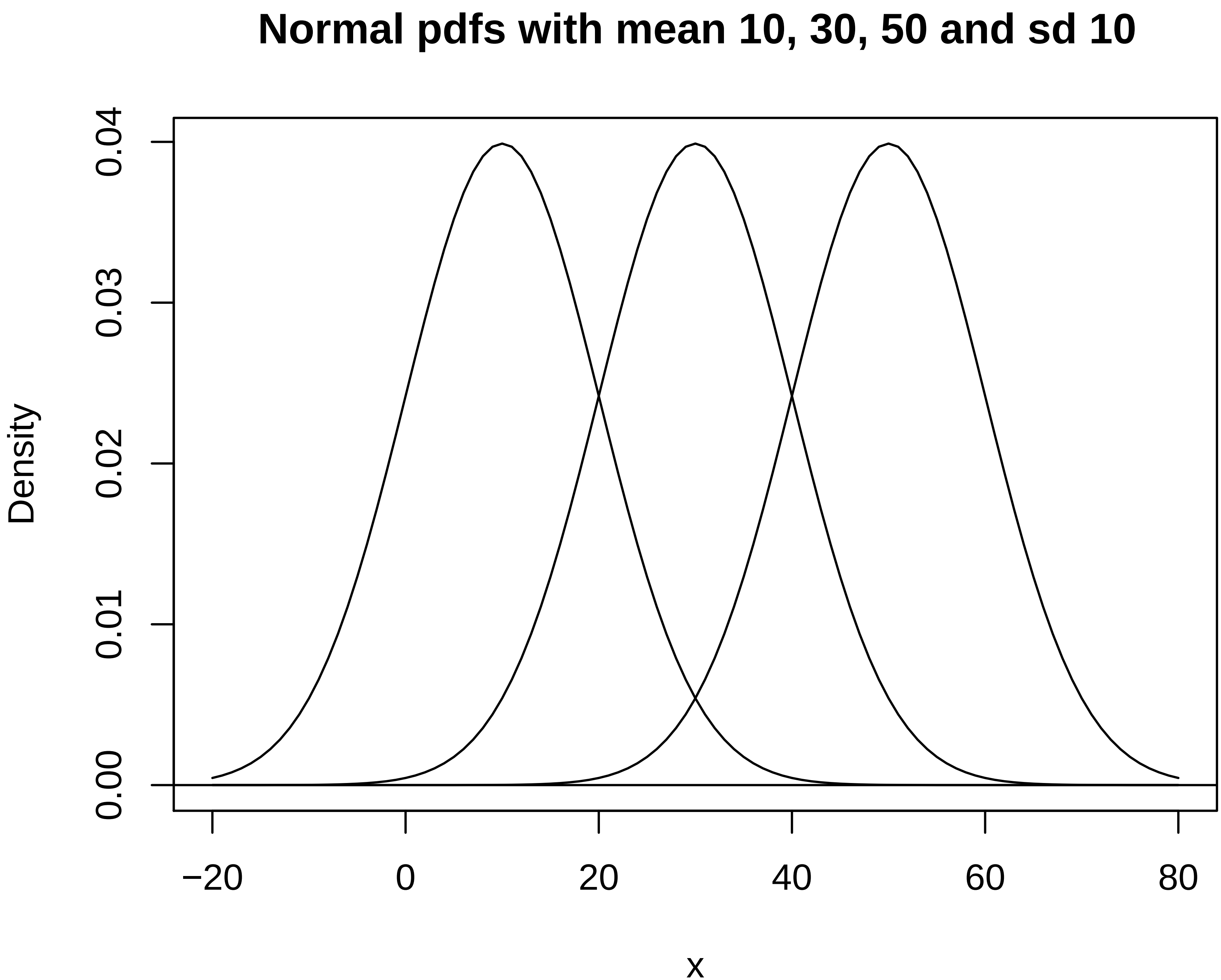
There are four important characteristics of the normal distribution.

- It is symmetrical about it's mean, μ.
- It's mean, median and mode all coincide.
- The area under the curve is equal to 1.
- The curve extends in both direction to infinity (∞).

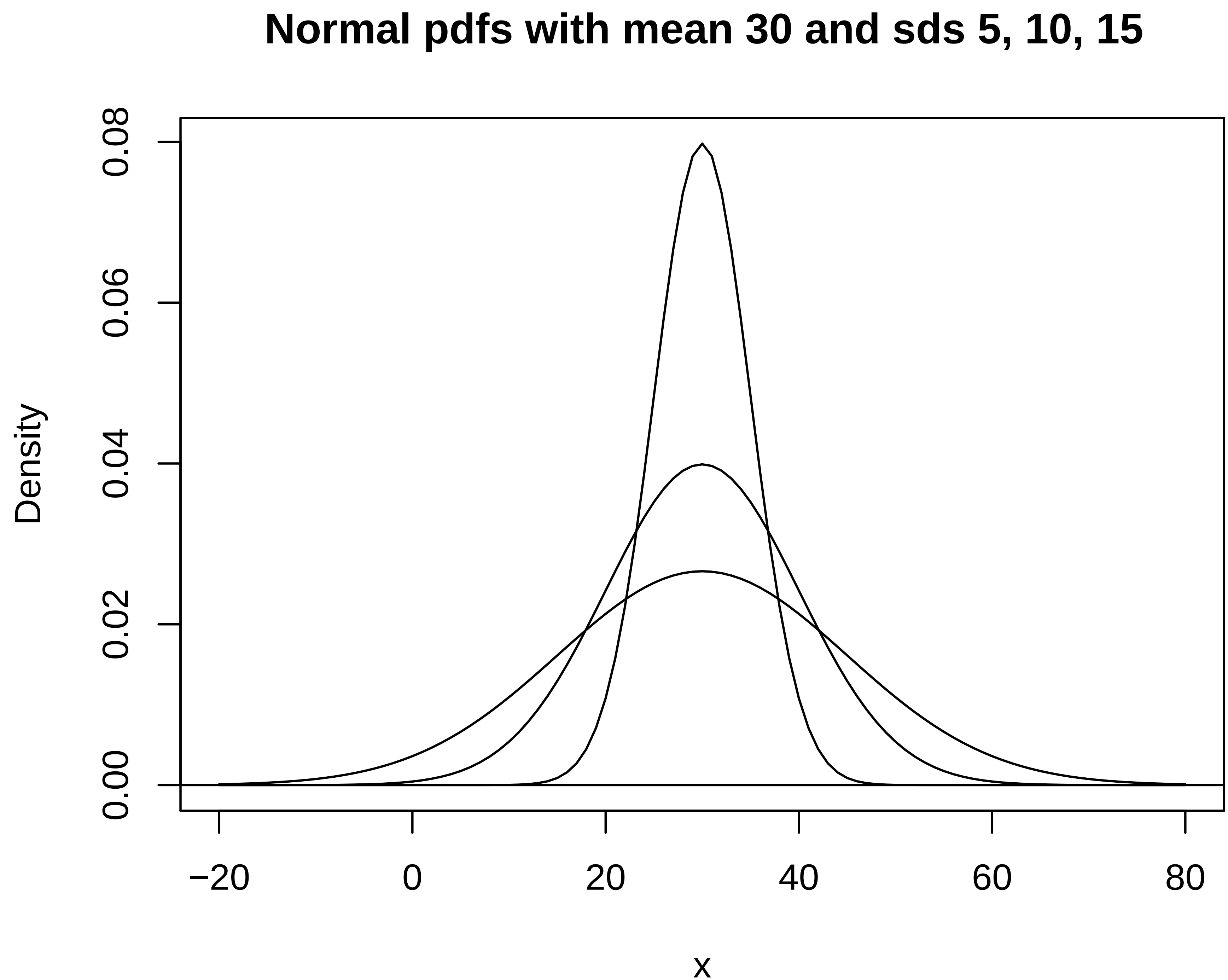

Normal pdf with mean 30 and sd 10
Density
0.00
0.01
0.02
0.03
0.04
0
10
20
30
40
50
60
x

Notation for the Normal Distribution

$$X \sim N\left(\mu, \sigma^2\right).$$

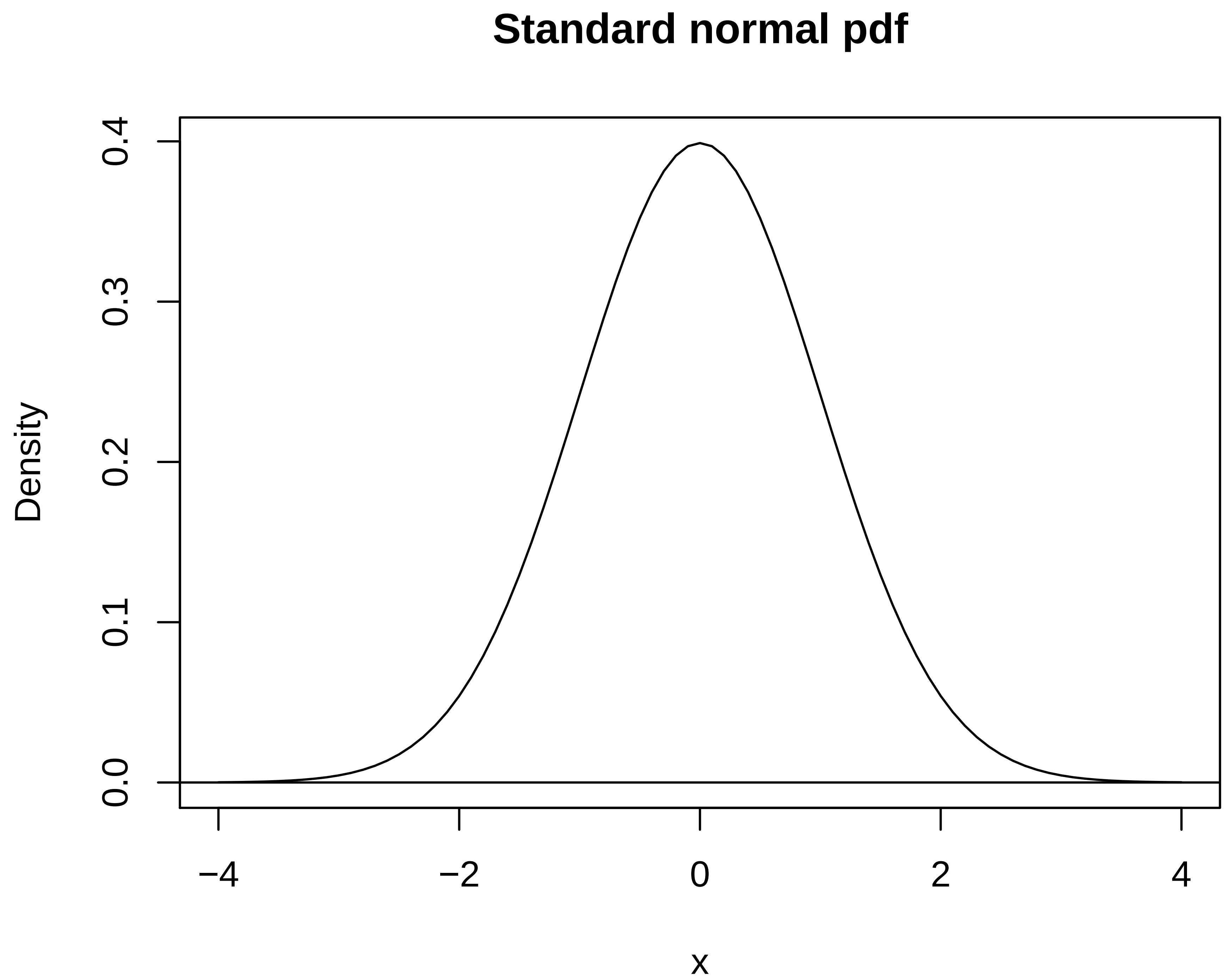


Normal pdfs with mean 10, 30, 50 and sd 10
Density
0.00
0.01
0.02
0.03
0.04
−20
0
20
40
60
80
x


Normal pdfs with mean 30 and sds 5, 10, 15
Density
0.00
0.02
0.04
0.06
0.08
−20
0
20
40
60
80
x

The Standard Normal Distribution

$$Z \sim N(0, 1).$$


Standard normal pdf
Density
0.0
0.1
0.2
0.3
0.4
−4
−2
0
2
4
x

Calculating Probabilities under the Standard Normal Distribution

1. Find the row of interest.

2. Find the column of interest.

3. Read off the probability.

Standard Normal Distribution Tables

$P(Z < z)$, where $Z \sim N(0, 1)$.

z	−0.09	−0.08	−0.07	−0.06	−0.05	−0.04	−0.03	−0.02	−0.01	0.00
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
−1.4	0.0681	0.0694	0.0708	0.0721	0.0735	0.0749	0.0764	0.0778	0.0793	0.0808
−1.3	0.0823	0.0838	0.0853	0.0869	0.0885	0.0901	0.0918	0.0934	0.0951	0.0968
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Standardising Random Variables

Any random variable X with mean, μ and variance, σ^2 can be standardised as follows

1. Take the variable X.

2. Subtract the mean, μ.

3. Divide by the standard deviation, σ.

Hence a random variable X becomes a standard normal random variable Z.

$$Z = \frac{X - \mu}{\sigma}$$

Example

The weights of packages have a normal distribution with mean 4kg and standard deviation 0.6kg. Find the probability that a package weighs more than 4.75kg.

<u>Solution</u>

$$X \sim N(4, 0.6^2)$$

$$\Pr(X > 4.75) = \Pr\left(\frac{X-4}{0.6} > \frac{4.75-4}{0.6}\right) = \Pr(Z > 1.25)$$

From tables $\Pr(Z < 1.25) = 0.8944$ so
$\Pr(Z > 1.25) = 1 - 0.8944 = 0.1056$.

So the probability that a package weighs more than 4.75 kg is 0.1056.

Example

The weights of the packages are independent. Find the probability that the total weight of 10 packages is less than 42kg.

<u>Solution</u>

Let the total weight be T.

$$T \sim N(10 \times 4,\ 10 \times 0.6^2)$$

That is

$$T \sim N(40,\ 3.6)$$

$$\Pr(T < 42) = \Pr\left(\frac{T - 40}{\sqrt{3.6}} < \frac{42 - 40}{\sqrt{3.6}}\right) = \Pr(Z < 1.054)$$

From tables $\Pr(Z < 1.05) = 0.8531$ and $\Pr(Z < 1.06) = 0.8554$ so $\Pr(Z < 1.054) \approx 0.8531 + 0.4 \times (0.8554 - 0.8531) = 0.8540$.

So the probability that the total weight is less than 42 kg is 0.8540.

Example

Find an approximate value for the probability that, out of a batch of 80 packages, fewer than 10 weigh more than 4.75kg.

Solution

The probability that a package weighs more than 4.75 kg is 0.1056. Let Y be the number in the batch which weigh more than 4.75kg. Then

$$Y \sim \text{Bin}(80, 0.1056)$$

Now $np = 80 \times 0.1056 = 8.448$ and $n(1-p) = 80 \times (1 - 0.1056) = 80 \times 0.8944 = 71.552$. Therefore we can use a normal approximation with $\mu = np = 8.448$ and $\sigma^2 = np(1-p) = 7.556$. The standard deviation is $\sqrt{7.556} = 2.749$.

The probability that fewer than 10 packages weigh more than 4.75kg is

$$\Pr\left(Z < \frac{9.5 - 8.448}{2.749}\right) = \Pr(Z < 0.3827)$$

From tables $\Pr(Z < 0.38) = 0.6480$ and $\Pr(Z < 0.39) = 0.6517$ so $\Pr(Z < 0.3827) = 0.6480 + 0.27 \times (0.6517 - 0.6480) = 0.6470.$

The required probability is approximately 0.647.